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The blocking lemma and group incentive compatibility for matching with contracts

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HIGHLIGHTS

- We show that the blocking lemma for matchings with contracts holds.
- The preexisting blocking lemmas for matchings without contracts are special cases of our result.
- As an immediate consequence of the blocking lemma, we show that the doctor-optimal stable mechanism is group strategy-proof for doctors.

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ABSTRACT

This paper considers a general class of two-sided many-to-one matching markets, so-called matching markets with contracts. We study the blocking lemma and group incentive compatibility for this class of matching markets. We first show that the blocking lemma for matching with contracts holds if hospitals' choice functions satisfy substitutes and the law of aggregate demand. The blocking lemma for one-to-one matching (Gale and Sotomayor, 1985) and that for many-to-one matching (Martínez et al., 2010) are special cases of this result. Then, as an immediate consequence of the blocking lemma, we show that the doctor-optimal stable mechanism is group strategy-proof for doctors if hospitals' choice functions satisfy substitutes and the law of aggregate demand. Hatfield and Kojima (2009) originally obtain this result by skillfully using the strategy-proofness of the doctor-optimal stable mechanism. In this paper we provide a different proof for the group incentive compatibility by applying the blocking lemma.

1. Introduction

Hatfield and Milgrom (2005) present a unified framework of matching with contracts,[1] which includes the two-sided matching model of Gale and Shapley (1962) and the labor market model of Kelso and Crawford (1982) as special cases. Matching with contracts model actually formulates a synthesis of many known results in a coherent framework, and it has given impetus to a good deal of theoretical research. Hatfield and Milgrom (2005) introduce the substitutes condition of contracts, which generalizes the substitutability condition in the matching literature (Roth and Sotomayor, 1990). They also introduce a new condition they call the law of aggregate demand, which states that hospitals choose more contracts as the set of contracts to choose from expands. They show that the doctor-offering generalized Gale–Shapley algorithm produces the doctor-optimal stable allocation if hospitals' preferences satisfy substitutes and the law of aggregate demand. They also obtain that the doctor-optimal stable mechanism is strategy-proof for doctors under these same hospitals' preference conditions. Furthermore, Hatfield and Kojima (2009) extend this incentive compatibility result. They show that the doctor-optimal stable mechanism is *group* strategy-proof for doctors if hospitals' preferences satisfy substitutes and the law of aggregate demand. That is, no group of doctors can make each of its members strictly better off by jointly misreporting their preferences. Indeed, this group incentive compatibility for matching with contracts generalizes all preexisting corresponding results for matching without contracts. Dubins and Freedman (1981) and Roth (1982) first obtain strategy-proofness result in one-to-one matching markets without contracts. In fact, Dubins and Freedman (1981) also show

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[1] For earlier literature, Roth (1984) and Fleiner (2003) have studied the matching model with contracts.

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group strategy-proofness for one-to-one matching without contracts. Martínez et al. (2004) obtain group strategy-proofness in many-to-one matching without contracts when hospitals' preferences satisfy substitutes and q-separability, a condition stronger than the law of aggregate demand.

For matching problems without contracts, the blocking lemma is an important result, which identifies a particular blocking pair for any unstable and individually rational matching that is preferred by some agents of one side of the market to their optimal stable matching. Using the blocking lemma for one-to-one matching,[2] Gale and Sotomayor (1985) give a short proof for the group strategy-proofness of the deferred acceptance algorithm. For many-to-one matching, the blocking lemma holds under responsive preferences.[3] The responsiveness seems too restrictive to be satisfied. For a weak preference restriction, Martínez et al. (2010) show that the corresponding blocking lemma holds under substitutable and quota-separable preference.[4] For matching with contracts, Hatfield and Kojima (2009) obtain the group incentive compatibility for doctors in a straightforward way (not relying on the corresponding blocking lemma). In the existing literature, there is no investigation on a version of the blocking lemma for the model with contracts. Our first result will fill this gap. We give a characterization of the blocking lemma for matching with contracts, which says that, if the set of doctors that strictly prefer an individually rational allocation $\widetilde{X}$ to the doctor-optimal stable allocation $\bar{X}$ is nonempty, then there must exist some particular set of contracts X' and hospital h such that h blocks $\widetilde{X}$ via X'. Indeed, the blocking lemma specifies the blocking set of contracts X' as the hospital h's chosen set from the union of $\widetilde{X}$ and a particular singleton (contract) set. This result actually states that only one additional contract is needed to block the given individually rational allocation. The blocking lemma for matchings with contracts generalizes the counterpart for matchings without contracts. We then show that the blocking lemma for matchings with contracts holds if hospitals' choice functions satisfy substitutes and the law of aggregate demand (Lemma 1). Since substitutes and the law of aggregate demand are very weak conditions and the preexisting blocking lemmas are obtained under much stronger preference condition, the existing blocking lemmas for matchings without contracts (for instance, Gale and Sotomayor, 1985 and Martínez et al., 2010) are special cases of our first result.

Furthermore, as a direct application of the blocking lemma, we prove that the doctor-optimal stable mechanism is group strategy-proof for doctors if hospitals' choice functions satisfy substitutes and the law of aggregate demand.

The remainder of the present paper is organized as follows. We present some preliminaries on the formal model in the next section. In Section 3 we study the blocking lemma for matching with contracts. In Section 4 we discuss the group incentive compatibility for matching with contracts by applying the blocking lemma. We give the concluding remark in Section 5. All proofs are provided in the Appendix.

2. The model

We mostly follow the notations of Hatfield and Milgrom (2005). There are finite sets D and H of doctors and hospitals. X is the set of contracts. Each contract $x \in X$ is associated with one doctor $x_D \in D$ and one hospital $x_H \in H$. We assume that each doctor can sign at most one contract. The null contract, meaning that the doctor has no contract, is denoted by $\varnothing$. For each $d \in D$, $\succ_d$ is a strict preference relation on $\{x \in X|x_D = d\} \cup \{\varnothing\}$. A contract is *acceptable* if it is preferred to the null contract and *unacceptable* if it is not preferred to the null contract. For any $x, x' \in X$ with $x_D = x'_D = d$, $x \succ_d x'$ denotes that doctor d prefers x to x' and, $x \succeq_d x'$ denotes either $x \succ_d x'$ or $x = x'$. For each $d \in D$ and $X_1 \subset X$, we define the *chosen set* $C_d(X_1)$ by

$$C_d(X_1) = \begin{cases} \varnothing & \text{if all } x \in X_1 \text{ such that } x_D = d \text{ are unacceptable to } d, \\ \max_{\succ_d}\{x \in X_1|x_D = d\} & \text{otherwise,} \end{cases}$$

for any $X_1 \subset X$. Let $C_D(X_1) = \bigcup_{d\in D} C_d(X_1)$ be the set of contracts chosen from X_1 by some doctor in D. The doctors' rejected set from X_1 is denoted by $R_D(X_1) = X_1 \setminus C_D(X_1)$.

We write $\succ \equiv (\succ_d)_{d\in D}$ to denote a preference profile of doctors. We also write $(\succ)_{-d}$ to denote $(\succ_{d'})_{d'\in D\setminus\{d\}}$ for $d \in D$, and $(\succ)_{D'}$ to denote $(\succ_d)_{d\in D'}$ and $(\succ)_{-D'}$ to denote $(\succ_d)_{d\in D\setminus D'}$ for $D' \subset D$.

We allow each hospital to sign multiple contracts. For hospitals, we follow a standard approach in recent literature to take the choice functions as primitives.[5] For each hospital h, its *chosen set* is a subset of the contracts that name it, that is, $C_h(X_1) \subset \{x \in X_1|x_H = h\}$. In addition, a hospital can sign only one contract with any given doctor: $(\forall h \in H)$ $(\forall X_1 \subset X)$ $(\forall x, x' \in C_h(X_1))$ $x \neq x' \Rightarrow x_D \neq x'_D$.[6] Let $C_H(X_1) = \bigcup_{h\in H} C_h(X_1)$ be the set of contracts chosen from X_1 by some hospital in H. The hospitals' rejected set is denoted by $R_H(X_1) = X_1 \setminus C_H(X_1)$.

A set of contracts $X_1 \subset X$ is *feasible* if $x, x' \in X_1$ and $x \neq x'$ imply $x_D \neq x'_D$. In words, a set of contracts is *feasible* if each doctor signs at most one contract. For two feasible sets of contracts X_1 and X_2, we say that doctor d prefers X_1 to X_2 if $C_d(X_1) \succ_d C_d(X_2)$. For a set of contracts $X_1 \subset X$, we denote by $|X_1|$ the number of non-null contracts in X_1. For the null contract $\varnothing$, we stipulate that $|\{\varnothing\}| = 0$ and $|X_1 \cup \{\varnothing\}| = |X_1|$.

Definition 1. A feasible set of contracts $X_1 \subset X$ is a *stable allocation* (or a *stable set of contracts*) if it is both

(1) *Individually Rational:* $C_D(X_1) = C_H(X_1) = X_1$, and
(2) *Unblocked:* there exists no hospital h and set of contracts $X_2 \neq C_h(X_1)$ such that $X_2 = C_h(X_1 \cup X_2) \subset C_D(X_1 \cup X_2)$.

[2] Gale and Sotomayor attribute the formulation of the lemma to J.S. Hwang.

[3] Roth (1985) introduces responsiveness of preference relations for college admission problems. Specifically, responsiveness means that, for any two subsets of doctors that differ in only one doctor, a hospital prefers the subset containing the most-preferred doctor. Formally, we say a hospital h's preference relation is responsive if for any d_1, d_2 and any S such that $d_1, d_2 \notin S$ and $|S| < q_h$, we have $S \cup \{d_1\} \succ_h S \cup \{d_2\}$ if and only if $\{d_1\} \succ_h \{d_2\}$, where d_1, d_2 are the partners of h and S is a set of partners of h. It is easy to obtain that the responsiveness is stronger than the substitutability.

[4] Barberà et al. (1991) propose the concept of separable preference, which has been extensively used in matching models, see, for instance, Alkan (2001), Dutta and Massó (1997), Ehlers and Klaus (2003), Martínez et al. (2000), Martínez et al. (2001), Martínez et al. (2004) and Papai (2000). Based on this condition, Martínez et al. (2010) propose a new concept called quota-separability. Formally, A hospital h's preference relation $\succ_h$ over sets of workers is quota q_h-separable if: (i) for all $S \subsetneq D$ such that $|S| < q_h$ and $d \notin S$, $(S \cup \{d\}) \succ_h S$ if and only if $\{d\} \succ_h \varnothing$. (ii) $\varnothing \succ_h S$ for all S such that $|S| > q_h$.

[5] Taking choice functions as primitives of a matching market has many advantages. For detailed illustration see, for instance, Chambers and Yenmez (2013).

[6] In order to guarantee the existence of stable matchings, economists usually impose the substitutes condition and irrelevance of rejected contracts (IRC) condition on hospitals' choice functions. Chambers and Yenmez (2013) use consistency instead of IRC, which says that: $C_h(Y) \subset Z \subset Y$ implies $C_h(Y) = C_h(Z)$. We develop our arguments under assumption of substitutes condition and the law of aggregate demand (LAD) on hospitals' choice functions. Aygün and Sönmez (2013) note that the substitutes condition and the law of aggregate demand together imply IRC condition.

When condition (2) is violated by some hospital h and X_2, we say that h *blocks* X_1 *via* X_2. For any $X_1 \subset X$ and $h \in H$, define the rejected set by $R_h(X_1) = X_1 \setminus C_h(X_1)$. $R_h(X_1)$ is the set of contracts in X_1 which h is willing to reject.

Definition 2. Contracts are *substitutes* for hospital h if we have $R_h(X_2) \subset R_h(X_1)$ for all $X_2 \subset X_1 \subset X$.

In other words, contracts are substitutes if any expansion of the choice set never induces a hospital to take a contract it previously rejected.

Definition 3. The choice functions of hospital $h \in H$ satisfy the *law of aggregate demand* if for all $X_2 \subset X_1 \subset X$, $|C_h(X_2)| \leq |C_h(X_1)|$.

According to this definition, if the set of possible contracts expands, then the total number of contracts chosen by hospital h either rises or stays the same.

Based on fixed point theory for finite lattices, Hatfield and Milgrom (2005) propose the generalized Gale–Shapley algorithm for matching with contracts to produce a stable allocation. In fact, the generalized Gale–Shapley algorithm is a cumulative offer process, which is defined as the iterated applications of a certain function. Let X_D denote the doctors' opportunity set and X_H the hospitals' opportunity set. We consider the doctor-offering algorithm, which starts at $(X_D, X_H) = (X, \varnothing)$. Let $X_H(t)$ be the cumulative set of contracts offered by the doctors to the hospitals through iteration t, and let $X_D(t)$ be the set of contracts that have not yet been rejected by the hospitals through iteration t. Then, the contracts "held" at the end of iteration t are precisely those that have been offered but not rejected, which are those in $X_D(t) \cap X_H(t)$. The doctor-offering algorithm initiates with $X_D(0) = X$ and $X_H(0) = \varnothing$ and proceeds by

$$X_D(t) = X \setminus [R_H(X_H(t-1))],$$
$$X_H(t) = X \setminus [R_D(X_D(t))].$$

Once a fixed point, denoted by $(\bar{X}_D, \bar{X}_H)$,[7] of this process is found, we have a stable set of contracts $\bar{X}_D \cap \bar{X}_H$. Hatfield and Milgrom (2005) obtain that $\bar{X} = \bar{X}_D \cap \bar{X}_H$ is the optimal stable allocation for doctors if hospitals' preferences satisfy substitutes.[8] Specifically,

Result 1 (*See Hatfield and Milgrom, 2005*)**.** *If hospitals' choice functions satisfy substitutes and the law of aggregate demand, then $\bar{X} = \bar{X}_D \cap \bar{X}_H$ is a stable allocation which every doctor weakly prefers to any other stable allocation.*

Hatfield and Kojima (2009) show that no individually rational allocation is preferred by all the doctors to the doctor-optimal stable allocation if hospitals' preferences satisfy substitutes and the law of aggregate demand. This result is known as "weak Pareto optimality" in the literature. Let $\bar{X}$ denote the doctor-optimal stable allocation. The "weak Pareto optimality" of $\bar{X}$ can be expressed as follows.

Result 2 (*See Hatfield and Kojima, 2009*)**.** *Suppose that hospitals' choice functions satisfy substitutes and the law of aggregate demand. Let $\bar{X}$ be the doctor-optimal stable allocation. For any individually rational allocation X_1 and any $D' \subset D$, if each doctor d in D' prefers $C_d(X_1)$ to $C_d(\bar{X})$, then $D \setminus D' \neq \emptyset$.*

7 Hatfield and Milgrom (2005) call it the highest fixed point.

8 Aygün and Sönmez (2013) note that the substitutes condition is not sufficient for the existence of stable allocations. They propose the irrelevance of rejected contracts (IRC) condition and obtain the existence of stable allocation under IRC and substitutes condition. They also note that the substitutes condition and the law of aggregate demand together imply IRC condition. Then the substitutes condition and the law of aggregate demand together ensure the existence of stable allocation.

3. The blocking lemma

For the matching problem without contracts, the blocking lemma identifies a particular blocking pair for any non-stable and individually rational matching that is preferred by some agents of one side of the market to their optimal stable matching. Gale and Sotomayor (1985) prove the blocking lemma for one-to-one matching. For the many-to-one matching problem, the blocking lemma can be easily obtained by the decomposition lemma (see Roth and Sotomayor, 1990) if the responsive preference condition is satisfied. Recently, Martínez et al. (2010) prove the same result under a weak preference condition (i.e., substitutable and quota-separable preference profiles).

We know that, for the matching problem with contracts, the preference restriction of substitutes and law of aggregate demand is weaker than responsiveness or substitutability plus quota-separability. A natural question, then, is for the matching problem with contracts, whether the blocking lemma holds under the substitutes condition and the law of aggregate demand. In this section, we give a positive answer to this question.

With the "weak Pareto optimality" of the doctor-optimal stable allocation, we can present the blocking lemma for matching with contracts. For purpose of comparison, we first recall the blocking lemma for matching without contracts as follows: If the set of doctors that strictly prefer an individually rational matching μ to the doctor-optimal stable matching μ_D is nonempty then, there must exist a blocking pair (d, h) of μ with the property that, under the matching μ, h is matched with some doctor who strictly prefers μ to μ_D, and d considers μ_D being at least as good as μ. As a generalization, we state the blocking lemma for matching problems with contracts as follows:

Definition 4 (*The Blocking Lemma*)**.** Let $\bar{X}$ be the doctor-optimal stable allocation and $\widetilde{X}$ be any individually rational allocation. We assume $D' \subset D$ is a nonempty set of all doctors who prefer their chosen contract in $\widetilde{X}$ to that in $\bar{X}$. We say that *the blocking lemma for matching with contracts holds* if there is a contract $x \in X$ and a hospital $h \in H$ such that

(1) $x_H = h$,
(2) h blocks $\widetilde{X}$ via $C_h(\widetilde{X} \cup \{x\})$,
(3) $C_h(\widetilde{X}) \cap C_{D'}(\widetilde{X}) \neq \emptyset$, that is, there is at least one doctor in D' who is matched to h under $\widetilde{X}$.
(4) $x_D \in D \setminus D'$.

It is easy to check that the blocking lemma for matching with contracts generalizes the counterpart of matching problem without contracts. The agents x_D and h mentioned in the above definition actually constitute a blocking pair for matching problem without contracts.

With the above preparation, we state our first result as follows.

Lemma 1. *If hospitals' choice functions satisfy substitutes and the law of aggregate demand, then the blocking lemma for matching with contracts holds.*

We give the proof of Lemma 1 in the Appendix.

Note that substitutes and the law of aggregate demand are very weak conditions and the preexisting blocking lemmas are obtained under much stronger preference condition, the existing blocking lemmas for matching without contracts (for instance, Gale and Sotomayor, 1985 and Martínez et al., 2010) are special cases of Lemma 1.

4. The group strategy-proofness

In this section, we will study the group incentive compatibility for matching with contracts. Hatfield and Kojima (2009) first investigate this problem and show that the doctor-optimal stable mechanism is group strategy-proof for doctors if the preferences of every hospital satisfy substitutes and the law of aggregate demand. Here we will give another proof for this property by applying the blocking lemma.

For completeness, we specify the definition of group strategy-proofness for doctors. Given the hospitals H, their choice functions and doctors D, a *mechanism* F is a function from any reported doctors' preference profile $\succ$ to a feasible set of contracts. The *doctor-optimal stable mechanism* is a mechanism which, for any reported preference profile $\succ$, produces the doctor-optimal stable allocation under $\succ$. A mechanism F is *group strategy-proof for doctors* if, for any preference profile $\succ$, there is no group of doctors $D' \subset D$ and a preference profile $(\succ')_{D'} \equiv (\succ'_d)_{d\in D'}$ such that $F((\succ')_{D'}, (\succ)_{-D'})$ is strictly preferred to $F(\succ)$ by all members of D'.

We now state and prove the last result.

Theorem 1. *Suppose that hospitals' choice functions satisfy substitutes and the law of aggregate demand. Then the doctor-optimal stable mechanism is group strategy-proof for doctors.*

Proof. Suppose that doctors' true preferences are given by the preference profile $\succ$. Let $\bar{X} = F(\succ)$ be the doctor-optimal stable allocation. We assume by contradiction that there is a non-empty set of doctors $D' \subset D$ and a preference profile $(\succ')_{D'} = (\succ'_d)_{d\in D'}$ such that $F((\succ')_{D'}, (\succ)_{-D'})$ is strictly preferred to $F(\succ)$ by all members of D'. Without loss of generality, we assume that D' contains all doctors $d \in D$ with $\succ'_d \neq \succ_d$. That is, D' contains all doctors who misreport their preferences and become strictly better off.

Let $\widetilde{X} \equiv F((\succ')_{D'}, (\succ)_{-D'})$. Clearly $\widetilde{X}$ is individually rational under the true preference profile $\succ$. The weak Pareto optimality of the doctor-optimal stable mechanism implies that D' is a proper subset of D. Applying Lemma 1, there exists a contract $x \in X$ and a hospital $h \in H$ satisfying the blocking lemma, i.e., (1) $x_H = h$, (2) h blocks $\widetilde{X}$ via $C_h(\widetilde{X} \cup \{x\})$, (3) $C_h(\widetilde{X}) \cap C_{D'}(\widetilde{X}) \neq \emptyset$, (4) $x_D \in D \setminus D'$.

According to that h blocks $\widetilde{X}$ via $C_h(\widetilde{X} \cup \{x\})$ under the true preference profile $\succ$, by definition we know that $C_h(\widetilde{X} \cup \{x\}) = C_h(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\})) \subset C_D(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}))$ holds under $\succ$. Since the hospital h's choice function is unchanged, $C_h(\widetilde{X} \cup \{x\}) = C_h(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}))$ still holds under the reported preference profile $((\succ')_{D'}, (\succ)_{-D'})$. $x_D \in D \setminus D'$ implies that x_D does not misreport her preference. Therefore, under both $\succ$ and $((\succ')_{D'}, (\succ)_{-D'})$ the chosen set $C_{x_D}(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}))$ keeps unchanged. From the fact that $\widetilde{X}$ is a stable allocation under $((\succ')_{D'}, (\succ)_{-D'})$, we can infer that there is at most one contract in $\widetilde{X}$ involving any doctor in D. Then for each doctor in $D \setminus \{x_D\}$, there is at most one contract in $\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\})$ to choose. Combining the fact that $\widetilde{X}$ is individually rational under $\succ$, we obtain that, under preference profile $\succ$ and $((\succ')_{D'}, (\succ)_{-D'})$, the set of contracts $C_{D\setminus\{x_D\}}(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}))$ keeps unchanged. And consequently, the chosen set $C_D(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\})) = C_{D\setminus\{x_D\}}(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\})) \cup C_{x_D}(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}))$ will be the same under both $\succ$ and $((\succ')_{D'}, (\succ)_{-D'})$. Then we have $C_h(\widetilde{X} \cup \{x\}) = C_h(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\})) \subset C_D(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}))$ still holds under $((\succ')_{D'}, (\succ)_{-D'})$. That is, h blocks $\widetilde{X}$ via $C_h(\widetilde{X} \cup \{x\})$ under the preference profile $((\succ')_{D'}, (\succ)_{-D'})$. This contradicts the assumption that $\widetilde{X}$ is the doctor-optimal stable allocation under $((\succ')_{D'}, (\succ)_{-D'})$. The proof is completed. □

5. Concluding remarks

This paper considers the matching markets with contracts. We study the blocking lemma and group incentive compatibility for this class of matching markets. We show that the blocking lemma holds if hospitals' choice functions satisfy the substitutes condition and the law of aggregate demand, which extends the corresponding result for matching without contracts to the matching with contracts setting. Then, as a direct consequence of the blocking lemma, we show that the doctor-optimal stable mechanism is group strategy-proof for doctors if hospitals' choice functions satisfy substitutes and the law of aggregate demand.

At the end of this paper, we note that the law of aggregate demand condition plays a crucial role for the blocking lemma and group incentive compatibility. Without the law of aggregate demand, the substitutes condition alone cannot guarantee the blocking lemma or group strategy-proofness for doctors. To see this, we consider the following example taken from Hatfield and Milgrom (2005).

Example 1. We assume there are two hospitals $H = \{h_1, h_2\}$ and three doctors $D = \{d_1, d_2, d_3\}$, where contracts are simply elements of $D \times H$ (which is essentially equivalent to the case without contracts). Let doctors' preferences be:

$$\succ_{d_1}: h_1 \succ h_2 \succ \varnothing, \qquad \succ_{d_2}: h_2 \succ h_1 \succ \varnothing,$$
$$\succ_{d_3}: h_2 \succ h_1 \succ \varnothing.$$

The hospitals h_1's choice function is given by

$C_{h_1}(\{d_1\}) = \{d_1\}$	$C_{h_1}(\{d_1, d_2\}) = \{d_1, d_2\}$	$C_{h_1}(\{d_1, d_2, d_3\}) = \{d_3\}$
$C_{h_1}(\{d_2\}) = \{d_2\}$	$C_{h_1}(\{d_1, d_3\}) = \{d_3\}$	
$C_{h_1}(\{d_3\}) = \{d_3\}$	$C_{h_1}(\{d_2, d_3\}) = \{d_3\}$	

The hospitals h_2's choice function is given by

$C_{h_2}(\{d_1\}) = \{d_1\}$	$C_{h_2}(\{d_1, d_2\}) = \{d_1\}$	$C_{h_2}(\{d_1, d_2, d_3\}) = \{d_1\}$
$C_{h_2}(\{d_2\}) = \{d_2\}$	$C_{h_2}(\{d_1, d_3\}) = \{d_1\}$	
$C_{h_2}(\{d_3\}) = \{d_3\}$	$C_{h_2}(\{d_2, d_3\}) = \{d_2\}$	

Using this example, Hatfield and Milgrom (2005) illustrate that the law of aggregate demand is indispensable for strategy-proofness. Thus the law of aggregate demand is also indispensable for group strategy-proofness. Indeed, the law of aggregate demand is also indispensable for the blocking lemma. Specifically, it is easy to check that the doctor-optimal stable allocation is $\bar{X} = \{(d_1, h_2), (d_3, h_1)\}$. For another individually rational allocation $\widetilde{X} = \{(d_1, h_1), (d_2, h_1), (d_3, h_2)\}$, one can see that each doctor in D prefers her corresponding contract in $\widetilde{X}$ to that in $\bar{X}$. Then $D' = D$. The blocking lemma does not hold. It is easy to check that both hospitals' choice functions satisfy substitutes, but h_1's choice function does not satisfy the law of aggregate demand. This example shows that the law of aggregate demand is an indispensable condition for the blocking lemma.

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Appendix

In order to complete the proof of Lemma 1, we first show the following result.

Proposition 1. *Suppose that hospitals' choice functions satisfy substitutes and the law of aggregate demand. Let $\bar{X}$ be the doctor-optimal stable allocation and $\widetilde{X}$ an individually rational allocation. Suppose the set of doctors $D' \subset D$ who prefer $\widetilde{X}$ to $\bar{X}$ is nonempty. Then there exist some set of contracts X' and hospital h such that*

(1) *h blocks $\widetilde{X}$ via X',*
(2) *$C_h(\widetilde{X}) \cap C_{D'}(\widetilde{X}) \neq \emptyset$, that is, h signs a contract with at least one doctor in D' under $\widetilde{X}$,*
(3) *there exists at least one contract $x \in X'$ such that $x_H = h$ and $x_D \in D \setminus D'$.*

Proof of Proposition 1. Let D denote the set of doctors, H the set of hospitals and X denote the set of contracts. We assume that $\bar{X}$ denotes the doctor-optimal stable allocation and $\widetilde{X}$ denotes the given individually rational allocation. We assume that the nonempty set $D' \subset D$ consists of all doctors who prefer $\widetilde{X}$ to $\bar{X}$. We construct two sets as follows.

$H_1 \equiv \{h \in H|$ there exists some contract $x \in \bar{X}$ such that $x_D \in D'$ and $x_H = h\}$ and,

$H_2 \equiv \{h \in H|$ there exists some contract $x \in \widetilde{X}$ such that $x_D \in D'$ and $x_H = h\}$.

In words, H_1 is the set of hospitals that are matched with at least one doctor belonging to D' under $\bar{X}$ and H_2 is the set of hospitals that are matched with at least one doctor belonging to D' under $\widetilde{X}$.

Let $\bar{X}' \equiv \{x \in \bar{X}|x_D \in D'\}$ and $\widetilde{X}' \equiv \{x \in \widetilde{X}|x_D \in D'\}$. Since each doctor d in D' prefers $\widetilde{X}$ to $\bar{X}$ and $\bar{X}$ is individually rational, under $\widetilde{X}$ each doctor $d \in D'$ signs a contract which is preferred to the null contract $\varnothing$. Thus we have $|\widetilde{X}'| \geq |\bar{X}'|$, that is, the number of (non-null) contracts signed by doctors in D' under $\widetilde{X}$ is no less than that under $\bar{X}$. Then, to complete the proof, we only need to consider the following two cases:

Case I. *There exists at least one hospital, say $h \in H$, such that*

$$|C_h(\bar{X}) \cap \bar{X}'| < |C_h(\widetilde{X}) \cap \widetilde{X}'|, \tag{1}$$

that is, h is matched with more contracts involving doctors in D' under $\widetilde{X}$ than that under $\bar{X}$.

Proof for Case I. Let $(\bar{X}_D, \bar{X}_H)$ be the highest fixed point of the generalized Gale–Shapley algorithm. We claim that there exists at least one contract $x \in X$ such that $x \in [C_h(\bar{X}_H \cup \widetilde{X})] \cap \{[C_h(\bar{X}) \cap (\bar{X} \setminus \bar{X}')] \setminus \widetilde{X}\}$.

Specifically, by the law of aggregate demand we have

$$|C_h(\bar{X}_H \cup \widetilde{X})| \geq |C_h(\widetilde{X})|. \tag{2}$$

It is easy to see that

$$|C_h(\widetilde{X})| = |C_h(\widetilde{X}) \cap \widetilde{X}'| + |C_h(\widetilde{X}) \cap [\widetilde{X} \setminus \widetilde{X}']|. \tag{3}$$

Then a combination of (1)–(3) derives

$$|C_h(\bar{X}_H \cup \widetilde{X})| > |C_h(\bar{X}) \cap \bar{X}'| + |C_h(\widetilde{X}) \cap [\widetilde{X} \setminus \widetilde{X}']|. \tag{4}$$

According to the substitutes condition, we obtain that

$$C_h(\bar{X}_H \cup \widetilde{X}) \subset \{[C_h(\bar{X}_H) \setminus \widetilde{X}] \cup C_h(\widetilde{X})\}. \tag{5}$$

It is known that $C_h(\bar{X}_H) = C_h(\bar{X})$, we can take a decomposition as follows

$$C_h(\bar{X}_H) \setminus \widetilde{X} = \{[C_h(\bar{X}) \cap \bar{X}'] \setminus \widetilde{X}\} \cup \{[C_h(\bar{X}) \cap (\bar{X} \setminus \bar{X}')] \setminus \widetilde{X}\}. \tag{6}$$

Since each doctor $d \in D'$ prefers $\widetilde{X}$ to $\bar{X}$, it follows by the generalized Gale–Shapley algorithm that $[C_h(\widetilde{X}) \cap \widetilde{X}'] \subset [\bar{X}_H \setminus C_h(\bar{X}_H)]$. Combining the substitutes condition, we have

$$[C_h(\widetilde{X}) \cap \widetilde{X}'] \subset [(\bar{X}_H \cup \widetilde{X}) \setminus C_h(\bar{X}_H \cup \widetilde{X})]. \tag{7}$$

Therefore, a combination of (5), (6), (7) and $C_h(\widetilde{X}) = [C_h(\widetilde{X}) \cap \widetilde{X}'] \cup \{C_h(\widetilde{X}) \cap [\widetilde{X} \setminus \widetilde{X}']\}$ derives the following inclusion relation

$$C_h(\bar{X}_H \cup \widetilde{X}) \subset \{[C_h(\bar{X}) \cap \bar{X}'] \setminus \widetilde{X}\} \cup \{[C_h(\bar{X}) \cap (\bar{X} \setminus \bar{X}')] \setminus \widetilde{X}\} \cup \{C_h(\widetilde{X}) \cap [\widetilde{X} \setminus \widetilde{X}']\}. \tag{8}$$

Which, together with (4) and $|[C_h(\bar{X}) \cap \bar{X}'] \setminus \widetilde{X}| \leq |C_h(\bar{X}) \cap \bar{X}'|$ implies that

$$[C_h(\bar{X}_H \cup \widetilde{X})] \cap \{[C_h(\bar{X}) \cap (\bar{X} \setminus \bar{X}')] \setminus \widetilde{X}\} \neq \emptyset. \tag{9}$$

Since $[C_h(\bar{X}_H \cup \widetilde{X})] \cap \{[C_h(\bar{X}) \cap (\bar{X} \setminus \bar{X}')] \setminus \widetilde{X}\}$ is non-empty, we can choose a contract, say x, in $[C_h(\bar{X}_H \cup \widetilde{X})] \cap \{[C_h(\bar{X}) \cap (\bar{X} \setminus \bar{X}')] \setminus \widetilde{X}\}$. Then $x \in C_h(\bar{X}_H \cup \widetilde{X})$, $x \in [C_h(\bar{X}) \cap (\bar{X} \setminus \bar{X}')]$ and $x \notin \widetilde{X}$. Let $X' \equiv C_h(\widetilde{X} \cup \{x\})$. We will show that h blocks $\widetilde{X}$ via X'.

Firstly, we show $X' \neq C_h(\widetilde{X})$. The substitutes condition and $x \in C_h(\bar{X}_H \cup \widetilde{X})$ imply $x \in C_h(\widetilde{X} \cup \{x\}) = X'$. On the other hand, $x \notin \widetilde{X}$ implies $x \notin C_h(\widetilde{X})$. Thus we obtain $X' \neq C_h(\widetilde{X})$. In addition, we note that the above-mentioned contract x is a desirable contract for Proposition 1. Indeed, it is easy to check that $x_H = h$ and $x_D \in D \setminus D'$.

Secondly, we show $X' = C_h(X' \cup \widetilde{X})$. By $x \in C_h(\widetilde{X} \cup \{x\}) = X'$, it is easy to show that $X' \cup \widetilde{X} = \widetilde{X} \cup \{x\}$. Then $C_h(X' \cup \widetilde{X}) = C_h(\widetilde{X} \cup \{x\}) = X'$.

Thirdly, we show $X' \subset C_D(X' \cup \widetilde{X})$. According to $X' \cup \widetilde{X} = \widetilde{X} \cup \{x\}$, we only need to show $X' \subset C_D(\{x\} \cup \widetilde{X})$. For arbitrary $z \in X'$, we consider the following two cases:

Case (1). $z_D \neq x_D$.

According to that $\widetilde{X}$ is an individually rational allocation, we know $\widetilde{X}$ is feasible. Then $z \in \widetilde{X}$ implies that z is the only available contract for z_D in $\widetilde{X}$. By $z_D \neq x_D$, then z is the only contract in $\{x\} \cup \widetilde{X}$ that is available for z_D. Then by the individual rationality of $\widetilde{X}$ and $z = C_{z_D}(\widetilde{X})$, we obtain that $z = C_{z_D}(\{x\} \cup \widetilde{X})$, and consequently, $z \in C_D(\{x\} \cup \widetilde{X})$.

Case (2). $z_D = x_D$.

Since $z \in X' = C_h(\widetilde{X} \cup \{x\})$ and h can sign only one contract with any given doctor, we have $z = x$. Now we show that $x \in C_D(\{x\} \cup \widetilde{X})$. Indeed, $x_D \in D \setminus D'$ and $x \notin \widetilde{X}$ imply that $x \succ_{x_D} C_{x_D}(\widetilde{X})$. Then $x = C_{x_D}(\widetilde{X} \cup \{x\})$, and consequently, $x \in C_D(\{x\} \cup \widetilde{X})$.

Then we obtain that $z \in X'$ implies $x \in C_D(\{x\} \cup \widetilde{X})$, that is, $X' \subset C_D(\{x\} \cup \widetilde{X})$. The proof is completed.

Case II. *For each hospital $h' \in H$, $|C_{h'}(\bar{X}) \cap \bar{X}'| = |C_{h'}(\widetilde{X}) \cap \widetilde{X}'|$.*

Proof for Case II. We assume that, for each hospital $h' \in H$, $|C_{h'}(\bar{X}) \cap \bar{X}'| = |C_{h'}(\widetilde{X}) \cap \widetilde{X}'|$. It is easy to see that $H_1 = H_2$. We denote $H' \equiv H_1 = H_2$. Since each doctor $d \in D'$ strictly prefers her corresponding contract in $\widetilde{X}$ to that in $\bar{X}$ and $\bar{X}$ is individually rational, we obtain that each doctor in D' signs a non-null contract under $\widetilde{X}$. This implies $|\widetilde{X}'| = |D'|$. And by assumption that $H_1 = H_2$ and for each hospital $h' \in H'$, $|C_{h'}(\bar{X}) \cap \bar{X}'| = |C_{h'}(\widetilde{X}) \cap \widetilde{X}'|$, we obtain $|\bar{X}'| = |\widetilde{X}'| = |D'|$. That is, each doctor in D' signs a non-null contract under $\bar{X}$.

We assume that, in the generalized Gale–Shapley algorithm, $x \in \bar{X}'$ is the last contract in $\bar{X}'$ proposed by its corresponding doctor (if there are more than one such contracts, we choose anyone among them). For simplicity, we assume x_D proposes x to $x_H = h$ at step k in the procedure of the generalized Gale–Shapley algorithm. Since each doctor in D' signs a non-null contract under $\bar{X}$, we know from the assumption on step k that, in the procedure of the generalized Gale–Shapley algorithm, each doctor $d \in D'$ is always matched with a contract $x' \in \bar{X}'$ (with $x'_D = d$) and there is no hospital which rejects any contract involving doctor in D' from step k to the termination of the algorithm.

Let $X_H(k)$ denote the hospitals' opportunity set at step k. We consider $X_H(k-1)$. Let $\widehat{X} \equiv C_H(X_H(k-1))$ and $\widehat{X}' \equiv \{x \in \widehat{X} | x_D \in D'\}$. In the following, we complete the proof by slightly changing the proof of Case I. Particularly, we only replace $\bar{X}_H$, $\bar{X}$ and $\bar{X}'$ in the proof of Case I (not all) with $X_H(k-1)$, $\widehat{X}$ and $\widehat{X}'$, respectively. For completeness, we give the details as follows.

For the above-mentioned contract x and $h = x_H$, we know $x \in C_h(\bar{X}) \cap \bar{X}'$, but $x \notin X_H(k-1)$. Then $x \notin C_h(X_H(k-1)) \cap \widehat{X}'$. Let $X'_H(k-1) \equiv \{x \in X_H(k-1) | x_D \in D'\}$ and $\bar{X}'_H \equiv \{x \in \bar{X}_H | x_D \in D'\}$. According to the substitutes condition and the assumption on step k of the generalized Gale–Shapley algorithm, we can infer that $R_h(\bar{X}_H) \cap \bar{X}'_H = R_h(X_H(k-1)) \cap X'_H(k-1)$. It follows that $|C_h(X_H(k-1)) \cap \widehat{X}'| < |C_h(\bar{X}) \cap \bar{X}'|$. And by $|C_h(\bar{X}) \cap \bar{X}'| = |C_h(\widetilde{X}) \cap \widetilde{X}'|$, we have

$$|C_h(X_H(k-1)) \cap \widehat{X}'| < |C_h(\widetilde{X}) \cap \widetilde{X}'|. \tag{10}$$

We claim that there exists at least one contract $x \in X$ such that $x \in [C_h(X_H(k-1) \cup \widetilde{X})] \cap \{[C_h(X_H(k-1)) \cap (\widehat{X} \setminus \widehat{X}')] \setminus \widetilde{X}\}$.

Specifically, by the law of aggregate demand we have

$$|C_h(X_H(k-1) \cup \widetilde{X})| \geq |C_h(\widetilde{X})|. \tag{11}$$

Then a combination of (3), (10) and (11) derives

$$|C_h(X_H(k-1) \cup \widetilde{X})| > |C_h(X_H(k-1)) \cap \widehat{X}'| + |C_h(\widetilde{X}) \cap [\widetilde{X} \setminus \widetilde{X}']|. \tag{12}$$

According to the substitutes condition, we obtain that

$$C_h(X_H(k-1) \cup \widetilde{X}) \subset \{[C_h(X_H(k-1)) \setminus \widetilde{X}] \cup C_h(\widetilde{X})\}. \tag{13}$$

We take the following decomposition:

$$C_h(X_H(k-1)) \setminus \widetilde{X} = \{[C_h(X_H(k-1)) \cap \widehat{X}'] \setminus \widetilde{X}\} \cup \{[C_h(X_H(k-1)) \cap (\widehat{X} \setminus \widehat{X}')] \setminus \widetilde{X}\}. \tag{14}$$

Since each doctor $d \in D'$ prefers $\widetilde{X}$ to $\bar{X}$, it follows by the generalized Gale–Shapley algorithm and the assumption on step k that $[C_h(\widetilde{X}) \cap \widetilde{X}'] \subset [X_H(k-1) \setminus C_h(X_H(k-1))]$. Combining the substitutes condition, we have

$$[C_h(\widetilde{X}) \cap \widetilde{X}'] \subset [(X_H(k-1) \cup \widetilde{X}) \setminus C_h(X_H(k-1) \cup \widetilde{X})]. \tag{15}$$

Therefore, a combination of (13), (14), (15) and $C_h(\widetilde{X}) = [C_h(\widetilde{X}) \cap \widetilde{X}'] \cup \{C_h(\widetilde{X}) \cap [\widetilde{X} \setminus \widetilde{X}']\}$ derives the following inclusion relation

$$\begin{aligned} C_h(X_H(k-1) \cup \widetilde{X}) \subset\ & \{[C_h(X_H(k-1)) \cap \widehat{X}'] \setminus \widetilde{X}\} \\ & \cup \{[C_h(X_H(k-1)) \cap (\widehat{X} \setminus \widehat{X}')] \setminus \widetilde{X}\} \\ & \cup \{C_h(\widetilde{X}) \cap [\widetilde{X} \setminus \widetilde{X}']\}. \end{aligned}$$

Which, together with (12) and $|[C_h(X_H(k-1)) \cap \widehat{X}'] \setminus \widetilde{X}| \leq |C_h(X_H(k-1)) \cap \widehat{X}'|$ implies that

$$[C_h(X_H(k-1) \cup \widetilde{X})] \cap \{[C_h(X_H(k-1)) \cap (\widehat{X} \setminus \widehat{X}')] \setminus \widetilde{X}\} \neq \emptyset. \tag{16}$$

Now let $X' \equiv C_h(\{C_h(X_H(k-1) \cup \widetilde{X}) \setminus [C_h(X_H(k-1)) \cap \widehat{X}']\} \cup \widetilde{X})$. We will show that h blocks $\widetilde{X}$ via X'.

Firstly, we show $X' \neq C_h(\widetilde{X})$. By (16) we can choose a contract, say x, in $[C_h(X_H(k-1) \cup \widetilde{X})] \cap \{[C_h(X_H(k-1)) \cap (\widehat{X} \setminus \widehat{X}')] \setminus \widetilde{X}\}$. Then $x \in C_h(X_H(k-1) \cup \widetilde{X})$, $x \in [C_h(X_H(k-1)) \cap (\widehat{X} \setminus \widehat{X}')]$ and $x \notin \widetilde{X}$. By $x \in [C_h(X_H(k-1)) \cap (\widehat{X} \setminus \widehat{X}')]$ we have $x \notin [C_h(X_H(k-1)) \cap \widehat{X}']$, and consequently, $x \in C_h(X_H(k-1) \cup \widetilde{X}) \setminus [C_h(X_H(k-1)) \cap \widehat{X}']$. Then by the substitutes condition and $[\{C_h(X_H(k-1) \cup \widetilde{X}) \setminus [C_h(X_H(k-1)) \cap \widehat{X}']\} \cup \widetilde{X}] \subset [X_H(k-1) \cup \widetilde{X}]$ we have $x \in C_h(\{C_h(X_H(k-1) \cup \widetilde{X}) \setminus [C_h(X_H(k-1)) \cap \widehat{X}']\} \cup \widetilde{X}) = X'$. On the other hand, we know $x \notin \widetilde{X}$, then $x \notin C_h(\widetilde{X})$. Thus we obtain $X' \neq C_h(\widetilde{X})$. In addition, we note that the above-mentioned contract x is a desirable contract for Proposition 1. Indeed, it is easy to check that $x_H = h$ and $x_D \in D \setminus D'$.

Secondly, we show $X' = C_h(X' \cup \widetilde{X})$. By the construction of X', we know $X' \subset [\{C_h(X_H(k-1) \cup \widetilde{X}) \setminus [C_h(X_H(k-1)) \cap \widehat{X}']\} \cup \widetilde{X}]$. Then $[X' \cup \widetilde{X}] \subset [\{C_h(X_H(k-1) \cup \widetilde{X}) \setminus [C_h(X_H(k-1)) \cap \widehat{X}']\} \cup \widetilde{X}]$. It follows by the substitutes condition that $X' \equiv C_h(\{C_h(X_H(k-1) \cup \widetilde{X}) \setminus [C_h(X_H(k-1)) \cap \widehat{X}']\} \cup \widetilde{X}) \subset C_h(X' \cup \widetilde{X})$. By the law of aggregate demand, we have $|X'| \geq |C_h(X' \cup \widetilde{X})|$. Then it holds $X' = C_h(X' \cup \widetilde{X})$.

Thirdly, we show $X' \subset C_D(X' \cup \widetilde{X})$. For arbitrary $x \in X'$, we consider the following two cases:

Case (1). $x_D = d \in D'$.

We claim that $x \in \widetilde{X}$. Suppose not, by the construction of X' we obtain that $x \in X_H(k-1)$ and $x \in C_h(X_H(k-1) \cup \widetilde{X})$. It follows by the substitutes condition that $x \in C_h(X_H(k-1))$. $x_D = d \in D'$ implies $x \in \widehat{X}'$. Thus $x \notin \{C_h(X_H(k-1) \cup \widetilde{X}) \setminus [C_h(X_H(k-1)) \cap \widehat{X}']\}$. Then we have $x \in \widetilde{X}$, as $x \in X'$. It is a contradiction. Since h can sign only one contract with any given doctor, $x \in X'$ implies that x is the only contract in X' that is available for d. According to that $\widetilde{X}$ is feasible, we know from $x \in \widetilde{X}$ that x is the only contract in $\widetilde{X}$ that is available for d. Therefore, x is the only contract in $X' \cup \widetilde{X}$ that is available for d. Then $x \in C_d(X' \cup \widetilde{X})$, and consequently, $x \in C_D(X' \cup \widetilde{X})$.

Case (2). $x_D = d \in D \setminus D'$.

There are two possible situations:

(a) There exists no contract $x' \in C_h(X_H(k-1))$ such that $x'_D = d$.

By assumption, it follows that $x \notin C_h(X_H(k-1))$. We assert that $x \in \widetilde{X}$. Specifically, if $x \notin X_H(k-1)$, then $x \in X'$ implies $x \in \widetilde{X}$. If $x \in X_H(k-1)$, by $x \notin C_h(X_H(k-1))$ and the substitutes condition we infer $x \notin C_h(X_H(k-1) \cup \widetilde{X})$. Then $x \in X'$ implies $x \in \widetilde{X}$. One can obtain that x is the only contract in $X' \cup \widetilde{X}$ that is available for d. Therefore, $x \in C_d(X' \cup \widetilde{X})$, and consequently $x \in C_D(X' \cup \widetilde{X})$.

(b) There exists some contract $x' \in C_h(X_H(k-1))$ such that $x'_D = d$.

We consider two cases: $x = x'$ and $x \neq x'$.

If $x = x'$, since $x' \in C_h(X_H(k-1))$ and $x'_D = x_D = d \in D \setminus D'$, we infer that $x = x' \succeq_d C_d(\bar{X}) \succeq_d C_d(\widetilde{X})$. By assumptions that $x \in X' = C_h(X' \cup \widetilde{X})$, $x_D = d$ and h can sign only one contract with d, we know that there exists no contract $x'' \in X'$ such that $x'' \neq x$ and $x''_D = d$. Then x is the most preferred contract of d in set $X' \cup \widetilde{X}$, and hence $x \in C_D(X' \cup \widetilde{X})$.

If $x' \neq x$, then $x \in \widetilde{X}$. Suppose not, then $x \in X_H(k-1) \setminus \widetilde{X}$. Since $x \neq x'$, $x_D = x'_D = d$ and $x' \in C_h(X_H(k-1))$, we have $x \in R_h(X_H(k-1))$. By substitutes condition, it follows that $x \notin C_h(X_H(k-1) \cup \widetilde{X})$. Then $x \notin X'$, as $x \notin \widetilde{X}$. We reach a contradiction. $x \in X'$ indicates that x is the only available contract in X' for d, and $x \in \widetilde{X}$ indicates that x is the only available contract in $\widetilde{X}$ for d. Then x is the only available contract in $X' \cup \widetilde{X}$ for d, and consequently, $x \in C_D(X' \cup \widetilde{X})$. The proof is completed.

A combination of the above two cases completes the proof of Proposition 1. □

Proof of Lemma 1. We use notations $x, h, \widetilde{X}$ as given in Proposition 1 and Definition 4. According to Proposition 1, we only need to prove that h blocks $\widetilde{X}$ via $C_h(\widetilde{X} \cup \{x\})$. We first show that $C_h(\widetilde{X} \cup \{x\}) = C_h(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}))$. It is easy to see that $C_h(\widetilde{X} \cup \{x\}) \subseteq \widetilde{X} \cup \{x\}$. Then $\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}) \subseteq \widetilde{X} \cup \{x\}$. The substitutes condition implies

$$C_h(\widetilde{X} \cup \{x\}) \subseteq C_h(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\})). \tag{17}$$

On the other hand, by the law of aggregate demand we have

$$|C_h(\widetilde{X} \cup \{x\})| \geq |C_h(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}))|. \tag{18}$$

Combining (17) and (18) we have $C_h(\widetilde{X} \cup \{x\}) = C_h(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}))$.

In the sequel we show that $C_h(\widetilde{X} \cup \{x\}) \subset C_D(\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}))$. According to Proposition 1, we know that $x \in X' = C_h(\widetilde{X} \cup X')$. Then $\widetilde{X} \cup \{x\} \subset \widetilde{X} \cup X'$, and consequently $x \in C_h(\widetilde{X} \cup \{x\})$ by substitutes condition. Therefore, we derive that $\widetilde{X} \cup C_h(\widetilde{X} \cup \{x\}) = \widetilde{X} \cup \{x\}$. Then we only need to prove $C_h(\widetilde{X} \cup \{x\}) \subseteq C_D(\widetilde{X} \cup \{x\})$. We consider the following two cases.

Case I. There is no contract $y \in \widetilde{X}$ such that $y_D = x_D$, that is, under $\widetilde{X}$ the doctor x_D does not sign any contract.

$\widetilde{X}$ is feasible, then for any doctor $d \in D$ there is at most one contract in $\widetilde{X}$ involving d. By the individual rationality of $\widetilde{X}$ we infer

$$C_{D \setminus \{x_D\}}(\widetilde{X} \cup \{x\}) = \widetilde{X}. \tag{19}$$

According to Lemma 1, $x \in X' \subset C_D(\widetilde{X} \cup X')$. This indicates that x is acceptable to x_D. Then we have

$$C_{x_D}(\widetilde{X} \cup \{x\}) = \{x\}. \tag{20}$$

Combining (19) and (20) one can obtain $C_D(\widetilde{X} \cup \{x\}) = \widetilde{X} \cup \{x\}$. Then it is obvious that $C_h(\widetilde{X} \cup \{x\}) \subseteq C_D(\widetilde{X} \cup \{x\})$. The proof is done.

Case II. There is some (unique) contract $y \in \widetilde{X}$ such that $y_D = x_D$, that is, under $\widetilde{X}$ the doctor x_D signs contract y.

$x \in C_D(\widetilde{X} \cup X')$ and $y \in \widetilde{X}$ imply $x \succ_{x_D} y$. Then

$$C_{x_D}(\widetilde{X} \cup \{x\}) = \{x\}. \tag{21}$$

Now taking a similar argument as in Case I we can obtain

$$C_{D \setminus \{x_D\}}(\widetilde{X} \cup \{x\}) = \widetilde{X} \setminus \{y\}. \tag{22}$$

Combining (21) and (22) one can obtain $C_D(\widetilde{X} \cup \{x\}) = [\widetilde{X} \cup \{x\}] \setminus \{y\}$. Since each hospital can sign only one contract with any given doctor, $x \in C_h(\widetilde{X} \cup \{x\})$ and $x_D = y_D$ imply $y \notin C_h(\widetilde{X} \cup \{x\})$. Then we can infer $C_h(\widetilde{X} \cup \{x\}) \subseteq [\widetilde{X} \cup \{x\}] \setminus \{y\} = C_D(\widetilde{X} \cup \{x\})$. The proof is completed. □

References

Alkan, A., 2001. On preferences over subsets and the lattice structure of stable matchings. Rev. Econ. Des. 6, 99–111.

Aygün, A., Sönmez, T., 2013. Matching with contracts: Comment. Am. Econ. Rev. 103, 2050–2051.

Barberà, S., Sonnenschein, H., Zhou, L., 1991. Voting by committees. Econometrica 59, 595–609.

Chambers, C.P., Yenmez, M.B., 2013. Choice and matching. Working Paper.

Dubins, L.E., Freedman, D.A., 1981. Machiavelli and the gale-shapley algorithm. Amer. Math. Monthly 88, 485–494.

Dutta, B., Massó, J., 1997. Stability of matchings when individuals have preferences over colleagues. J. Econom. Theory 75, 464–475.

Ehlers, L., Klaus, B., 2003. Coalitional strategy-proof and resource-monotonic solutions for multiple-assignment problems. Soc. Choice Welf. 21, 265–280.

Fleiner, T., 2003. A fixed-point approach to stable matchings and some applications. Math. Oper. Res. 28, 103–126.

Gale, D., Shapley, L., 1962. College admissions and the stability of marriage. Amer. Math. Monthly 69, 9–15.

Gale, D., Sotomayor, M., 1985. Some remarks on the stable matching problem. Discrete Appl. Math. 11, 223–232.

Hatfield, J.W., Kojima, F., 2009. Group incentive compatibility for matching with contracts. Games Econom. Behav. 67, 745–749.

Hatfield, J.W., Milgrom, P.R., 2005. Matching with contracts. Am. Econ. Rev. 95, 913–935.

Kelso, A., Crawford, V., 1982. Job matching, coalition formation, and gross substitutes. Econometrica 50, 1483–1504.

Martínez, R., Massó, J., Neme, A., Oviedo, J., 2000. Single agents and the set of many-to-one stable matchings. J. Econom. Theory 91, 91–105.

Martínez, R., Massó, J., Neme, A., Oviedo, J., 2001. On the lattice structure of the set of stable matchings for a many-to-one model. Optimization 50, 439–457.

Martínez, R., Massó, J., Neme, A., Oviedo, J., 2004. On group strategy-proof mechanisms for a many-to-one matching model. Internat. J. Game Theory 33, 115–128.

Martínez, R., Massó, J., Neme, A., Oviedo, J., 2010. The Blocking Lemma for a many-to-one matching model. J. Math. Econom. 46, 937–949.

Papai, S., 2000. Strategy-proof multiple assignment using quotas. Rev. Econ. Des. 5, 91–105.

Roth, A.E., 1982. The economics of matching: stability and incentives. Math. Oper. Res. 7, 617–628.

Roth, A.E., 1984. Stability and polarization of interests in job matching. Econometrica 52, 47–57.

Roth, A.E., 1985. The college admissions problem is not equivalent to the marriage problem. J. Econom. Theory 36, 277–288.

Roth, A.E., Sotomayor, A.M., 1990. Two-Sided Matching: A Study in Game-theoretic Modeling and Analysis. Cambridge University Press, Cambridge.