Games and Economic Behavior 102 (2017) 44–55

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

The Blocking Lemma and strategy-proofness in many-to-many matchings

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ARTICLE INFO

Article history:
Received 28 March 2016
Available online 17 November 2016

JEL classification:
C78
D71
D78

Keywords:
Many-to-many matching
Blocking Lemma
Max–min preferences
Deferred acceptance algorithm
Strategy-proofness

ABSTRACT

This paper considers the incentive compatibility in many-to-many two-sided matching problems. We first show that the Blocking Lemma holds for many-to-many matchings under the extended max–min preference criterion and quota-saturability condition. This result extends the Blocking Lemma for one-to-one matching and for many-to-one matching to many-to-many matching problem. It is then shown that the deferred acceptance mechanism is strategy-proof for agents on the proposing side under the extended max–min preference criterion and quota-saturability condition. Neither the Blocking Lemma nor the incentive compatibility can be guaranteed if the preference condition is weaker than the extended max–min criterion.

1. Introduction

Many-to-many two-sided matching models study assignment problems where agents can be divided into two disjoint sets: the set of firms and the set of workers. Each firm wishes to hire a set of workers, and each worker wishes to work for a set of firms. Firms have preferences over the possible sets of workers, and workers have preferences over the possible sets of firms. The assumption that workers may work in more than one firm is not unusual. A physician may have a medical position at a hospital and a teaching position at some university. A faculty member in college may have a part-time position in different places. A many-to-many two-sided assignment problem is to match each agent (a firm or a worker) with a subset of agents from the other side of the market. If a firm hires a worker, we say that the two agents form a partnership. A set of partnerships is called a matching.

The many-to-many matching problem is a natural extension of the one-to-one marriage problem and the many-to-one college admissions problem of Gale and Shapley (1962) with general quotas. The notions with respect to the college admissions problem are commonly generalized to the many-to-many matching model. For a matching problem, the stability of matching is of primary importance. A matching is pairwise-stable if all partnerships occur between acceptable partners (in-

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[1] We thank an advisory editor and two anonymous referees for their helpful comments and suggestions. Jiao acknowledges financial support from the Zhejiang Provincial Natural Science Foundation of China (Grant No. LY16G010008) and China Postdoctoral Science Foundation. Tian acknowledges financial support from the National Natural Science Foundation of China (NSFC-71371117) and the Key Laboratory of Mathematical Economics (SUFE) at Ministry of Education of China. All errors are our own.

http://dx.doi.org/10.1016/j.geb.2016.10.015
0899-8256/

dividual rationality) and there is no unmatched worker–firm pair that mutually prefer each other to their assigned partners (pairwise blocking). For the marriage problem and the college admissions problem, Gale and Shapley's deferred acceptance algorithms yield stable matchings. Moreover, the stable matching produced by the deferred acceptance procedure is also optimal for agents on the proposing side. That is, every agent on the proposing side is at least as well off under the assignment given by the deferred acceptance procedure as he would be under any other stable assignment. Roth (1984) adapts the deferred acceptance algorithm to the many-to-many matching market and obtains the corresponding optimal stable assignment.

The incentive compatibility of matching is also an important problem. For one-to-one matching problem, Roth (1982) investigates the marriage problem and obtains that the men (resp. women)-optimal matching is strategy-proof for men (resp. women).[2] For a unified model, Hatfield and Milgrom (2005) study the incentive property for matching with contracts. They obtain that the doctor-optimal matching is strategy-proof for doctors under very weak preference assumption (hospitals' preferences satisfy substitutability and the law of aggregate demand). Under the same framework, Hatfield and Kojima (2009) show that the doctor-optimal matching is in fact group strategy-proof for doctors. For many-to-one matching problem, Roth (1985) studies the college admissions problem and shows that, when colleges have responsive preferences, the colleges-proposing deferred acceptance algorithm may not be strategy-proof for colleges, while the students-proposing deferred acceptance algorithm is strategy-proof for students. It is interesting to study the incentive compatibility for agents with multi-unit demand.

For incentive compatibility in many-to-many matching problems, Baïou and Balinski (2000) claim that their reduction algorithm is stable and strategy-proof for agents on one side of the matching market under the max–min criterion condition. Unfortunately, their claim is incomplete. Hatfield et al. (2014) show that the max–min preference criterion is not sufficient for the existence of a stable and strategy-proof matching mechanism even in many-to-one matching markets. As such, it is still an unanswered question on the strategy-proofness in many-to-many matching problems.

This paper considers the (group) strategy-proofness in many-to-many two-sided matching problems under the extended max–min preference criterion and quota-saturability condition. The extended max–min criterion indicates that agents always want to match with as many acceptable partners as possible within their quotas, which means that agents should use their capacity of resources as they can, and focus on the worst partners when ranking different sets of partners. The firms-quota-saturability says that, there is a sufficiently large number of available and acceptable workers in the market such that every firm can hire as many workers as its quota. We show that the firms-proposing deferred acceptance algorithm is (group) strategy-proof for firms if all agents (firms and workers) have the extended max–min preferences and the firms-quota-saturability is satisfied.

In order to obtain the result of strategy-proofness, we first extend the Blocking Lemma to many-to-many matching markets. For one-to-one and many-to-one matching problems, the Blocking Lemma is an important instrumental result, which identifies a particular blocking pair for any unstable and individually rational matching that is preferred by some agents of one side of the market to their optimal stable matching. Its interest lies in the fact that it has been used to derive some key conclusions on matching. Using the Blocking Lemma for one-to-one matching,[3] Gale and Sotomayor (1985) give a short proof for the group strategy-proofness of the deferred acceptance algorithm. For many-to-one matching, the Blocking Lemma holds under responsive preference profile.[4] The responsiveness seems too restrictive to be satisfied. For a weak preference restriction, Martínez et al. (2010) show that the corresponding Blocking Lemma for workers (who have unit demand) holds under substitutable and quota-separable preference.[5] They also note that the Blocking Lemma for firms (which have multi-unit demand) does not hold even under responsive preference. Under the extended max–min preference criterion and quota-saturability condition, Jiao and Tian (2015a) obtain the Blocking Lemma for agents with multi-unit demand in many-to-one matching markets, and then show the strategy-proofness of the deferred acceptance algorithm for agents on the proposing side. It is then interesting to investigate the Blocking Lemma in many-to-many matching markets.

In this paper we show that the extended max–min preference restriction, together with the quota-saturability condition, establishes the Blocking Lemma for many-to-many matchings. As an immediate consequence of the Blocking Lemma, we obtain the strategy-proofness of the deferred acceptance algorithm in many-to-many matchings. In addition, we note by

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[2] Dubins and Freedman (1981) show that, under the men (resp. women)-proposing deferred acceptance algorithm, there exists no coalition of men (resp. women) that can simultaneously improve the assignment of all its members if those outside the coalition state their true preferences. This result implies the property of strategy-proofness.

[3] Gale and Sotomayor attribute the formulation of the lemma to J.S. Hwang.

[4] Roth (1985) introduces responsiveness of preference relations for college admissions problems. Specifically, responsiveness means that, for any two subsets of workers that differ in only one worker, a firm prefers the subset containing the most-preferred worker. Formally, we say a firm f's preference relation is responsive if for any w_1, w_2 and any S such that $w_1, w_2 \notin S$ and $|S| < q_f$, we have $S \cup \{w_1\} P(f) S \cup \{w_2\}$ if and only if $\{w_1\} P(f) \{w_2\}$, where w_1, w_2 are the partners of f and S is a set of partners of f. It is easy to obtain that the responsiveness is stronger than the substitutability.

[5] Barberà et al. (1991) propose another concept of separable preference (different from that used by Sotomayor, 1999), which has been extensively used in matching models. See, for instance, Alkan (2001), Dutta and Massó (1997), Ehlers and Klaus (2003), Martínez et al. (2000, 2001, 2004b), Papai (2000), and Sönmez (1996). Based on this condition, Martínez et al. (2010) propose a new concept called quota-separability. Formally, a firm f's preference relation $P(f)$ over sets of workers is quota q_f-separable if: (i) for all $S \subsetneq W$ such that $|S| < q_f$ and $w \notin S$, it implies $(S \cup \{w\}) P(f) S$ if and only if $\{w\} P(f) \varnothing$; (ii) $\varnothing P(f) S$ for all S such that $|S| > q_f$. One can check that the extended max–min criterion introduced in this paper implies (i). The definition of a matching requires that $|\mu(f)| \leq q_f$ for all $f \in F$, and consequently condition (ii) is satisfied. That is, in our setting, the extended max–min criterion is stronger than quota-separability.

example (Example 3) that both the Blocking Lemma and incentive compatibility for many-to-many matching may not be true when the preference condition is weaker than the extended max–min criterion condition.

The remainder of the present paper is organized as follows. We present some preliminaries on the formal model in the next section. In Section 3 we study the Blocking Lemma for a many-to-many matching model. In Section 4 we study the incentive compatibility of the firms-proposing deferred acceptance algorithm. We conclude in Section 5. All proofs are provided in Appendix A.

2. Preliminaries

2.1. The model

For concreteness, we use the language of firms–workers matching model. The agents in our model consist of two disjoint sets, the set of *firms* $F = \{f_1, \cdots, f_n\}$ and the set of *workers* $W = \{w_1, \cdots, w_m\}$. Generic agents are denoted by $v \in V \equiv F \cup W$ while generic firms and workers are denoted by f and w, respectively. Each firm wishes to hire a set of workers, and each worker wishes to work for a set of firms. Each $f \in F$ has a strict, complete and transitive preference relation $\succ_f$ over $W \cup \{\varnothing\}$, and each $w \in W$ has a strict, complete and transitive preference relation $\succ_w$ over $F \cup \{\varnothing\}$. The notation $\varnothing$ denotes the prospect of a position remaining unfilled.

We call a firm f is *acceptable to* w if $f \succ_w \varnothing$, and a worker w is *acceptable to* f if $w \succ_f \varnothing$. The preference relations of firms and workers can be represented by order lists. For example, $\succ_f$: $w_2, w_1, \varnothing, w_3, \cdots$ denotes that firm f prefers to hire w_2 rather than w_1, that it prefers to hire either one of them rather than leave a position unfilled, and that all other workers are unacceptable, in the sense that it would be preferable to leave a position unfilled rather than to fill it with, say, worker w_3. Similarly, for the preference relation of a worker, $\succ_w$: $f_1, f_3, f_2, \varnothing, \cdots$ indicates that the only positions the worker would accept are those offered by f_1, f_3 and f_2, in that order. We will write $f_i \succ_w f_j$ to indicate that worker w prefers f_i to f_j, and $f_i \succeq_w f_j$ to indicate that either $f_i \succ_w f_j$ or $f_i = f_j$. Similarly, we can give corresponding notations on preference relations of firms.

We are allowing for the possibility that worker w may prefer to remain unemployed rather than work for an unacceptable firm and that firm f may prefer not to hire any worker rather than hire any unacceptable worker. Each agent v has a *quota* q_v which is the maximum number of partnerships she or it may enter into. Let $q \equiv (q_v)_{v \in V}$ denote the vector of quotas.

Definition 1. A *matching* is a mapping $\mu : F \cup W \to 2^{F \cup W}$ such that

(1) $\mu(f) \in 2^W$ and $|\mu(f)| \le q_f$ for all $f \in F$,
(2) $\mu(w) \in 2^F$ and $|\mu(w)| \le q_w$ for all $w \in W$,
(3) $w \in \mu(f)$ if and only if $f \in \mu(w)$ for all $f \in F$ and $w \in W$.

We denote by $\mu(f) \equiv \{w \in W | f \in \mu(w)\}$ the set of workers who are assigned to f and $\mu(w) \equiv \{f \in F | w \in \mu(f)\}$ the set of firms which are assigned to w. We will denote by $\min(\mu(v))$ the least preferred agent of v in $\mu(v)$. If $w \in \mu(f)$, then we write $(f, w) \in \mu$ and say that there is a partnership between w and f under matching μ.

Note that, for $v \in F \cup W$, we stipulate $|\mu(v)| = 0$ if $\mu(v) = \varnothing$. For this condition we say agent v is single.

A matching μ is *blocked by an individual* $i \in F \cup W$ if there exists some player $j \in \mu(i)$ such that $\varnothing \succ_i j$. A matching is *individually rational* if it is not blocked by any individual. A matching μ is *blocked by a pair* $(f, w) \in F \times W$ if they are not matched under μ, and further

(1) f is acceptable to w and w is acceptable to f,
(2) $|\mu(f)| < q_f$ or $w \succ_f w'$ for some $w' \in \mu(f)$,
(3) $|\mu(w)| < q_w$ or $f \succ_w f'$ for some $f' \in \mu(w)$.

For many-to-many matching, there are many ways to define the stability of a matching. For considering group strategy-proofness, researchers typically adopt the notion of group stability (or setwise stability). However, for many-to-many matching under the extended max–min preferences (the definition is given below, see Definition 5), Jiao and Tian (2015b) obtain the equivalence between pairwise-stability and setwise stability (or group stability). As a result, we can just focus on the pairwise-stability throughout this paper.

Definition 2. A matching μ is *pairwise-stable* if it is not blocked by any individual nor any firm–worker pair.

In many-to-many matching problems, since every agent (a firm or a worker) wishes to match with a subset of agents from the other side of the market, we also need to specify each firm f's (resp. worker w's) preference relation, denoted by $P(f)$ (resp. $P(w)$), over the set of potential partner groups 2^W (resp. 2^F). For each $v \in V$, we assume the preference relation $P(v)$ is strict and transitive, but the completeness is not required. Notice that, over the set of singleton subsets, the preference relation $P(v)$ coincides with $\succ_v$. That is, $\{w_1\}P(f)\{w_2\}$ if and only if $w_1 \succ_f w_2$ and $\{f_1\}P(w)\{f_2\}$ if and only if $f_1 \succ_w f_2$ for any $f, f_1, f_2 \in F$ and $w, w_1, w_2 \in W$. We can also present $P(f)$ and $P(w)$ by order lists. Since only acceptable sets of partners will matter, we only write the lists involving acceptable partners. For instance,

$$P(f)\colon w_1w_2, w_1, w_2, w_3, \text{ and}$$

$$P(w)\colon f_1f_3, f_1, f_2$$

indicate that $\{w_1, w_2\}P(f)\{w_1\}P(f)\{w_2\}P(f)\{w_3\}P(f)\varnothing$ and $\{f_1f_3\}P(w)\{f_1\}P(w)\{f_2\}P(w)\varnothing$, respectively.

For any two matchings μ and μ', the notation $\mu P(f)\mu'$ means $\mu(f)P(f)\mu'(f)$, $\mu R(f)\mu'$ means $\mu P(f)\mu'$ or $\mu(f) = \mu'(f)$, $\mu R(F)\mu'$ means $\mu R(f)\mu'$ for all $f \in F$, and, $\mu P(F)\mu'$ means $\mu R(F)\mu'$ and $\mu \neq \mu'$.

Some restrictions on firm f's preference relation $P(f)$ are usually imposed, among which substitutability is often adopted.[6] In the language of firms–workers matching model, substitutability of firm f's preferences requires: "if hiring w is optimal when certain workers are available, hiring w must still be optimal when a subset of workers are available." Formally,

Definition 3. An agent f's preference relation $P(f)$ satisfies *substitutability* if, for any sets S and S' with $S \subseteq S'$ and $w \in S$, $w \in Ch(S', P(f))$ implies $w \in Ch(S, P(f))$, where $Ch(S, P(f))$ denotes agent f's most-preferred subset of S according to f's preference relation $P(f)$.

In literature, there is a different way to define substitutability. For instance, Hatfield and Milgrom (2005) define the substitutability by rejection function. Specifically, the definition of substitutability by rejection function can be expressed as: f's preference relation $P(f)$ satisfies substitutability if for any sets S and S' with $S \subseteq S' \subseteq W$, we have $R(S, P(f)) \subseteq R(S', P(f))$, where $R(S, P(f)) \equiv S \setminus Ch(S, P(f))$ denotes agent f's rejection function. It is easy to check that these two ways are in fact equivalent.

The substitutable preference condition, (together with other condition, such as the law of aggregate demand), ensures the stability of matching problem and incentive compatibility of agents with unit demand. However, for incentive compatibility of agents with multi-unit demand, we need a stronger preference condition. Throughout this paper, we assume that each agent v's preference relation $P(v)$ satisfies the extended max–min criterion, which is a slight modification of Baïou and Balinski's (2000) max–min criterion.[7] For completeness, we first state the definition of max–min preference as follows:

Definition 4 *(Max–min criterion).* The firm f's preference relation $P(f)$ is said to satisfy the *max–min criterion* if for any two sets of acceptable workers $S_1, S_2 \in 2^W$ with $|S_1| \le q_f$ and $|S_2| \le q_f$,

(i) The strict preference relation $P(f)$ over 2^W is defined as: $S_1P(f)S_2$ if $|S_1| \ge |S_2|$ and $\min(S_1) \succ_f \min(S_2)$ (i.e., f strictly prefers the least preferred worker in S_1 to the least preferred worker in S_2), where $\min(S_i)$ denotes the least preferred worker of f in S_i;
(ii) The weak preference relation, denoted by $R(f)$, is defined as: $S_1R(f)S_2$ if $S_1P(f)S_2$ or $S_1 = S_2$.

The preference relation $P(w)$ of worker $w \in W$ is said to satisfy the max–min criterion if the corresponding condition is met.

Note that preference relation over groups of workers by max–min criterion is not complete. For instance, a set of workers S_1 and its proper subset S_2 may not be comparable. Here, we extend the max–min preference relation over groups of workers by assuming that any group of workers S_1 is always comparable with, and actually strictly preferable to, its proper subgroup S_2. Formally, we have the following notion of extended max–min criterion.

Definition 5 *(Extended max–min criterion).* The firm f's preference relation $P(f)$ is said to satisfy the *extended max–min criterion* if for any two sets of acceptable workers $S_1, S_2 \in 2^W$ with $|S_1| \le q_f$ and $|S_2| \le q_f$,

(i) The strict preference relation $P(f)$ over 2^W is defined as: $S_1P(f)S_2$ if one of the following two conditions holds: (1) S_2 is a proper subset of S_1, (2) $|S_1| \ge |S_2|$ and $\min(S_1) \succ_f \min(S_2)$;
(ii) The weak preference relation, denoted by $R(f)$, is defined as: $S_1R(f)S_2$ if and only if $S_1P(f)S_2$ or $S_1 = S_2$.

[6] Kelso and Crawford (1982) introduce the substitutability condition on hospital preferences in a matching model with wages. Roth (1984) adapts the deferred acceptance algorithm to the many-to-many matching model with substitutable preference and obtains the corresponding optimal stable matching. Martínez et al. (2004a, 2004b), Echenique and Oviedo (2006) and Klaus and Walzl (2009) study the stability problem of many-to-many matching under substitutable preference. Hatfield and Kojima (2008, 2009, 2010) and Hatfield and Milgrom (2005) study the matching with contracts under substitutable hospitals' preferences.

[7] Similar conditions were studied by Echenique and Oviedo (2006), Kojima (2007) and Sotomayor (2004).

The preference relation $P(w)$ of worker $w \in W$ is said to satisfy the extended max–min criterion if the corresponding condition is met. We say that the *extended max–min criterion* is satisfied if the preferences of all agents (firms and workers) satisfy the extended max–min criterion.

Thus, we modify the conventional max–min criterion by adding that $S_1 P(f) S_2$ if S_2 is a proper subset of S_1, i.e., hiring more acceptable workers is always more preferable to firms, which is clearly very reasonable for capacity utilization of resources. As such, for the extended max–min criterion, "max" indicates that firms always want to hire as many workers as possible, and "min" indicates that firms focus on the worst worker in ranking different sets of workers and firms would like to hire as preferable workers (in the sense of preferences over individual worker) as possible. It is easy to check there is no implication relationship between the extended max–min criterion and the conventional one.

It is well known that responsiveness implies substitutability. Here we note that the extended max–min criterion also implies substitutability.[8] In addition, we note that there is no implication relationship between the extended max–min criterion and responsiveness. Indeed, firstly, the extended max–min criterion is not stronger than responsiveness. For example, suppose $w_1 \succ_f w_2 \succ_f w_3$ and $q_f = 2$. Then $\{w_1, w_3\} P(f) \{w_2, w_3\}$ if $P(f)$ is responsive. However, $\{w_1, w_3\} P(f) \{w_2, w_3\}$ does not hold if $P(f)$ satisfies the extended max–min criterion. Secondly, responsiveness is not stronger than the extended max–min criterion. For example, we assume $w_1 \succ_f w_2 \succ_f w_3 \succ_f w_4$ and $q_f = 2$. Then $\{w_2, w_3\} P(f) \{w_1, w_4\}$ if $P(f)$ satisfies the extended max–min criterion. However, $\{w_2, w_3\} P(f) \{w_1, w_4\}$ does not necessarily hold if $P(f)$ is responsive.

Denote the preferences profile of all agents by $P \equiv (P(v))_{v \in V}$, and a firms–workers matching problem by a four-tuple $(F, W; q; P)$. Given a matching market $(F, W; q; P)$, it is assumed that a firm can provide at most one position to any given worker.[9]

2.2. Deferred acceptance algorithm

The deferred acceptance algorithm was first proposed by Gale and Shapley (1962) to find a stable assignment for the marriage problem (one-to-one matching) and college admissions problem (many-to-one matching). If every player has the extended max–min preference, then the deferred acceptance algorithm can be used to find a stable assignment for a many-to-many matching.

Specifically, the Firms-Proposing Deferred Acceptance Algorithm proceeds as follows:

Step 1. (a). Each firm f proposes to its most-preferred q_f acceptable workers (and if it has fewer acceptable choices than q_f, then it proposes to all its acceptable choices).

(b). Each worker w then places on her waiting list the q_w firms which rank highest, or all firms if there are fewer than q_w firms, and rejects the rest.

In general, at

Step k. (c). Any firm f that was rejected at step $(k-1)$ by any worker proposes to its most-preferred q_f acceptable workers who have not yet rejected it (and if there are fewer than q_f remaining acceptable workers, then it proposes to all).

(d). Each worker w selects the top q_w — or all firms if there are fewer than q_w firms—from among the new firms and those on her waiting list, puts them on her new waiting list, and rejects the rest.

Since no firm proposes twice to the same worker, this algorithm always terminates in a finite number of steps. The algorithm terminates when there are no more rejections. Each worker is matched with firms on her waiting list in the last step.

It can be shown that, under the extended max–min preference restriction, the matching produced by the firms-proposing deferred acceptance algorithm, denoted by μ_F, is stable (see, for instance, Jiao and Tian, 2015b). Symmetrically, the matching produced by the workers-proposing deferred acceptance algorithm, denoted by μ_W, is also stable. Furthermore, we have the following proposition that shows μ_F is the (unique) optimal stable matching for firms.

Proposition 1. *For the firms–workers matching market $(F, W; q; P)$, if the extended max–min criterion is satisfied, then μ_F is the optimal stable assignment for the firms; that is, for any other stable matching μ, $\mu_F(f) R(f) \mu(f)$ for every $f \in F$.*

The proof of Proposition 1 is given in Appendix A. Symmetrically, μ_W is the unique optimal stable assignment for the workers if the extended max–min criterion is satisfied.

3. The Blocking Lemma

For the firms–workers matching model with the extended max–min preferences, the Blocking Lemma for firms can be expressed as: if the set of firms that strictly prefer an individually rational matching μ to μ_F is nonempty, then there must exist a blocking pair (f, w) of μ with the property that under the matching μ, w works for a firm strictly preferring μ to μ_F, and f considers μ_F being at least as good as μ. Formally,

[8] See Jiao and Tian (2015b) for detailed discussion. In addition, Echenique and Oviedo (2006) propose the concept of strong substitutability and show that strong substitutability is stronger than substitutability. It is easy to verify that the extended max–min criterion is stronger than strong substitutability, and consequently, the extended max–min criterion implies substitutability.

[9] For matching with contracts, Hatfield and Milgrom (2005) make a similar assumption.

Definition 6 *(The Blocking Lemma).* Let μ be any individually rational matching and F' be the nonempty set of all firms that prefer μ to μ_F. We say that the *Blocking Lemma* (for firms) holds if there exists $f \in F \backslash F'$ and $w \in \mu(F')$ such that the pair (f, w) blocks μ, where $\mu(F') \equiv \bigcup_{f \in F'} \mu(f)$.

Gale and Sotomayor (1985) prove the Blocking Lemma for one-to-one matching. For many-to-one matching problem, the Blocking Lemma (for agents with unit demand) can be easily obtained by the decomposition lemma (see Roth and Sotomayor, 1990) if the responsive preference condition is satisfied. Recently, Martínez et al. (2010) prove the same result under a weak preference condition (i.e., substitutable and quota-separable preference profiles). They also note that the Blocking Lemma for agents with multi-unit demand may not hold under responsive preference profiles. In fact, the following example shows that, under any reasonable preference assumption (including responsiveness, separability and even extended max–min criterion), we cannot expect to obtain the Blocking Lemma for agents with multi-unit demand.

Example 1. There are two firms f_1, f_2 with $q_{f_1} = 1, q_{f_2} = 2$, and two workers w_1, w_2 with $q_{w_1} = 1 = q_{w_2}$. The preferences are as follows:

$$P(w_1) \colon f_1,\ f_2 \qquad P(f_1):\ w_2,\ w_1$$

$$P(w_2) \colon f_2,\ f_1 \qquad P(f_2):\ w_1 w_2,\ w_1,\ w_2\ .$$

The matching produced by the firms-proposing deferred acceptance algorithm is:

$$\mu_F = \begin{pmatrix} f_1 & f_2 \\ w_1 & w_2 \end{pmatrix}.$$

An individually rational matching μ is:

$$\mu = \begin{pmatrix} f_1 & f_2 \\ w_2 & w_1 \end{pmatrix}.$$

One can see that $P(f_2)$ satisfies responsiveness, separability and extended max–min criterion. It is easy to check that both f_1 and f_2 prefer μ to μ_F. Thus $F' = \{f_1, f_2\}$ is nonempty and $F' = F$. The conclusion of the Blocking Lemma does not hold.

In Example 1, we can see that firm f_2's preference relation is reasonable, but the number of workers is less than the aggregate quota of firms. This may potentially cause the failure of the Blocking Lemma. To avoid this possible environment, we propose the following condition which says that the workers' side of the market is sufficiently "large" such that each firm can hire as many acceptable workers as its quota if it wants. That is, in the matching market, from the view of firms the worker resource is not scarce. Formally,

Definition 7 *(Quota-saturability condition).* For the firms–workers matching problem, let $\widetilde{W}$ be a subset of W such that every firm $f \in F$ and every worker $w \in \widetilde{W}$ are acceptable to each other. We say that *the firms-quota-saturability condition* holds if there exists some $\widetilde{W} \subset W$ such that $|\widetilde{W}|$ is large enough to satisfy $|\widetilde{W}| \geq \sum_{f \in F} q_f$, i.e., the number of available and acceptable workers is not less than the aggregate quota of firms.

In other words, firms-quota-saturability means that there are enough available and acceptable workers such that each firm f can be assigned q_f acceptable workers if it wants. Thus, the quota-saturability condition presents the relative size of the agents on the two matching market sides. An intuitive interpretation of quota-saturability would be a certain "excess supply" of workers in the market that allows all firms to fill their quotas if they want. Similarly, we can define the workers-quota-saturability condition.

For the firms–workers model, we know that $|\mu(f)| \leq q_f$ for each firm $f \in F$. If the firms-quota-saturability condition holds and firms have the extended max–min preferences, then every firm wants to hire as many workers as its quota and there is a sufficiently large number of workers in the market. We can infer that the matching μ must be unstable if $|\mu(f)| < q_f$ for any $f \in F$ because f has empty position and there are workers who want to enter firm f. Therefore, one can obtain that $|\mu(f)| = q_f$ for any $f \in F$ if μ is stable.

With the above preparation, we now state the following Blocking Lemma for many-to-many matchings.

Theorem 1 *(Blocking Lemma). For the firms–workers matching model, if the extended max–min criterion and the firms-quota-saturability condition are satisfied, then the Blocking Lemma for firms holds.*

The proof of Theorem 1 is given in Appendix A. Symmetrically, the Blocking Lemma for workers holds if every agent has the extended max–min preference and the workers-quota-saturability condition is satisfied.

Note that both the quota-saturability condition and the extended max–min criterion are crucial for the Blocking Lemma. In Example 1, the firms-quota-saturability condition is not satisfied and the Blocking Lemma fails. The following example shows the importance of the extended max–min criterion even if the firms-quota-saturability condition is satisfied.

Example 2. There are three firms f_1, f_2, f_3 with $q_{f_1} = 2, q_{f_2} = 1, q_{f_3} = 1$, and four workers w_1, w_2, w_3, w_4 with $q_{w_1} = q_{w_2} = q_{w_3} = q_{w_4} = 1$. The preferences are as follows:

$$P(f_1)\colon w_1w_2,\ w_1w_3,\ w_1w_4,\ w_2w_3,\ w_2w_4,\ w_3w_4,\ w_1,\ w_2,\ w_3,\ w_4$$
$$P(f_2):\ w_3,\ w_1,\ w_2,\ w_4 \qquad P(f_3):\ w_1,\ w_2,\ w_3,\ w_4$$
$$P(w_1):\ f_2,\ f_1,\ f_3 \qquad P(w_2):\ f_3,\ f_1,\ f_2$$
$$P(w_3):\ f_1,\ f_2,\ f_3 \qquad P(w_4):\ f_1,\ f_3,\ f_2.$$

The matching produced by the firms-proposing deferred acceptance algorithm is:

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3w_4 & w_1 & w_2 \end{pmatrix}.$$

An individually rational matching μ is as follows:

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_2w_4 & w_3 & w_1 \end{pmatrix}.$$

One can see that the firms-quota-saturability condition holds (the number of workers equals the sum of firms' quotas), but firm f_1's preference does not satisfy the extended max–min criterion. It is easy to check that all of the three firms prefer μ to μ_F, so $F' = \{f_1, f_2, f_3\}$ is nonempty and $F' = F$. Therefore, the Blocking Lemma does not hold. This example shows that, if the extended max–min criterion is not satisfied, the Blocking Lemma does not hold even if the firms-quota-saturability condition holds and each agent has responsive preference.

This example also shows that the extended max–min criterion cannot be weakened to substitutability criterion for the Blocking Lemma, as we can check that $P(f_1)$ is responsive and responsiveness implies substitutability. Furthermore, we note that the Blocking Lemma fails to hold even if we slightly weaken the requirement of extended max–min preference. To see this, we consider the following example.

Example 3. There are three firms f_1, f_2, f_3 with $q_{f_1} = 3, q_{f_2} = 1, q_{f_3} = 1$, and five workers w_1, w_2, w_3, w_4, w_5. The preferences are as follows:

$$\succ_{w_1}:\ f_2,\ f_1,\ f_3,\ \varnothing \qquad \succ_{w_5}\colon\ f_1,\ f_3,\ f_2,\ \varnothing$$
$$\succ_{w_2}\colon\ f_3,\ f_1,\ f_2,\ \varnothing \qquad \succ_{f_1}\colon\ w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ \varnothing$$
$$\succ_{w_3}\colon\ f_1,\ f_2,\ f_3,\ \varnothing \qquad \succ_{f_2}\colon\ w_3,\ w_1,\ \cdots$$
$$\succ_{w_4}\colon\ f_1,\ f_3,\ f_2,\ \varnothing \qquad \succ_{f_3}\colon\ w_4,\ w_2,\ \cdots.$$

Then the firms-optimal matching is

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3w_4w_5 & w_1 & w_2 \end{pmatrix}.$$

For firm f_1's preference over groups of workers, we slightly weaken the requirement of the extended max–min criterion.

Case I: We assume that f_1 prefers $\{w_1, w_2\}$ to $\{w_3, w_4, w_5\}$. Sometimes, such assumption seems to be reasonable. One can easily check that this assumption violates the extended max–min criterion, as $|\{w_1, w_2\}| \geq |\{w_3, w_4, w_5\}|$ does not hold although $\min(\{w_1, w_2\}) \succ_{f_1} \min(\{w_3, w_4, w_5\})$. Then, for the following individually rational matching

$$\mu_1 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1w_2 & w_3 & w_4 \end{pmatrix},$$

we can see that all of the three firms prefer μ to μ_F, so $F' = \{f_1, f_2, f_3\}$ is nonempty and $F' = F$. Therefore, the Blocking Lemma does not hold.

Case II: We assume that f_1 prefers $\{w_1, w_2, w_5\}$ to $\{w_3, w_4, w_5\}$. Generally, such assumption is natural and reasonable. One can easily check that this assumption violates the extended max–min criterion, as $\min(\{w_1, w_2, w_5\}) \succ_{f_1} \min(\{w_3, w_4, w_5\})$ does not hold although $|\{w_1, w_2, w_5\}| \geq |\{w_3, w_4, w_5\}|$. Then, for the following individually rational matching

$$\mu_2 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1w_2w_5 & w_3 & w_4 \end{pmatrix},$$

we can see that all of the three firms prefer μ to μ_F, so $F' = \{f_1, f_2, f_3\}$ is nonempty and $F' = F$. Therefore, the Blocking Lemma does not hold.

This example indicates that one cannot expect to obtain the Blocking Lemma for agents with multi-unit demand under a preference restriction which is weaker than the extended max–min criterion.

4. Strategy-proofness in many-to-many matching problem

We know that the Blocking Lemma is an important instrumental result for one-to-one and many-to-one matching problem. For one-to-one matching, Gale and Sotomayor (1985) give a short proof for the group strategy-proofness of the deferred acceptance algorithm with the help of the Blocking Lemma. Under responsive preference, we can extend the corresponding result to many-to-one matching (strategy-proofness for agents with unit demand). In this section, we show the strategy-proofness of the firms-proposing deferred acceptance algorithm, which heavily relies on the Blocking Lemma obtained in the previous section.

Given the firms F and workers W, a *mechanism* φ is a function from any stated preferences profile P and quota-vector q to a matching. A mechanism is *stable* if the outcome of that mechanism, denoted by $\varphi(F, W; q; P)$, is a stable matching (with respect to P) for any reported preferences profile $P \equiv (P(v))_{v \in F \cup W}$ and quota-vector $q \equiv (q_v)_{v \in F \cup W}$.

Definition 8. A mechanism φ is *group strategy-proof for firms* if, for every preference profile $P = (P(v))_{v \in W \cup F}$ and every quota-vector $q = (q_v)_{v \in W \cup F}$, there is no group of firms $F' \subset F$, a preference profile $(P'(f))_{f \in F'}$ and quota-vector $(q'_f)_{f \in F'}$ such that each $f \in F'$ strictly prefers $\varphi(F, W; (q'_f)_{f \in F'}, (q_f)_{f \in F \setminus F'}, (q_w)_{w \in W}; (P'(f))_{f \in F'}, (P(f))_{f \in F \setminus F'}, (P(w))_{w \in W})$ to $\varphi(F, W; q; P)$.

Now we state the incentive compatibility for many-to-many matching as follows. We actually obtain a much stronger result: the group strategy-proofness for many-to-many matching.

Theorem 2. *For the firms–workers matching model, if the extended max–min criterion and the firms-quota-saturability condition are satisfied, then the firms-proposing deferred acceptance algorithm is group strategy-proof for firms.*

We note that this group incentive compatibility heavily depends on the Blocking Lemma. Since the conditions of the extended max–min preference and firms-quota-saturability are crucial for the Blocking Lemma, they play an important role for the group strategy-proofness of firms-proposing deferred acceptance algorithm. As in the case for the Blocking Lemma, one cannot expect to obtain the group incentive compatibility for firms under a preference restriction which is weaker than the extended max–min criterion. Specifically, we continue to see Example 3.

Example 3 *(Continued).* Consider the sets of firms and workers, and their preferences and quotas as given in Example 3. The firms-optimal matching is

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3 w_4 w_5 & w_1 & w_2 \end{pmatrix}.$$

For firm f_1's preference over groups of workers, we slightly weaken the requirement of the extended max–min criterion.

Case I: We assume that f_1 prefers $\{w_1, w_2\}$ to $\{w_3, w_4, w_5\}$, which violates the extended max–min criterion. Then firm f_1 can receive a strictly better assignment by misreporting its preferences and quota. That is, if it reports $q_{f_1} = 2$ and w_1, w_2 are its only acceptable workers, then the matching produced by the firms-proposing deferred acceptance algorithm is:

$$\mu_1 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1 w_2 & w_3 & w_4 \end{pmatrix},$$

and f_1 will become strictly better off than to report truthfully. Consequently, μ_F is not strategy-proof.

Case II: We assume that f_1 prefers $\{w_1, w_2, w_5\}$ to $\{w_3, w_4, w_5\}$, which violates the extended max–min criterion. Then firm f_1 can receive a strictly better assignment by misreporting its preferences and quota. That is, if it reports $q_{f_1} = 3$ and w_1, w_2, w_5 are its only acceptable workers, then the matching produced by the firms-proposing deferred acceptance algorithm is:

$$\mu_2 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1 w_2 w_5 & w_3 & w_4 \end{pmatrix},$$

and f_1 will become strictly better off than to report truthfully. Consequently, μ_F is not strategy-proof.

Since strategy-proofness is a special case of group strategy-proofness, we immediately have the following corollary.

Corollary 1. *For the firms–workers matching model, if the extended max–min criterion and the firms-quota-saturability condition are satisfied, then the firms-proposing deferred acceptance algorithm is strategy-proof for firms.*

5. Conclusion

We obtain the Blocking Lemma and the group incentive compatibility for agents with multi-unit demand in many-to-many matchings, which extend the corresponding results for many-to-one matchings (see Jiao and Tian, 2015a) to the case of many-to-many matchings. We show that the deferred acceptance mechanism is group strategy-proof for agents on the proposing side under the extended max–min preference criterion and quota-saturability condition in many-to-many matching markets. The corresponding results fail to hold even if we just slightly weaken the requirement of the extended max–min preference criterion.

Appendix A

Proof of Proposition 1. Define a firm f and a worker w to be *achievable* for each other in a matching problem $(F, W; q; P)$ if f and w are matched at some stable assignment. We will show that in the deferred acceptance algorithm with firms proposing, no firm is ever rejected by an achievable worker. Consequently the stable matching μ_F is produced by assigning each firm to its most-preferred achievable workers. Then μ_F is the optimal stable matching for every firm.

We argue by contradiction. Suppose that there exists some firm which is ever rejected by one of its achievable workers. Let k $(\in \{1, 2, \cdots\})$ be the first step at which at least one firm, say f, is rejected by its achievable worker, say w. (So no firm is rejected by any of its achievable workers from step 1 through step $(k-1)$ in the algorithm.) If w rejects f as unacceptable, then w is not achievable for f, and we are done. If f is acceptable to w, then w rejects f because she has better (than f) choices in hand. By the extended max–min criterion, there must be q_w firms which are better than f to w on her waiting list at the end of step k. We denote the set of these q_w firms by $S(q_w)$. We show that w is not achievable for f.

Specifically, one can know that any firm $\widetilde{f} \in S(q_w)$ proposes to w at step k or at some previous step. By assumption we know that $\widetilde{f}$ has not been rejected by any worker who is achievable to it before step k. Then on $\widetilde{f}$'s preference list there are fewer than $q_{\widetilde{f}}$ workers who are achievable to $\widetilde{f}$ and are better than w for $\widetilde{f}$. We suppose w and f are achievable for each other. By definition there exists some stable matching $\widetilde{\mu}$ such that $f \in \widetilde{\mu}(w)$. Since $|\widetilde{\mu}(w)| \le q_w$, $|S(q_w)| = q_w$, $f \in \widetilde{\mu}(w)$ and $f \notin S(q_w)$, we can infer that $S(q_w) \backslash \widetilde{\mu}(w)$ is nonempty. There exists some firm, say $f' \in S(q_w) \backslash \widetilde{\mu}(w)$. Since there are fewer than $q_{f'}$ workers who are achievable to f' and are better than w for f' on the preference list of f', one can infer that either $|\widetilde{\mu}(f')| < q_{f'}$ or there exists some worker $w' \in W$ such that $w' \in \widetilde{\mu}(f')$ and $w \succ_{f'} w'$. This, together with $f \in \widetilde{\mu}(w)$ and $f' \succ_w f$, implies that $\widetilde{\mu}$ is blocked by (f', w), which contradicts the stability of $\widetilde{\mu}$. The proof is completed. □

Proof of Theorem 1. For $F' \subset F$, $\mu(F') \equiv \bigcup_{f \in F'} \mu(f)$. Let $|\mu(F')| \equiv |\bigcup_{f \in F'} \mu(f)|$ denote the number of workers in $\mu(F')$ and $||\mu(F')|| \equiv \sum_{f \in F'} |\mu(f)|$ denote the number of partnerships between firms in F' and workers in $\mu(F')$ under matching μ. We prove the theorem by considering two cases.

Case I. $\mu(F') \neq \mu_F(F')$.

For any firm $\widetilde{f} \in F'$, by the quota-saturability condition, we have $|\mu_F(\widetilde{f})| = q_{\widetilde{f}}$. The assumption that $\widetilde{f}$ prefers μ to μ_F, together with the extended max–min criterion, implies $|\mu(\widetilde{f})| \ge |\mu_F(\widetilde{f})| = q_{\widetilde{f}}$. Since $\tilde{f}$ cannot employ more workers than its quota, we have $|\mu(\widetilde{f})| \le q_{\widetilde{f}}$. Thus we obtain $|\mu(\widetilde{f})| = q_{\widetilde{f}}$. Then it holds $||\mu(F')|| \equiv \sum_{f \in F'} |\mu(f)| = \sum_{f \in F'} |\mu_F(f)| \equiv ||\mu_F(F')||$.

(i) $\mu(F')$ is a proper subset of $\mu_F(F')$.

We construct two sets as follows. Let $M_1 \equiv \{(f, w) \in \mu | f \in F', w \in \mu(F')\}$ and $M_2 \equiv \{(f, w) \in \mu_F | f \in F', w \in \mu(F')\}$. It holds $M_1 = (M_1 \cap M_2) \cup (M_1 \setminus M_2)$. One can easily check that $|M_1| = ||\mu(F')||$ and $|M_2| < ||\mu_F(F')||$ by the constructions of M_1 and M_2, and the assumption of $\mu(F')$ being a proper subset of $\mu_F(F')$. Together with $||\mu(F')|| = ||\mu_F(F')||$, we have $|M_2| < |M_1|$. Then $M_1 \setminus M_2 \neq \emptyset$. For each pair $(\widetilde{f}, \widetilde{w}) \in M_1 \setminus M_2$, one can infer that $\widetilde{w} \succ_{\widetilde{f}} \min(\mu_F(\widetilde{f}))$ by the quota-saturability condition, the extended max–min criterion, and $\mu(\widetilde{f}) \succ_{\widetilde{f}} \mu_F(\widetilde{f})$. Then $\widetilde{f}$ must ever propose to $\widetilde{w}$ in the procedure of the deferred acceptance algorithm of μ_F. By the individual rationality of μ, $(\widetilde{f}, \widetilde{w}) \in \mu$ indicates that $\widetilde{f}$ is acceptable to $\widetilde{w}$. Then $(\widetilde{f}, \widetilde{w}) \notin \mu_F$ implies that $\widetilde{w}$ has $q_{\widetilde{w}}$ choices which are better than $\widetilde{f}$ and then rejects $\widetilde{f}$. We have $|\mu_F(\widetilde{w})| = q_{\widetilde{w}}$. Thus, by $\widetilde{f} \in \mu(\widetilde{w}) \setminus \mu_F(\widetilde{w})$ and $|\mu_F(\widetilde{w})| = q_{\widetilde{w}} \ge |\mu(\widetilde{w})|$, we obtain that there must exist some $f' \in F$ such that $(f', \widetilde{w}) \in \mu_F \setminus \mu$. Now we claim the following statement.

Claim 1. *There is at least one pair $(f_1, w_1) \in M_1 \setminus M_2$ and one firm $f_2 \in F \setminus F'$ such that $f_2 \in \mu_F(w_1) \setminus \mu(w_1)$.*

We prove Claim 1 by contradiction. Suppose not. Then for any $(\widetilde{f}, \widetilde{w}) \in M_1 \setminus M_2$, there always exists some firm $f \in F'$ such that $f \in \mu_F(\widetilde{w}) \setminus \mu(\widetilde{w})$. That is, every pair $(\widetilde{f}, \widetilde{w}) \in M_1 \setminus M_2$ corresponds to one pair $(f, \widetilde{w}) \in M_2 \setminus M_1$. This implies

$|M_2 \setminus M_1| \geq |M_1 \setminus M_2|$, and hence $|M_2| = |(M_2 \setminus M_1) \cup (M_1 \cap M_2)| \geq |(M_1 \setminus M_2) \cup (M_1 \cap M_2)| = |M_1|$, which contradicts $|M_2| < |M_1|$. The proof of the claim is completed.

For w_1 and f_2 as given in Claim 1, we show that (f_2, w_1) is a blocking pair of μ satisfying the Blocking Lemma. Indeed, one can know that f_2 and w_1 are unmatched under μ by $f_2 \notin \mu(w_1)$. We have shown that the condition $(f_1, w_1) \in M_1 \setminus M_2$ implies that, in the procedure of the deferred acceptance algorithm of μ_F, f_1 proposes to w_1 but w_1 has q_{w_1} choices which are better than f_1 and then rejects f_1. Combining $f_2 \in \mu_F(w_1)$, one can obtain $f_2 \succ_{w_1} f_1$. However, $(f_1, w_1) \in M_1$ indicates that f_1 and w_1 are matched together under μ. On the other hand, considering $f_2 \in F \setminus F'$, we know that $\mu_F(f_2) R(f_2) \mu(f_2)$. If $|\mu(f_2)| < q_{f_2}$, then (f_2, w_1) blocks μ. If $|\mu(f_2)| = q_{f_2}$, the extended max–min criterion implies $\min(\mu_F(f_2)) \succeq_{f_2} \min(\mu(f_2))$. Together with $w_1 \in \mu_F(f_2)$ and $w_1 \notin \mu(f_2)$, we can obtain $w_1 \succ_{f_2} \min(\mu(f_2))$. Thus, (f_2, w_1) blocks μ.

(ii) $\mu(F')$ is not a proper subset of $\mu_F(F')$.

In view of $\mu(F') \neq \mu_F(F')$, one can obtain $\mu(F') \backslash \mu_F(F') \neq \emptyset$. Choose w in $\mu(F') \backslash \mu_F(F')$ such that $w \in \mu(f')$ for some $f' \in F'$. The individual rationality of μ implies that f' is acceptable to w. Since f' prefers $\mu(f')$ to $\mu_F(f')$, $w \in \mu(f')$ implies $w \succeq_{f'} \min(\mu(f')) \succ_{f'} \min(\mu_F(f'))$ by the extended max–min criterion. This indicates that, in the procedure of the deferred acceptance algorithm, f' had ever proposed to w before it proposed to $\min(\mu_F(f'))$. Since f' is acceptable to w, $w \notin \mu_F(f')$ implies that w has q_w choices which are better than f' and then rejects f'. Then we have $|\mu_F(w)| = q_w$. Combining conditions $f' \in \mu(w)$, $f' \notin \mu_F(w)$, $|\mu_F(w)| = q_w$ and $|\mu(w)| \leq q_w$, one can infer that there must exist some firm, say f, in $\mu_F(w)$, such that $f \notin \mu(w)$.

It is easy to see that (f, w) blocks μ. Firstly, we know that $f' \in \mu(w)$ and $f \notin \mu(w)$. $f \in \mu_F(w)$ and w rejecting f' in the procedure of the deferred acceptance algorithm imply that $f \succ_w f'$. Secondly, by $w \notin \mu_F(F')$ and $w \in \mu_F(f)$ we have $f \in F \backslash F'$, and consequently $\mu_F R(f) \mu$. If $|\mu(f)| < q_f$, we are done. If $|\mu(f)| = q_f$, the extended max–min criterion implies $\min(\mu_F(f)) \succeq_f \min(\mu(f))$. Together with $w \in \mu_F(f)$ and $w \notin \mu(f)$, we can obtain $w \succ_f \min(\mu(f))$. So (f, w) blocks μ with $f \in F \backslash F'$ and $w \in \mu(F')$.

Case II. $\mu(F') = \mu_F(F') \equiv W'$.

For any $w' \in W'$, the notation $\mu(w') \cap F'$ will denote the set of firms which are matched to w' and belong to F'.

(iii) There exists some worker $w \in W'$ such that $|\mu(w) \cap F'| > |\mu_F(w) \cap F'|$.

We can find some firm, say, $f' \in F'$, such that $f' \in [\mu(w) \cap F'] \backslash [\mu_F(w) \cap F']$. Thus $w \in \mu(f') \backslash \mu_F(f')$. f' preferring μ to μ_F implies $\min(\mu(f')) \succ_{f'} \min(\mu_F(f'))$. Thus we have $w \succ_{f'} \min(\mu_F(f'))$. This indicates that f' must propose to w in the procedure of the deferred acceptance algorithm of μ_F. $f' \notin \mu_F(w)$ implies that $|\mu_F(w)| = q_w$ and w rejects f'. By $|\mu(w) \cap F'| > |\mu_F(w) \cap F'|$, we infer that $|\mu_F(w) \cap F'| < q_w$. Hence, there is at least one firm, say, $f \in F \backslash F'$, satisfying $f \in \mu_F(w) \backslash \mu(w)$. Clearly, $f \succ_w f'$. Also by $f \in F \backslash F'$, we have $\mu_F R(f) \mu$. By a similar argument as given in (i) or (ii), it is easy to show (f, w) blocks μ.

(iv) There exists some worker $w' \in W'$ such that $|\mu(w') \cap F'| < |\mu_F(w') \cap F'|$.

By conditions $\mu(F') = \mu_F(F')$ and $\sum_{f \in F'} |\mu(f)| = \sum_{f \in F'} |\mu_F(f)|$, we infer that there must be some worker $w \in W'$ such that $|\mu(w) \cap F'| > |\mu_F(w) \cap F'|$. The proof is done by (iii).

(v) $|\mu(w') \cap F'| = |\mu_F(w') \cap F'|$ for all $w' \in W'$.

By the firms-quota saturability condition, one can obtain $|\mu_F(f)| = q_f$ for all $f \in F'$. The condition $W' \equiv \mu_F(F')$ implies that each worker in W' ever received at least one proposal proposed by some firm in F'. Since both F' and W' are finite sets and no firm proposes twice to the same worker, one can infer that all members in F' propose to workers in W' finite times in the procedure of the deferred acceptance algorithm of μ_F. Then there exists a worker, say w, in W' such that w receives the last proposal among all proposals offered by firms in F' in the procedure of the deferred acceptance algorithm of μ_F (if there is more than one such worker, we choose anyone among them). For the sake of simplicity, we assume that this proposal was offered by firm $f' \in F'$ at step k.[10] Now for further argument, we show the following claim.

Claim 2. *If the assumption is as given above, then in the procedure of the deferred acceptance algorithm of μ_F, each firm $\tilde{f} \in F'$ is always matched with $q_{\tilde{f}}$ workers and there is no worker who will reject any firm in F' from step k to the termination of the algorithm.*

Proof of Claim 2. If there is some firm, say $\hat{f} \in F'$, and worker, say $\hat{w} \in W$, such that $\hat{w}$ rejects $\hat{f}$ at step $\hat{k}$ $(\geq k)$, then $\hat{f}$ has vacancy at the end of step $\hat{k}$. By the firms-quota saturability condition, one can obtain $|\mu_F(\hat{f})| = q_{\hat{f}}$. Therefore, there exists at least one worker, say $\bar{w}$, in $\mu_F(\hat{f})$ such that $\hat{f}$ proposes to $\bar{w}$ after step k. By $\mu_F(\hat{f}) \subseteq \mu_F(F') = W'$ we have $\bar{w} \in W'$. Then w receives the proposal offered by f' before $\bar{w}$ receives $\hat{f}$'s proposal, which contradicts that w receives the last proposal among all proposals offered by firms in F' in the procedure of the deferred acceptance algorithm of μ_F. Then we obtain that there is no worker who will reject any firm in F' from step k to the termination of the algorithm. Since at step k, w receives the last proposal among all proposals offered by firms in F' in the procedure of the deferred acceptance algorithm of μ_F, it is easy to see that, no firm in F' proposes to new worker after step k in the procedure of the deferred

[10] If both f_1 and f_2 (in F') propose to workers in W' at step k, then we can choose either f_1 or f_2 as f'. If f' proposes to more than one worker at step k, then we choose the worker who has the lowest preference order (with respect to f's preference) among them as w.

acceptance algorithm of μ_F. According to the firms-quota saturability condition, we have $|\mu_F(\tilde{f})| = q_{\tilde{f}}$ for all $\tilde{f} \in F'$. Then we obtain that each firm $\tilde{f} \in F'$ is always matched with $q_{\tilde{f}}$ workers from step k to the termination of the algorithm. The proof of the claim is completed.

Now we proceed with the proof of Subcase (v). By $|\mu_F(f')| = q_{f'}$ and the assumption that w receives the last proposal offered by firm $f' \in F'$ in the procedure of the deferred acceptance algorithm of μ_F, one can infer that w finally accepts f'. Otherwise, f' must propose to other workers, which contradicts that w receives the last proposal offered by firm $f' \in F'$ in the procedure of the deferred acceptance algorithm of μ_F. Then we have $(f', w) \in \mu_F$. Since w receives the last proposal offered by firm f', one can infer that $w = \min(\mu_F(f'))$.

Since f' prefers μ to μ_F, it implies $\min(\mu(f')) \succ_{f'} \min(\mu_F(f'))$ by the extended max–min criterion. Then we know that $w = \min(\mu_F(f')) \notin \mu(f')$. Also, since $w \in \mu_F(F')$ and $\mu(F') = \mu_F(F')$, we have $w \in \mu(F')$. Together with $|\mu(w) \cap F'| = |\mu_F(w) \cap F'|$, we obtain that there is some firm $f'' \in F'$ such that $w \in \mu(f'')\backslash\mu_F(f'')$. f'' preferring μ to μ_F implies $\min(\mu(f'')) \succ_{f''} \min(\mu_F(f''))$. Thus we have $w \succ_{f''} \min(\mu_F(f''))$. This indicates that f'' must propose to w in the procedure of the deferred acceptance algorithm of μ_F. $f'' \notin \mu_F(w)$ implies that $|\mu_F(w)| = q_w$ and w rejects f''. By Claim 2 we know that w rejects f'' at some step before step k. This indicates that w already has q_w (better than f'') choices from step $(k-1)$. Since f' proposes to w and is accepted by w at step k and w does not reject any firm belonging to F' at this step, w must reject some firm $f \in F\backslash F'$ and accept f' at step k.

Then, (a) If $(f, w) \notin \mu$, we can prove (f, w) is a blocking pair of μ. Indeed, on the one hand, since w rejects f'' before she rejects f in the deferred acceptance algorithm of μ_F, we have $f \succ_w f''$. However, $(f, w) \notin \mu$ and $(f'', w) \in \mu$. On the other hand, $f \in F\backslash F'$ implies $\mu_F R(f)\mu$. If $|\mu(f)| < q_f$, then (f, w) blocks μ. If $|\mu(f)| = q_f$, $\mu_F R(f)\mu$ implies $\min(\mu_F(f)) \succeq_f \min(\mu(f))$. Since w rejects f in the procedure of the deferred acceptance algorithm of μ_F, we have $w \succ_f \min(\mu_F(f))$ and in turn $w \succ_f \min(\mu(f))$. So (f, w) blocks μ with $f \in F\backslash F'$ and $w \in \mu(F')$.

(b) If $(f, w) \in \mu$, there must exist at least one firm, say, $f_0 \in F \setminus F'$ satisfying $f_0 \in \mu_F(w) \setminus \mu(w)$, as $|\mu_F(w)| = q_w$, $|\mu(w) \cap F'| = |\mu_F(w) \cap F'|$, $f \in F\backslash F'$, $f \notin \mu_F(w)$ and $f \in \mu(w)$. Then we can show (f_0, w) is a blocking pair of μ. To see this, on the one hand, since $f_0 \in \mu_F(w)$ and w rejects f'' in the deferred acceptance algorithm of μ_F, we have $f_0 \succ_w f''$. However, $(f_0, w) \notin \mu$ and $(f'', w) \in \mu$. On the other hand, $f_0 \in F\backslash F'$ implies $\mu_F R(f_0)\mu$. If $|\mu(f_0)| < q_{f_0}$, then (f_0, w) blocks μ. If $|\mu(f_0)| = q_{f_0}$, $\mu_F R(f_0)\mu$ implies $\min(\mu_F(f_0)) \succeq_{f_0} \min(\mu(f_0))$. Together with $w \in \mu_F(f_0)$ and $w \notin \mu(f_0)$, we can obtain $w \succ_{f_0} \min(\mu(f_0))$. Thus, (f_0, w) blocks μ. The proof is completed. □

Proof of Theorem 2. The proof is precisely the same as Gale and Sotomayor's (1985) proof. The only difference is to replace their Blocking Lemma for one-to-one matching with the Blocking Lemma for many-to-many matching obtained in this paper. We refer the readers to Gale and Sotomayor (1985) for detailed proof. □

References

Alkan, A., 2001. On preferences over subsets and the lattice structure of stable matchings. Rev. Econ. Design 6, 99–111.
Baïou, M., Balinski, M., 2000. Many-to-many matching: stable polyandrous polygamy (or polygamous polyandry). Discrete Appl. Math. 101, 1–12.
Barberà, S., Sonnenschein, H., Zhou, L., 1991. Voting by committees. Econometrica 59, 595–609.
Dubins, L.E., Freedman, D.A., 1981. Machiavelli and the Gale–Shapley algorithm. Amer. Math. Monthly 88, 485–494.
Dutta, B., Massó, J., 1997. Stability of matchings when individuals have preferences over colleagues. J. Econ. Theory 75, 464–475.
Echenique, F., Oviedo, J., 2006. A theory of stability in many-to-many matching markets. Econ. Theory 1, 233–273.
Ehlers, L., Klaus, B., 2003. Coalitional strategy-proof and resource-monotonic solutions for multiple-assignment problems. Soc. Choice Welfare 21, 265–280.
Gale, D., Shapley, L., 1962. College admissions and the stability of marriage. Amer. Math. Monthly 69, 9–15.
Gale, D., Sotomayor, M., 1985. Some remarks on the stable matching problem. Discrete Appl. Math. 11, 223–232.
Hatfield, J.W., Kojima, F., 2008. Matching with contracts: comment. Amer. Econ. Rev. 98, 1189–1194.
Hatfield, J.W., Kojima, F., 2009. Group incentive compatibility for matching with contracts. Games Econ. Behav. 67, 745–759.
Hatfield, J.W., Kojima, F., 2010. Substitutes and stability for matching with contracts. J. Econ. Theory 145, 1704–1723.
Hatfield, J.W., Kojima, F., Narita, Y., 2014. Many-to-many matching with max–min preferences. Discrete Appl. Math. 179, 235–240.
Hatfield, J.W., Milgrom, P., 2005. Matching with contracts. Amer. Econ. Rev. 95, 913–935.
Jiao, Z., Tian, G., 2015a. Efficiency, strategy-proofness and monotonicity for colleges in college admissions problem. Working paper.
Jiao, Z., Tian, G., 2015b. The stability of many-to-many matching with max–min preferences. Econ. Letters 129, 52–56.
Kelso, A., Crawford, V., 1982. Job matching, coalition formation, and gross substitutes. Econometrica 50, 1483–1504.
Klaus, B., Walzl, M., 2009. Stable many-to-many matchings with contracts. J. Math. Econ. 45, 422–434.
Kojima, F., 2007. When can manipulations be avoided in two-sided matching markets? Maximal domain results. Contrib. Theor. Econ. 7. Article 32.
Martínez, R., Massó, J., Neme, A., Oviedo, J., 2000. Single agents and the set of many-to-one stable matchings. J. Econ. Theory 91, 91–105.
Martínez, R., Massó, J., Neme, A., Oviedo, J., 2001. On the lattice structure of the set of stable matchings for a many-to-one model. Optimization 50, 439–457.
Martínez, R., Massó, J., Neme, A., Oviedo, J., 2004a. On group strategy-proof mechanisms for a many-to-one matching model. Int. J. Game Theory 33, 115–128.
Martínez, R., Massó, J., Neme, A., Oviedo, J., 2004b. An algorithm to compute the full set of many-to-many stable matchings. Math. Soc. Sci. 47, 187–210.
Martínez, R., Massó, J., Neme, A., Oviedo, J., 2010. The Blocking Lemma for a many-to-one matching model. J. Math. Econ. 46, 937–949.
Papai, S., 2000. Strategy-proof multiple assignment using quotas. Rev. Econ. Design 5, 91–105.
Roth, A.E., 1982. The economics of matching: stability and incentives. Math. Oper. Res. 7, 617–628.
Roth, A.E., 1984. Stability and polarization of interests in job matching. Econometrica 52, 47–57.
Roth, A.E., 1985. The college admissions problem is not equivalent to the marriage problem. J. Econ. Theory 36, 277–288.
Roth, A.E., Sotomayor, M., 1990. Two-sided matching: a study in game-theoretic modeling and analysis. In: Econometric Society Monographs. Cambridge University Press.

Sotomayor, M., 1999. Three remarks on the many-to-many stable matching problem. Math. Soc. Sci. 38, 55–70.
Sotomayor, M., 2004. Implementation in the many-to-many matching market. Games Econ. Behav. 46, 199–212.
Sönmez, T., 1996. Strategy-proofness in many-to-one matching problems. Econ. Design 1, 365–380.