Economics Letters 129 (2015) 52–56

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

The stability of many-to-many matching with max–min preferences

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HIGHLIGHTS

- The equivalence between the pairwise-stability and the setwise-stability is obtained.
- We show that the pairwise-stability implies the strong corewise-stability.
- We show that the strong core may be a proper subset of the core.
- We show that the deferred acceptance algorithm yields a pairwise-stable matching.

ARTICLE INFO

Article history:
Received 6 November 2014
Received in revised form
24 January 2015
Accepted 6 February 2015
Available online 14 February 2015

JEL classification:
C78

Keywords:
Many-to-many matching
Pairwise-stability
Core
Setwise-stability
Max–min preferences

ABSTRACT

This paper investigates the two-sided many-to-many matching problem, where every agent has max–min preference. The equivalence between the pairwise-stability and the setwise-stability is obtained. It is shown that the pairwise-stability implies the strong corewise-stability and the former may be strictly stronger than the latter. We also show that the strong core may be a proper subset of the core. The deferred acceptance algorithm yields a pairwise-stable matching. Thus the set of stable matchings (in all four senses) is non-empty.

1. Introduction

For a matching problem, researchers primarily concern the existence and the structure of stable assignments. There are several concepts on stability of matching: pairwise-stability, corewise-stability, strong corewise-stability and setwise-stability (or group stability). Gale and Shapley (1962, henceforth GS) originally study the stable matching between men and women, and, between students and colleges. They give the definition of (pairwise-)stable matching and propose the deferred acceptance algorithm to yield the optimal stable matching. Roughly speaking, pairwise-stability means that the matching is individually rational and there exists no pair of unmatched players who can become better off if they are matched together. For one-to-one matching problem, it is enough to explore its pairwise-stability. If every player has strict preference, the above-mentioned four different concepts of stability are equivalent for the one-to-one case.

The concepts of the core and the strong core originate from the cooperative game theory. In matching theory, corewise-stability (resp. strong corewise-stability) describes the condition of no subset of players, who by forming *all* their partnerships among themselves, can *all* obtain a strictly preferred set of partners (resp. can *all* obtain a weakly preferred set of partners and *at least one of them* becomes strictly better off). Clearly, the strong corewise-stability strengthens the requirement of the corewise-stability.

Roth (1985) proposes the concept of group-stability in the context of the college admissions problem. For many-to-many matching problem, Sotomayor (1999) called the notion of group-stability as setwise-stability, which characterizes the condition that there is no subset of players who by forming *new* partnerships only among themselves, possibly dissolving some partnerships of the given

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http://dx.doi.org/10.1016/j.econlet.2015.02.009
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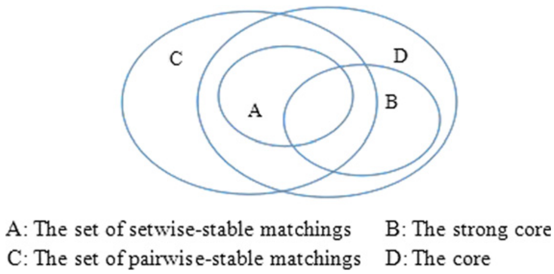

Fig. 1. The relationships between different concepts of stability under separable preference.

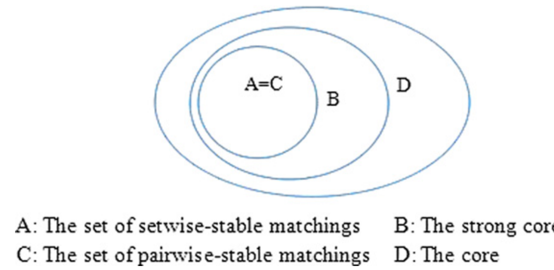

Fig. 2. The relationships between different concepts of stability under max–min preference.

matching to remain within their quotas and possibly keeping other ones, can *all* obtain a strictly preferred set of partners. The concept of setwise-stability generalizes that of pairwise-stability. Obviously, the setwise-stability implies both pairwise-stability and corewise-stability.

Matching theory proceeds normally by imposing hypotheses on agents' preferences. For many-to-many matchings, under the assumption of separable preferences,[1] Sotomayor (1999) shows that pairwise-stability is independent of the corewise-stability and setwise-stability may be strictly stronger than the requirement of pairwise-stability plus corewise-stability. She also constructs an artful example such that both the core and the set of pairwise-stable matchings are nonempty, but their intersection set is empty. Consequently, Sotomayor obtains that there may be no setwise-stable assignment for many-to-many matching with separable preferences. Since separability implies responsiveness and consequently substitutability,[2]the above-mentioned results hold for these three kinds of preferences. More intuitively, the relationship between different concepts of stability obtained by Sotomayor can be expressed by a Venn diagram (see Fig. 1).

This paper investigates the stability of many-to-many matching with max–min preference. For a many-to-many matching problem, we prove that, if every agent has max–min preference, then the deferred acceptance algorithm yields a pairwise-stable assignment. We also show that the pairwise-stability is equivalent to the setwise-stability and the pairwise-stability may be strictly stronger than the strong corewise-stability. Thus it implies that both the core and the strong core are nonempty. Summarily, we obtain the relationship between different concepts of stable matching under max–min preference as in Fig. 2.

2. The model

Let R denote a finite set of *row-players* and C a finite set of *column-players*.[3] Each $r \in R$ has a strict, complete and transitive preference relation $\succ_r$ over C and a quota p_r. The weak preference relation associated with $\succ_r$ is denoted by $\succeq_r$. For any $c_1, c_2 \in C$, $c_1 \succeq_r c_2$ means either $c_1 \succ_r c_2$ or $c_1 = c_2$. The notation $c \succ_r \emptyset$ means that the column player c is *acceptable* to r and $\emptyset \succ_r c$ denotes that c is *unacceptable* to r. For column-players, we can define corresponding notation and denote the quota of column-player c by q_c.

For many-to-many matching, we also need to consider agents' preferences over groups of players on the opposite side. We assume these preferences are transitive, but the completeness is not required. Throughout this paper, we assume that the preference relation of every row-player satisfies the following property:

Weak monotonicity in population: For each row-player r, for any $S \in 2^C$ with $|S| < p_r$ and any column-players c not in S, r prefers $S \cup \{c\}$ to S if and only if c is acceptable to r, where the notation $|S|$ denotes the number of elements in S.[4]

Corresponding weak monotonicity in population for the preference relation of column-player is also required.

Given a set of agent r's partners S, let $\min(S)$ denote the least preferred partner of r in S.

Baïou and Balinski (2000) propose the max–min preference and study the Pareto efficiency and incentives properties for many-to-many matching when every player has max–min preference. In the framework of matching markets and the setting of max–min preference, Hatfield et al. (2014) obtain some negative results on the Pareto efficiency and incentives properties of many-to-many matching by constructing an ingenious example. In this paper, we follow the definition of max–min preference introduced by Baïou and Balinski as below:

Definition 1. The preference relation of row-player $r \in R$ is said to satisfy the *max–min criterion* if the following condition is met: for any two sets of acceptable column-players $S_1, S_2 \in 2^C$ with $|S_1| \leq p_r$ and $|S_2| \leq p_r$,

(i) The strict preference relation $\succ_r$ over groups of column-players is defined as: $S_1 \succ_r S_2$ if and only if S_2 is a proper subset of S_1 or, $|S_1| \geq |S_2|$ and r strictly prefers the least preferred column-player in S_1 to the least preferred column-player in S_2.
(ii) The weak preference relation $\succeq_r$ over groups of column-players is defined as: $S_1 \succeq_r S_2$ if and only if $S_1 \succ_r S_2$ or $S_1 = S_2$.

The preference relation of column-player $c \in C$ satisfies the max–min criterion if the corresponding condition is met.

We note that there is no implication relationship between max–min preference and responsive preference. In fact, firstly, max–min criterion is not stronger than responsiveness. For example, we assume $c_1 \succ_r c_2 \succ_r c_3$, $p_r = 2$ and $S = \{c_3\}$. Under max–min criterion we cannot achieve $S \cup \{c_1\} \succ_r S \cup \{c_2\}$. Secondly, responsiveness is not stronger than max–min criterion. For example, we assume $c_1 \succ_r c_2 \succ_r c_3 \succ_r c_4$ and $p_r = 2$. Under responsiveness we cannot infer $\{c_2, c_3\} \succ_r \{c_1, c_4\}$.

1 Separable preference is defined as follows: let u_{ij} denote the utility that i can get in case i and j form a partnership. For any two sets of partners of i, S, S' with $|S| < q_i$ and $|S'| < q_i$, the player i prefers S to S' if and only if $\sum_{j \in S} u_{ij} > \sum_{j \in S'} u_{ij}$.

2 Responsive preference is defined as follows: For any i, j and any S such that $i, j \notin S$ and $|S| < q_k$, $S \cup \{i\} \succ_k S \cup \{j\}$ if and only if $i \succ_k j$, where i, j are the partners of k and S is a set of partners of k. Separability implies responsiveness because that, for any i, j and any S such that $i, j \notin S$ and $|S| < q_k$, if $\succ_k$ is separable, then by definition it is easy to obtain $S \cup \{i\} \succ_k S \cup \{j\}$ if and only if $i \succ_k j$.

3 For example, row- and column-players may correspond to firms and workers.

4 Row-players' preferences are strongly monotonic in population if $\forall r \in R, \forall S, S' \in 2^C$, $|S'| < |S| \leq p_r$ implies $S \succ_r S'$ (see Konishi and Ünver, 2006). Obviously, strong monotonicity implies weak monotonicity.

As we know, for many-to-one or many-to-many matching problem, substitutable preference[5] is often adopted by researchers. Here we note that max–min criterion is stronger than substitutability. Indeed, we know that responsiveness implies substitutability and responsiveness does not imply max–min criterion. Then we obtain that substitutability does not imply max–min criterion. It is easy to show that max–min criterion implies substitutability. Specifically, we assume agent r's preference relation $\succ_r$ satisfies max–min criterion. Let S and S' be sets of column-players, with $S \subseteq S'$. Suppose $c \in Ch(S' \cup \{c\}, \succ_r)$. We shall prove $c \in Ch(S \cup \{c\}, \succ_r)$. Consider the following two cases: Case (I) $|Ch(S \cup \{c\}, \succ_r)| < p_r$. The condition $c \in Ch(S' \cup \{c\}, \succ_r)$ implies that c is acceptable to r. Then, by the definition of max–min criterion, we have $c \in Ch(S \cup \{c\}, \succ_r)$. Case (II) $|Ch(S \cup \{c\}, \succ_r)| = p_r$. We have $|Ch(S' \cup \{c\}, \succ_r)| = p_r$ according to max–min criterion. Since $S \cup \{c\} \subseteq S' \cup \{c\}$, we have $\min(Ch(S' \cup \{c\}, \succ_r)) \succeq_r \min(Ch(S \cup \{c\}, \succ_r))$. The condition $c \in Ch(S' \cup \{c\}, \succ_r)$ implies $c \succeq_r \min(Ch(S' \cup \{c\}, \succ_r))$. Thus we infer $c \succeq_r \min(Ch(S \cup \{c\}, \succ_r))$. By max–min criterion, $c \in Ch(S \cup \{c\}, \succ_r)$.

2.1. Definitions of stable matching

A *matching* is a correspondence $\mu : R \cup C \to 2^{R \cup C}$ such that

(1) $\mu(r) \subseteq C$ and $|\mu(r)| \leq p_r$ for all $r \in R$,
(2) $\mu(c) \subseteq R$ and $|\mu(c)| \leq q_c$ for all $c \in C$,
(3) $r \in \mu(c)$ if and only if $c \in \mu(r)$ for all $r \in R$ and $c \in C$.

We denote by $\mu(r) \equiv \{c \in C | r \in \mu(c)\}$ the set of column-players who are assigned to r and $\mu(c) \equiv \{r \in R | c \in \mu(r)\}$ the set of row-players who are assigned to c. We will denote by $\min(\mu(r))$ the least preferred column-player for r in $\mu(r)$.

For any two matchings μ and μ', the notation $\mu \succ_r \mu'$ means $\mu(r) \succ_r \mu'(r)$, $\mu \succeq_r \mu'$ means $\mu \succ_r \mu'$ or $\mu(r) = \mu'(r)$, $\mu \succeq_R \mu'$ means $\mu \succeq_r \mu'$ for any $r \in R$, and, $\mu \succ_R \mu'$ means $\mu \succeq_R \mu'$ and $\mu \neq \mu'$.

A matching μ is *blocked by an individual* $i \in R \cup C$ if there exists some player $j \in \mu(i)$ such that $\emptyset \succ_i j$. A matching is *individually rational* if it is not blocked by any individual. A matching μ is *blocked by a pair* $(r, c) \in R \times C$ if they are not matched together under μ and

(1) c is acceptable to r and r is acceptable to c,
(2) $|\mu(r)| < p_r$ or $c \succ_r c'$ for some $c' \in \mu(r)$,
(3) $|\mu(c)| < q_c$ or $r \succ_c r'$ for some $r' \in \mu(c)$.

Definition 2. A matching μ is *pairwise-stable* if it is not blocked by any individual or a pair.

An individually rational matching μ' *dominates* another matching μ via a coalition S contained in $R \cup C$ if for all row players r and column-players c in S,

(i) If $C' = \mu'(r)$, then $C' \subseteq S$, and if $R' = \mu'(c)$, then $R' \subseteq S$; and
(ii) $\mu'(r) \succ_r \mu(r)$, and $\mu'(c) \succ_c \mu(c)$.

Definition 3. A matching μ is *in the core (corewise-stable)* if it is not dominated by any other matching.

Similarly, an individually rational matching μ' *weakly dominates* another matching μ via a coalition S contained in $R \cup C$ if for all row players r and column-players c in S,

(i) If $C' = \mu'(r)$ then $C' \subseteq S$, and if $R' = \mu'(c)$ then $R' \subseteq S$;
(ii) $\mu'(r) \succeq_r \mu(r)$, and $\mu'(c) \succeq_c \mu(c)$ and
(iii) $\mu'(s) \succ_s \mu(s)$ for some $s \in S$.

Definition 4. A matching μ is *in the strong core (strong corewise-stable)* if it is not weakly dominated by any other matching.

Clearly, the strong core is a subset of the core.

In order to give a more general concept of stability, we first define the notion of setwise-instability. Precisely, we will call a matching μ *setwise-instable* if there exist some individually rational matching μ' and a subset $S \subseteq R \cup C$ such that for all $i \in S$,

(1) $\mu'(i) \succ_i \mu(i)$, and
(2) $j \in \mu'(i)$ implies $j \in S \cup \mu(i)$.

Definition 5. A matching μ is *setwise-stable* if it is not setwise-instable.

By definition, one can easily see that setwise-stability implies not only pairwise-stability but also corewise-stability.

2.2. Deferred acceptance algorithm

The deferred acceptance algorithm was first proposed by GS to find a stable assignment for the marriage problem (one-to-one matching) and college admissions problem (many-to-one matching). If every player has max–min preference, then the deferred acceptance algorithm can be used to find a stable assignment for a many-to-many matching.

Specifically, the Row-Players-Proposing Deferred Acceptance Algorithm proceeds as follows:

Step 1. (a) Each row-player r proposes to her top p_r acceptable column-players (and if she has fewer acceptable choices than p_r, then she proposes to all her acceptable choices).

(b) Each column-player c then places on his waiting list the q_c row-players who rank highest, or all row-players if there are fewer than q_c, and rejects the rest.

In general, at

Step k. (c) Any row-player r who was rejected at step $(k-1)$ by any column-player proposes to her most-preferred p_r acceptable column-players who have not yet rejected her (and if there are fewer than p_r remaining acceptable column-players, then she proposes to all).

(d) Each column-player c selects the top q_c – or all row-players if there are fewer than q_c – from among the new row-players and those on his waiting list, puts them on his new waiting list, and rejects the rest.

Since no row-player proposes twice to the same column-player, this algorithm always terminates in a finite number of steps. The algorithm terminates when there are no more rejections. Each column-player is matched with row-players on his waiting list in the last step.

In the next section, we will show that the matching produced by the row-players-proposing deferred acceptance algorithm, denoted by μ_R, is pairwise-stable. Symmetrically, the matching produced by the column-players-proposing deferred acceptance algorithm, denoted by μ_C, is also pairwise-stable.

3. Relationship between different concepts of stability

We first show that the deferred acceptance algorithm produces a pairwise-stable assignment and consequently the set of pairwise-stable matchings is nonempty.

[5] Kelso and Crawford (1982) first introduce the concept of substitutability. In the language of firms–workers matching model, substitutability of firm r's preferences requires: "if hiring c is optimal when certain workers are available, hiring c must still be optimal when a subset of workers are available". Formally, an agent r's preference relation $\succ_r$ satisfies substitutability if, for any sets S and S' with $S \subseteq S'$, $c \in Ch(S' \cup \{c\}, \succ_r)$ implies $c \in Ch(S \cup \{c\}, \succ_r)$, where $Ch(S \cup \{c\}, \succ_r)$ denotes agent r's most-preferred subset of $S \cup \{c\}$ according to r's preference relation $\succ_r$.

Proposition 1. *For a many-to-many matching problem, if every player has max–min preference, then the deferred acceptance algorithm yields a pairwise-stable matching.*

Proof. Suppose there is a pair (c, r) such that c is acceptable to r and r is acceptable to c, but $(r, c) \notin \mu_R$. By the procedure of the deferred acceptance algorithm, there are two possible cases:

(i) r does not propose to c. This indicates that r is matched to p_r column-players who are better than c for r. Then (r, c) is not a blocking pair.

(ii) r proposes to c, but c rejects r. This indicates that c is matched to q_c column-players who are better than r for c. Then (r, c) is not a blocking pair. The proof is completed. □

Remark. For many-to-many matching problem, Echenique and Oviedo (2006) and Klaus and Walzl (2009) prove that the set of pairwise-stable matchings in nonempty under substitutable preferences. Since max–min criterion implies substitutability, one can know that the set of pairwise-stable matchings is nonempty under max–min preferences. Proposition 1 lies in specifying a pairwise-stable matching outcome by the deferred acceptance algorithm. In the sense of the existence of pairwise-stable matching, there is coincidence between Proposition 1 and the above-mentioned result of Echenique and Oviedo (2006) and Klaus and Walzl (2009).

The following result shows the equivalence between the pairwise-stability and the setwise-stability.

Proposition 2. *For a many-to-many matching problem, if every player has max–min preference, then the pairwise-stability is equivalent to the setwise-stability.*

Proof. By definition we know that the setwise-stability implies the pairwise-stability. We show the opposite implication. Let μ be a pairwise-stable assignment. We want to show μ is setwise-stable. We argue by contradiction. Suppose μ is not setwise-stable. There exist some individually rational matching μ' and a subset of agents $S \subseteq R \cup C$ such that for all $i \in S$, (1) $\mu'(i) \succ_i \mu(i)$ and, (2) $j \in \mu'(i)$ implies $j \in S \cup \mu(i)$. Without loss of generality, we assume there exists some row-player $r \in S$ such that $\mu'(r) \succ_r \mu(r)$. According to the definition of max–min preference, it implies that there must exist some column-player, say, c, such that $c \in \mu'(r) \setminus \mu(r)$, and consequently $c \in S$. Then it is easy to show (r, c) blocks μ. In fact, we consider the following four cases: (i) $|\mu(r)| < p_r$ and $|\mu(c)| < q_c$. Since $(r, c) \in \mu'$ implies that r is acceptable to c and c is acceptable to r, (r, c) blocks μ. (ii) $|\mu(r)| < p_r$ and $|\mu(c)| = q_c$. $c \in S$ implies $\mu'(c) \succ_c \mu(c)$. By max–min preference, we have $|\mu'(c)| = q_c$ and $\min(\mu'(c)) \succ_c \min(\mu(c))$. Since $r \in \mu'(c)$, $r \succ_c \min(\mu(c))$. Then (r, c) blocks μ. (iii) $|\mu(r)| = p_r$ and $|\mu(c)| < q_c$. $\mu'(r) \succ_r \mu(r)$ implies $|\mu'(r)| = p_r$ and $\min(\mu'(r)) \succ_r \min(\mu(r))$. Since $c \in \mu'(r)$, $c \succ_r \min(\mu(r))$. Then (r, c) blocks μ. (iv) $|\mu(r)| = p_r$ and $|\mu(c)| = q_c$. By (ii) and (iii) we know $c \succ_r \min(\mu(r))$ and $r \succ_c \min(\mu(c))$. Then (r, c) blocks μ. The proof is completed. □

In the following, we investigate the relationship between the set of pairwise stable matchings and the core (the strong core).

Proposition 3. *For a many-to-many matching problem, if every player has max–min preference, then the pairwise-stability implies the strong corewise-stability.*

Proof. Let μ be a pairwise-stable assignment. We want to show μ is strong corewise-stable. We argue by contradiction. Suppose μ is not in the strong core. Then there exists an individually rational matching μ' that weakly dominates μ via a coalition S contained in $R \cup C$. Without loss of generality, we assume there exists some row-player $r \in S$ such that $\mu'(r) \succ_r \mu(r)$. This implies that there must exist some column-player, say, c, such that $c \in \mu'(r) \setminus \mu(r)$. Then it is easy to show (r, c) blocks μ. In fact, we consider the following four cases: (i) $|\mu(r)| < p_r$ and $|\mu(c)| < q_c$. Since r is acceptable to c and c is acceptable to r, (r, c) blocks μ. (ii) $|\mu(r)| < p_r$ and $|\mu(c)| = q_c$. $c \in \mu'(r)$ implies $c \in S$. Then $\mu'(c) \succeq_c \mu(c)$. Since $r \in \mu'(c) \setminus \mu(c)$ implies $\mu'(c) \neq \mu(c)$, we have $\mu'(c) \succ_c \mu(c)$ and consequently $\min(\mu'(c)) \succ_c \min(\mu(c))$. Together with the condition $r \in \mu'(c)$ we have $r \succ_c \min(\mu(c))$. Then (r, c) blocks μ. (iii) $|\mu(r)| = p_r$ and $|\mu(c)| < q_c$. By max–min criterion, $\mu'(r) \succ_r \mu(r)$ implies $|\mu'(r)| = p_r$ and $\min(\mu'(r)) \succ_r \min(\mu(r))$. Since $c \in \mu'(r)$, $c \succ_r \min(\mu(r))$. Then (r, c) blocks μ. (iv) $|\mu(r)| = p_r$ and $|\mu(c)| = q_c$. By (ii) and (iii) we know $c \succ_r \min(\mu(r))$ and $r \succ_c \min(\mu(c))$. Then (r, c) blocks μ. The proof is completed. □

Since the strong corewise-stability implies the corewise-stability, together with Proposition 3, we have the following corollary.

Corollary. *For a many-to-many matching problem, if every player has max–min preference, then the pairwise-stability implies the corewise-stability.*

The following two results show that the set of pairwise-stable matchings may be a proper subset of the strong core and the strong core may be a proper subset of the core.

Proposition 4. *For a many-to-many matching problem, if every player has max–min preference, then the pairwise-stability may be strictly stronger than the strong corewise-stability.*

Proof. We consider the following example. Suppose there are three row-players r_1, r_2, r_3 and three column-players c_1, c_2, c_3. The quotas are $p_{r_1} = p_{r_2} = q_{c_2} = q_{c_3} = 2$ and $p_{r_3} = q_{c_1} = 1$. The preferences are as follows:

$\succ_{r_1}: c_3\, c_2\, c_1$ $\quad \succ_{r_2}: c_2\, c_3\, c_1$ $\quad \succ_{r_3}: c_3\, c_2\, c_1$

$\succ_{c_1}: r_1\, r_2\, r_3$ $\quad \succ_{c_2}: r_2\, r_1\, r_3$ $\quad \succ_{c_3}: r_3\, r_1\, r_2$.

We assume a matching μ is defined as $\mu(r_1) = \{c_1, c_2\}$; $\mu(r_2) = \{c_2, c_3\}$; $\mu(r_3) = \{c_3\}$. Obviously, (r_1, c_3) blocks μ, so μ is not pairwise-stable. It is easy to check that μ is strong corewise-stable. □

Proposition 5. *For a many-to-many matching problem, if every player has max–min preference, then the strong corewise-stability may be strictly stronger than the corewise-stability.*

Proof. We consider the following example. Suppose there are three row-players r_1, r_2, r_3 and three column-players c_1, c_2, c_3. The quotas are $p_{r_2} = p_{r_3} = q_{c_1} = q_{c_2} = 1$ and $p_{r_1} = q_{c_3} = 2$. The preferences are as follows:

$\succ_{r_1}: c_1\, c_3\, c_2$ $\quad \succ_{r_2}: c_3\, c_2\, c_1$ $\quad \succ_{r_3}: c_3\, c_1\, c_2$

$\succ_{c_1}: r_1\, r_2\, r_3$ $\quad \succ_{c_2}: r_1\, r_3\, r_2$ $\quad \succ_{c_3}: r_3\, r_1\, r_2$.

We assume a matching μ is defined as $\mu(r_1) = \{c_1, c_2\}$; $\mu(r_2) = \{c_3\}$; $\mu(r_3) = \{c_3\}$. For a subset of agents $S = \{r_1, r_3, c_1, c_3\}$, we assume an assignment μ' is given as $\mu'(r_1) = \{c_1, c_3\}$; $\mu'(r_3) = \{c_3\}$. The matching μ is not strong corewise-stable because μ' weakly dominates μ via $S = \{r_1, r_3, c_1, c_3\}$, and r_1 and c_3 strictly prefer μ' to μ. Since r_3 and c_1 are indifferent between μ and μ', the coalition S does not block μ. It is easy to check that μ is corewise-stable. □

The results of this section can be summarized as follows:

(1) Pairwise-stability $\Leftrightarrow$ setwise-stability $\Rightarrow$ strong corewise-stability $\Rightarrow$ corewise-stability and neither of the two one-directional arrows can be reversed.

(2) The deferred acceptance algorithm yields a pairwise-stable matching outcome and the set of stable matchings (in all four senses) is non-empty.

Acknowledgments

The authors would like to thank the editor and an anonymous referee for their helpful comments and suggestions. Financial support from the National Natural Science Foundation of China (NSFC-71371117), the Key Laboratory of Mathematical Economics (SUFE) at Ministry of Education of China and the Natural Science Foundation of Zhejiang Province (No. Y15G010045) is gratefully acknowledged.

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