On the Efficiency of Wage-Setting Mechanisms with Search Frictions and Human Capital Investment*

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Abstract

A challenge facing labor economists is to explain the existence of different wage-setting mechanisms. This paper investigates the relative efficiency of competitive wage, wage bargaining and wage posting. In a search-theoretical model with human capital investment we establish the conditions, which are associated to the cost structures of human capital investment and job vacancy creation, for one of them to prevail. The insight is that a mechanism generates the highest level of equilibrium welfare by achieving the best balance between aggregate output and aggregate cost. Moreover, we find under the Hosios condition and for a broad range of cost parameter values that: if workers' matching contribution is sufficiently larger than their output contribution, then wage posting prevails; if their output contribution is sufficiently larger than their matching contribution, then wage bargaining prevails; if the two contributions are sufficiently close to each other, then competitive wage prevails. These findings justify the survey evidence reported by Hall and Krueger (2012).

Keywords: Competitive wage; Wage bargaining; Wage posting; Welfare comparison; Human capital investment; Search frictions.

JEL classification codes: D40; D61; J63; J64.

1 Introduction

For any equilibrium search theory involving labor market, Rogerson et al. (2005) identify two paramount factors from a huge body of literature: search friction and wage-setting mechanism. Hall and Krueger (2012) provide the survey evidence that wage bargaining and wage posting coexist in the US labor market. Some other studies (e.g., Kiyotaki and Moore, 2012; Aruoba et al., 2011), however, highlight the advantage

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of competitive wage in standard macro models. We hence address the question: which one induces the highest level of social welfare? To the best of our knowledge, our study represents the first attempt to explore optimal wage-setting arrangements in a search theoretical model with human capital investment. Instead of showing which one better matches the data, we document their efficiency differences, theoretically and numerically. We are not motivated to propose a substitutive mechanism but to further enhance our understanding of their relative advantage. As such, our theory not only helps to explain the existence of all three mechanisms but also characterizes the links of their relative efficiency with the underlying economic environments.

We consider a situation where workers must make human capital investment before finding a job and search frictions prevent ex ante contracting[1] between workers and firms. Thus, workers may underinvest or overinvest, and the entry of firms may be too high or too low, resulting in a too low or too high unemployment rate. In the presence of search frictions, on the one hand, both wages to be received in a match and the job finding probability shape a worker's incentive of human capital investment; on the other hand, both wages to be paid out and the productivity of the worker to be employed shape a firm's incentive of job vacancy creation. The incentives are intertwined through wages, so it is not surprising that we are interested in exploring optimal wage-setting arrangements.

We compare the three wage-setting mechanisms regarding the steady-state social welfare under the assumption of power production function and Cobb-Douglas matching function.[2] As the welfare function is nonlinear with respect to three interdependent variables determined by three nonlinear equations, we can just establish necessary conditions when comparing competitive wage and wage bargaining and also need to impose the restriction that a worker's marginal contribution rates to production and matching are equal when comparing bargaining and posting. These restrictions must be tolerated for deriving the formal results.

First, competitive wage induces a higher firm entry rate, a lower search unemployment rate and more human capital investments than does wage posting, thereby exhibiting an advantage in aggregate output while a disadvantage in aggregate cost compared to posting. If the marginal cost of job creation is bounded above, then its advantage outweighs its disadvantage, resulting in a higher level of social welfare than under posting. If, however, the marginal cost of job creation is bounded below, then its advantage is dominated by its disadvantage, resulting in a lower level of social welfare than under posting.

Second, suppose Hosios condition[3] holds, then for competitive wage to induce a higher firm entry rate, a lower search unemployment rate and more human capital investments than does wage bargaining, it is necessary that either the worker or the firm has an output share (under competitive wage) greater

[1] There is a natural incompleteness of contracts: before workers finish education investments, workers and firms cannot write labor contracts based on education.

[2] Note that when capital input is fixed, Cobb-Douglas or CES production function reduces to power function in labor input.

[3] That is, the elasticity of the matching function with respect to the number of unemployed is equal to the share of workers in the surplus of a match (see Hosios, 1990).

than the corresponding bargaining share. Thus, if the marginal cost of job creation is bounded above while the average cost of human capital investment is bounded below, to induce a higher level of social welfare under competitive wage than under bargaining it is necessary that either the worker or the firm has a stronger incentive under competitive wage than under bargaining. If both costs are bounded above, to induce a higher level of social welfare under bargaining than under competitive wage it is necessary that firms' bargaining share is greater than its output share under competitive wage.

Third, suppose Hosios condition holds and the average cost of human capital investment is bounded below, then posting induces a higher firm entry rate, a lower search unemployment rate and more human capital investments than does bargaining. Thus, if the marginal cost of job creation is bounded above, posting's relative advantage in aggregate output outweighs its relative disadvantage in aggregate cost, resulting in a higher level of social welfare than under bargaining. If, however, the marginal cost of job creation is bounded below, posting's relative advantage is dominated by its relative disadvantage, resulting in a lower level of social welfare than under bargaining. These conclusions reveal the substantial relevance of cost structures in mutual welfare comparison.

And fourth, we find under the Hosios condition and for a broad range of cost parameter values that: if workers' matching contribution (measured by matching elasticity) is sufficiently larger than their output contribution (measured by output elasticity), then wage posting prevails; if their output contribution is sufficiently larger than their matching contribution, then wage bargaining prevails; if the two contributions are sufficiently close to each other, then competitive wage prevails. So, each one of them can be dominant in equilibrium social welfare within a typical region of the two-dimensional parameter space, and a typical wage-setting mechanism should be adopted for labor markets (or jobs) located within the typical region.

We also numerically calculate this model by calibrating it to match some relevant characteristics of the US labor market. We get the following findings.

Firstly, competitive wage produces the smallest unemployment rate, even smaller than is socially desirable; bargaining produces the largest one, significantly larger than is socially desirable; posting falls in between, still larger than is socially desirable.

Secondly, competitive wage produces the largest aggregate output, even larger than is socially desirable; bargaining produces the smallest one, smaller than is socially desirable; posting falls in between, still smaller than is socially desirable. For this difference, the contribution from equilibrium unemployment rate is magnified by another finding that workers overinvest under competitive wage while underinvesting under both bargaining and posting. If the average stock of human capital is too small (resp. large), the probability for a firm to find a high-productivity worker is very low (resp. high), resulting in expected profits that are too small (resp. large) such that the incentive of job vacancy creation is very weak (resp. strong). This explains why a low unemployment rate is accompanied by overinvestments while a high unemployment rate is accompanied by underinvestments.

Thirdly, posting produces the largest social welfare that is either slightly smaller than or approximately equal to the efficient level. The steady-state welfare is defined as aggregate output minus the total costs of human capital investments and job vacancy creation. Under posting, the loss in aggregate output and the gain in total costs saved almost balance each other, resulting in a value of social welfare close to the socially optimal level. Under competitive wage, significant overinvestments in human capital and too many job openings cost the society too much such that the gain in aggregate output is unable to sufficiently cover the increase in total costs, resulting in an inefficiently low level of social welfare. Under bargaining, significant underinvestments in human capital and too few job openings generate an aggregate output that is too small so that the gain in total costs saved is unable to sufficiently cover the loss in aggregate output, resulting in an inefficiently low level of social welfare. The immediate implication is that, without government intervention, either too thick or too thin labor markets may be socially undesirable.

These findings are shown to be robust to the choice of market structure used for calibration. We also verify the robustness of the above rankings with respect to the bargaining share of workers. These results show that different wage-setting mechanisms have different implications for the nature of equilibrium. They help to sort out when the inefficiency arising from search-theoretical models of labor market is due to features of the environment (such as preferences and information) and when it is due to the assumed wage-setting mechanism, which is especially informative for policymakers. Moreover, as our results are derived based on a laissez-faire economy, they help to identify along which dimension and to what extent government intervention should be. For example, we conjecture that, ceteris paribus, schooling subsidies[4], which lift up the threat point of workers and alleviate the tension of holdups, should be more desirable under bargaining while laws guaranteeing the enforcement of labor contracts is more desirable under posting, owing to that wage commitment is a key feature in realizing efficiency advantage.

Our paper is related to the literature studying the (in)efficiency of (laissez-faire) wage determination in the presence of both human capital investment and labor search frictions. Under competitive wage, Masters (1998) shows that efficiency can be restored, but he assumes a constant wage rate and also that the social planner imposes a specific weight on workers in the social welfare function. Under wage bargaining, many studies (e.g., Acemoglu, 1996; Moen, 1999; Burdett and Smith, 2002; Charlot and Decreuse, 2005, 2010; Charlot et al., 2005; de Meza and Lockwood, 2010) show that efficiency cannot be restored. Under wage posting, Masters (2011) shows that efficiency can be restored whereas Kaas and Zink (2011) show that inefficiency occurs despite competitive search. Similar to the literature, we also check whether or not efficiency can be restored under each wage-setting mechanism. It turns out that, in general, none of them can restore efficiency in our model.

Departing from the literature, we focus on mutual welfare comparison between any two of the three wage-setting mechanisms, which helps us to figure out which one may prevail in equilibrium and under what conditions. This is technically nontrivial as the equilibrium welfare is a nonlinear function of three

[4]In fact, Flinn and Mullins (2015) find that policies like schooling subsidies attempting to redistribute the surplus between firms and workers tend to promote more schooling investments and lead to efficiency gains.

interdependent variables determined by three nonlinear equations. Even so, we derive explicit conditions enabling us to predict when one of them is most efficient. In particular, our theoretical results imply that wage posting does not always dominate the other two and wage bargaining is not necessarily dominated by the other two. Given that the literature has not yet provided a formal theory on why we should adopt a typical wage-setting mechanism (other than the alternatives), our contribution is to not only rationalize the existence of all three types of wage-setting mechanisms but also identify the links of their relative advantage with the economic environments under consideration.

The remainder of the paper is organized as follows. Section 2 presents basic assumptions and characterizes local efficiency. Section 3 derives decentralized equilibrium under alternative wage-setting mechanisms. Section 4 proceeds to a comparative analysis. Section 5 illustrates using numerical examples. Section 6 concludes. All proofs are relegated to Appendix.

2 The Model

2.1 Basic Assumptions

The economy is populated by a continuum of workers and a continuum of firms who are risk-neutral. Workers invest in human capital that is essential for production, while firms create jobs and organize production. The measure of workers is normalized to one. All agents live forever in continuous time and discount the future at the common rate $r > 0$; that is, the rate of time preference is equal to the interest rate under risk neutrality. There is a complete capital market, and all agents have access to this market.

Workers search for a job after they have acquired some human capital $h \geq 0$ at marginal cost $p > 0$, which can be interpreted as the average annual tuition fees. Each firm can create at most one job, and jobs are either filled or vacant. If a firm employs a worker, it produces a flow of output $y = f(h)$ with the price normalized to one, in which f is continuously differentiable, strictly increasing, concave and satisfies Inada conditions. We also let $f(0) = 0$ and $\lim_{h \downarrow 0} hf'(h) = 0$, which are satisfied by power functions such as $f(h) = h^{\alpha}$ for $\alpha \in (0, 1)$.

The source of job turnover is exogenous and follows a Poisson process with a constant arrival rate $\delta > 0$. Unemployed workers receive a flow payoff of zero. That is, let any unemployment benefits from home production and leisure be normalized to zero. We impose the free entry assumption on firms so they exhaust the rents from job creation in the long run. Additionally, once meetings occur, all payoff relevant characteristics of the other party are revealed so that there is no private information within each match. Here we hold factors such as individual ability and education quality constant and just use the number of years of education to measure the amount of human capital. As it is observable in reality, this assumption regarding information structure is without further loss of generality.

2.2 Random Matching

Unemployed workers and firms with vacancies come together in pairs via a matching technology $M(u, v)$, where u is the unemployment rate, v is the measure of vacancies, and M is concave and homogeneous of degree one in (u, v) with continuous derivatives. This enables us to write the flow rate of match for a vacancy as $M(u, v)/v \equiv q(\theta)$, where $q'(\theta) < 0$ and $\theta \equiv v/u$ is the labor market tightness (or the inverse of queue length). Assume that the absolute value of the elasticity of $q(\theta)$ satisfies $-q'(\theta)\theta/q(\theta) = \eta$ for a constant $\eta \in (0, 1)$. The flow rate of match for an unemployed worker is $M(u, v)/u \equiv \theta q(\theta)$. In general, $q(\theta)$, $\theta q(\theta) < \infty$; thus, it takes time for them to find production partners.[5] We place Inada-type assumptions on M such that $\theta q(\theta)$ is strictly increasing in θ, $\lim_{\theta\uparrow\infty} q(\theta) = \lim_{\theta\downarrow 0} \theta q(\theta) = 0$, and $\lim_{\theta\downarrow 0} q(\theta) = \lim_{\theta\uparrow\infty} \theta q(\theta) = \infty$. Intuitively, when market tightness goes to zero, the arrival rates of trading partners for firms and workers go to infinity and zero, respectively; when θ goes to infinity, the opposite holds.

Matches are consummated only if the joint surplus is nonnegative. We need more notations: U is the value of unemployment, $W(h)$ is the value of employment for a worker with human capital h, V is the value of a vacancy, and $J(h)$ is the value to a firm of filling a job. Therefore, a match is formed only if

$$W(h) + J(h) \geq U + V, \tag{1}$$

for any $h \geq 0$. In fact, inequality (1) can be interpreted as the group rationality constraint.

2.3 Asset Values

We restrict attention to a steady state. The asset value of unemployment, U, satisfies Bellman equation:

$$rU = \max_{h\geq 0} \{-ph + \theta q(\theta)[W(h) - U]\}. \tag{2}$$

Hence the expected income flow when unemployed is equal to the current income, $-ph$, plus the expected gain from job search, $\theta q(\theta)[W(h) - U]$. Similarly, the value of employment, $W(h)$, is defined by Bellman equation:

$$rW(h) = w(h) + \delta[U - W(h)], \tag{3}$$

where $w(h)$ denotes wages satisfying $w(0) = 0$ which is a normalization, $w'(\cdot) > 0$ which rationalizes the activity of human capital investment and $w''(\cdot) \leq 0$ which can be interpreted as a regularity constraint. To accept a wage offer, individual rationality requires that $w(h) \geq rU$, in which rU can be seen as reservation wages.

We next give the value equations of firms. The value of a vacancy satisfies Bellman equation:

$$rV = -c + q(\theta)[J(h) - V], \tag{4}$$

[5]The expected duration of time spent waiting for a match for any agent is the inverse of her arrival rate.

where $c > 0$ denotes a constant flow cost of renting a cite. So, the expected income flow associated with a vacancy is equal to the current income flow, $-c$, plus the expected gain from search, $q(\theta)[J(h) - V]$. The value to a firm having a vacancy filled is given by:

$$rJ(h) = f(h) - w(h) + \delta[V - J(h)]. \tag{5}$$

To employ a worker with human capital h, individual rationality requires $f(h) - w(h) \geq -c$. We further assume $f(h) \geq w(h)$ (i.e., filled vacancies earn non-negative profits) and $f'(h) > w'(h)$ (i.e., profits are strictly increasing in the human capital input) for $\forall h > 0$, in which the former is not restrictive at all while the latter motivates firms to hire workers with higher productivity.

Applying equations (3)-(5) to inequality (1), a match is formed only if

$$f(h) \geq rU \text{ for } \forall h \geq 0, \tag{6}$$

where we have used the free entry assumption $V = 0$.

2.4 Local Efficiency

An allocation is (constrained) efficient if it maximizes the net output subject to search restrictions. Following Pissarides (2000), a social planner weights all firms and workers equally and chooses the time path of human capital investment $h(t)$, market tightness $\theta(t)$, and unemployment rate $u(t)$ to maximize the discounted value of net output subject to the same matching constraint as workers and firms. That is:

$$\max_{h,\theta,u\geq 0} \int_0^\infty e^{-rt} \left\{ f\left[h(t)\right]\left[1 - u(t)\right] - ph(t)u(t) - c\theta(t)u(t) \right\} dt \tag{7}$$

subject to

$$\dot{u}(t) = \delta\left[1 - u(t)\right] - \theta(t)q\left[\theta(t)\right]u(t). \tag{8}$$

The constraint (8) is a standard differential equation describing the evolution of unemployment rate. In (7), the first term is the total output produced by all filled jobs, the second term is the net payoff flows of all unemployed workers, and the third term is the total cost of vacancies. The social planner is not interested in wages because they only determine the distribution of output, and distributional considerations are excluded from the social welfare function by assumption. Also, similar to the assumption used by Acemoglu and Shimer (1999), we restrict the planer to choose the same investment for all workers. Since there is no (ex ante) heterogeneity among workers, treating them in this way is without great loss of generality and meanwhile greatly simplifies the tractability.

The following lemma characterizes local efficiency.

Lemma 2.1. *For economic environments under consideration, the following statements are true:*

(i) The locally efficient steady-state allocation, written as $\{h^*, \theta^*, u^*\}$, satisfies:

$$f'(h)\left[\frac{\theta q(\theta)}{\delta}\right] = p, \tag{9}$$

$$c(r+\delta) = \{(1-\eta)\,[f(h)+ph] - \eta c\theta\}\, q(\theta), \tag{10}$$

and

$$u = \frac{\delta}{\delta + \theta q(\theta)}. \tag{11}$$

(ii) If the following conditions are satisfied:

$$\begin{cases} \Gamma'(\theta) < 0, \\ \lim_{\theta \downarrow 0} \Gamma(\theta) > c(r+\delta), \\ \lim_{\theta \uparrow \infty} \Gamma(\theta) < c(r+\delta), \end{cases} \tag{12}$$

where $\Gamma(\theta) \equiv (1-\eta)\left\{ f\left((f')^{-1}\left[\frac{\delta p}{\theta q(\theta)}\right]\right) + p(f')^{-1}\left[\frac{\delta p}{\theta q(\theta)}\right]\right\} q(\theta)$*, then* $\{h^*, \theta^*, u^*\}$ *determined by equations (9)-(11) is unique, and it is also an interior solution.*

(iii) If $q(\theta) = \theta^{-\eta}$ *and* $f(h) = h^{\alpha}$ *for a constant* $\alpha \in (0,1)$*, then (12) is satisfied for* $\forall \eta \in \left[\frac{1}{2-\alpha}, 1\right)$*.*

(iv) If $\eta \geq f'(h^*)\, h^* / f(h^*)$*, then the instantaneous welfare evaluated at the interior steady-state allocation is positive; that is,* $\mathcal{W}^* \equiv f(h^*)(1-u^*) - ph^*u^* - c\theta^* u^* > 0$ *for* $\forall h^* > 0$ *and* $\forall \theta^* > 0$*.*

Proof. See Appendix A. □

First, efficient investment should solve equation (9) so that the marginal social cost of investment, p, is equal to the marginal social benefit of investment, $f'(h)$, times the wedge $\theta q(\theta)/\delta$ which captures search frictions and the exogenous shock of job separation. Second, plugging equation (9) in equation (10) gives $c = \frac{(1-\eta)q(\theta)}{r+\delta+\eta\theta q(\theta)}\left\{1 + \left[\frac{f'(h)h}{f(h)}\right]\left[\frac{\theta q(\theta)}{\delta}\right]\right\} f(h)$. Efficient entry should solve this equation so that the marginal social cost of entry, c, is equal to the marginal social benefit of entry, $f(h)$, times a wedge term which captures search frictions, the exogenous shock of job separation, and the time-discounting friction. Third, efficient unemployment rate is a solution of equation (11). We also provide the sufficient conditions such that efficient allocation is the unique interior solution of system (9)-(11) and the corresponding value of social welfare is strictly positive.

3 Decentralized Equilibrium Derivation

3.1 Competitive Wage

A steady-state competitive equilibrium must satisfy five conditions: (1) matches are mutually acceptable, so inequality (6) must be satisfied; (2) wages are competitively determined (i.e., agents maximize payoff by taking wages as given and wages are adjusted to clear market); (3) individual rationality constraints facing workers and firms are satisfied; (4) firms creating a job vacancy earn zero profits, and hence $V = 0$; and (5) the flow of workers into and out of unemployment must be equal (i.e., total employment remains constant at the steady state).

The following lemma establishes the steady-state equilibrium:

Lemma 3.1. *For economic environments under consideration, the following statements are true:*

(i) The steady-state competitive equilibrium, written as $\{h^C, \theta^C, u^C\}$, *satisfies:*

$$f'(h)\left[\frac{\theta q(\theta)}{r+\delta}\right] = p, \tag{13}$$

$$c(r+\delta) = [f(h) - w(h)]\, q(\theta), \tag{14}$$

and

$$u = \frac{\delta}{\delta + \theta q(\theta)}. \tag{15}$$

(ii) For $\Psi(\theta) \equiv \left\{f\left((f')^{-1}\left[\frac{(r+\delta)p}{\theta q(\theta)}\right]\right) - \left[\frac{(r+\delta)p}{\theta q(\theta)}\right](f')^{-1}\left[\frac{(r+\delta)p}{\theta q(\theta)}\right]\right\} q(\theta)$ *and* $w(h) = f'(h)h$, *if either*

$$\begin{cases} f'(h)\frac{h}{f(h)} > \eta, \\ \lim_{\theta\downarrow 0} \Psi(\theta) < c(r+\delta), \\ \lim_{\theta\uparrow\infty} \Psi(\theta) > c(r+\delta) \end{cases} \quad \text{or} \quad \begin{cases} f'(h)\frac{h}{f(h)} < \eta, \\ \lim_{\theta\downarrow 0} \Psi(\theta) > c(r+\delta), \\ \lim_{\theta\uparrow\infty} \Psi(\theta) < c(r+\delta) \end{cases} \tag{16}$$

holds, then $\{h^C, \theta^C, u^C\}$ *determined by equations (13)-(15) is unique and also an interior solution.*

(iii) If $q(\theta) = \theta^{-\eta}$ *and* $f(h) = h^\alpha$ *for a constant* $\alpha \in (0,1)$, *then (16) can be satisfied for* $\alpha \neq \eta$.

Proof. See Appendix A. □

3.2 Wage Bargaining

A steady-state search equilibrium must satisfy six conditions: (1) matches are mutually acceptable, so inequality (6) must be satisfied; (2) workers make payoff-maximizing investments; (3) wages are determined by bilateral bargaining between matched workers and firms; (4) individual rationality constraints facing workers and firms are satisfied; (5) firms creating a job vacancy earn zero profits, and hence $V = 0$; and (6) the flow of workers into and out of unemployment must be equal.

The following lemma establishes the steady-state equilibrium:

Lemma 3.2. *For economic environments under consideration, the following statements are true:*

(i) The steady-state search equilibrium, written as $\{h^B, \theta^B, u^B\}$, *satisfies:*

$$f'(h)\left[\frac{\beta\theta q(\theta)}{r+\delta}\right] = p, \tag{17}$$

$$c(r+\delta) = \{(1-\beta)\,[f(h) + ph] - \beta c\theta\}\, q(\theta), \tag{18}$$

and

$$u = \frac{\delta}{\delta + \theta q(\theta)}. \tag{19}$$

(ii) If the following conditions are satisfied:

$$\begin{cases} \Phi'(\theta) < 0, \\ \lim_{\theta \downarrow 0} \Phi(\theta) > c(r+\delta), \\ \lim_{\theta \uparrow \infty} \Phi(\theta) < c(r+\delta), \end{cases} \tag{20}$$

where $\Phi(\theta) \equiv (1-\beta)\left\{ f\left((f')^{-1}\left[\frac{(r+\delta)p}{\beta\theta q(\theta)}\right]\right) + p(f')^{-1}\left[\frac{(r+\delta)p}{\beta\theta q(\theta)}\right]\right\} q(\theta)$*, then* $\{h^B, \theta^B, u^B\}$ *determined by equations (17)-(19) is unique, and it is also an interior solution.*

(iii) If $q(\theta) = \theta^{-\eta}$ *and* $f(h) = h^{\alpha}$ *for a constant* $\alpha \in (0,1)$*, then (20) is satisfied for* $\forall \eta \in \left[\frac{1}{2-\alpha}, 1\right)$*.*

Proof. See Appendix A. □

3.3 Wage Posting

Firms commit to and post wage contracts, denoted (h, w), before meeting workers in an effort to attract applicants, while workers can observe all posted wages and then decide which of these to seek. So, firms are market makers who can open submarkets via posting wages, while workers are allowed to adjust their application decisions in response to wage differentials across submarkets. We still use θ to denote the inverse of queue length in submarkets. Since matching frictions still exist within each submarket, workers in a submarket offering a wage of w are hired with probability $\theta(w)q[\theta(w)]$. In equilibrium, the set of submarkets is complete in the sense that there is no submarket that could be opened that makes some firms and workers better off.

Events proceed as follows: workers first make investments; after observing these investments, each firm posts a wage contract, taking as given the wage contracts of its competitors; then each worker chooses the submarkets to seek, taking as given wage contracts and the search strategies of other workers.

A steady-state competitive search equilibrium must satisfy eight conditions: (1) matches within each submarket are mutually acceptable, so inequality (6) must be satisfied for any posted w; (2) workers make payoff-maximizing investments; (3) wage commitments are profit-maximizing; (4) firms creating a job vacancy earn zero profits, and hence $V = 0$; (5) workers direct their search toward the wages that maximize their expected payoff; (6) $\theta(w)$ is consistent with rational expectations beginning at any decision node (namely sequentially rational); (7) individual rationality constraints facing workers and firms are satisfied; and (8) the flow of workers into and out of unemployment must be equal.

The following lemma establishes the steady-state equilibrium:

Lemma 3.3. *For economic environments under consideration, the following statements are true:*

(i) The steady-state competitive search equilibrium, written as $\{h^P, \theta^P, u^P\}$*, satisfies:*

$$f'(h)\left[\frac{\theta q(\theta)}{r+\delta}\right] = p, \tag{21}$$

$$c(r+\delta) = \{(1-\eta)[f(h)+ph] - \eta c\theta\} q(\theta), \tag{22}$$

and

$$u = \frac{\delta}{\delta + \theta q(\theta)}. \tag{23}$$

(ii) If the following conditions are satisfied:

$$\begin{cases} \Xi'(\theta) < 0, \\ \lim_{\theta \downarrow 0} \Xi(\theta) > c(r+\delta), \\ \lim_{\theta \uparrow \infty} \Xi(\theta) < c(r+\delta), \end{cases} \tag{24}$$

where $\Xi(\theta) \equiv (1-\eta)\left\{f\left((f')^{-1}\left[\frac{(r+\delta)p}{\theta q(\theta)}\right]\right) + p(f')^{-1}\left[\frac{(r+\delta)p}{\theta q(\theta)}\right]\right\} q(\theta)$, *then* $\{h^P, \theta^P, u^P\}$ *determined by equations (21)-(23) is unique, and it is also an interior solution.*

(iii) If $q(\theta) = \theta^{-\eta}$ *and* $f(h) = h^\alpha$ *for a constant* $\alpha \in (0,1)$, *then (24) is satisfied for* $\forall \eta \in \left[\frac{1}{2-\alpha}, 1\right)$.

Proof. See Appendix A. □

In equilibrium, all firms offer the same wages and all workers make the same investments.

4 Steady-State Welfare Comparison

Before proceeding to the comparative analysis among the three wage-setting mechanisms, we first establish the proposition:

Proposition 4.1. *None of the three wage-setting mechanisms can implement the efficient allocation.*

Proof. See Appendix B. □

In fact, only asymptotic efficiency can be achieved in our model. We next compare the three wage-setting mechanisms regarding the steady-state welfare[6]:

$$\mathcal{W}^j \equiv f\left(h^j\right)\left(1-u^j\right) - ph^j u^j - c\theta^j u^j,$$

for $\forall j \in \{C, B, P\}$. Though no one achieves the efficient allocation, it is still informative to know which one is better than others, and under what conditions. To assure attractability, we need

Assumption 4.1. $f(h) = h^\alpha$ *and* $q(\theta) = \theta^{-\eta}$ *for parameters* $\alpha, \eta \in (0,1)$.

Also, for our interest, we impose another assumption[7]:

Assumption 4.2. $\eta = \beta \in (0,1)$, *i.e., the Hosios condition holds.*

[6]Under risk-neutral preferences, steady-state welfare is defined as net output of the society.

[7]In fact, we have analyzed the general case without using Assumption 4.2 and find that nothing essentially new arises from relaxing this assumption. Also, Assumption 4.2 is used only when wage bargaining is involved.

First, we get the following four lemmas.

Lemma 4.1. *Suppose Assumption 4.1 holds and* $\alpha \neq \eta$*, then we have* $\theta^C > \theta^P$*,* $u^C < u^P$ *and* $h^C > h^P$*.*

Proof. See Appendix A. □

As already shown in Lemma 3.1, condition $\alpha \neq \eta$ assures the existence of competitive equilibrium under Assumption 4.1. Lemma 4.1 states that workers make more human capital investments, firm entry rate is higher and search unemployment rate is lower under competitive wage than under wage posting.

Lemma 4.2. *Suppose Assumptions 4.1-4.2 hold, then we have: (i) If* $1-\alpha \geq (1-\beta)\beta^{\alpha/(1-\alpha)}$*, then* $\theta^C \neq \theta^B$ *and* $u^C \neq u^B$*; (ii) For* $\theta^C < \theta^B$ *and* $u^C > u^B$*, it is necessary that* $\alpha(1-\beta)\beta^{\alpha/(1-\alpha)} < 1-\alpha < 1-\beta$ *holds; (iii) For* $\theta^C > \theta^B$*,* $u^C < u^B$ *and* $h^C > h^B$*, it is necessary that either* $1-\alpha < (1-\beta)\beta^{\alpha/(1-\alpha)}$ *or* $1-\alpha > 1-\beta$ *holds; (iv)* $h^C < h^B$ *if and only if* $\theta^B/\theta^C > \beta^{1/(\beta-1)}$*.*

Proof. See Appendix A. □

To intuitively interpret Lemma 4.2, we impose a constant wage rate under competitive wage and hence α denotes the output share of a matched worker while $1-\alpha$ for a matched firm. Part (i) gives the sufficient condition under which competitive wage and wage bargaining diverge on market tightness and search unemployment. For more specific comparison stated in parts (ii)-(iii), we can just establish necessary conditions under the current assumptions. To induce a lower firm entry rate and a higher search unemployment rate under competitive wage than under wage bargaining, part (ii) shows that it is necessary that the firm's output share under competitive wage is strictly smaller than its bargaining share under wage bargaining. To induce a higher firm entry rate, a lower search unemployment rate and more human capital investments under competitive wage than under wage bargaining, part (iii) shows that it is necessary that either the firm or the worker is provided a stronger incentive under competitive wage than under wage bargaining. Part (iv) states that a worker makes more human capital investments under wage bargaining than under competitive wage if and only if firm entry rate under the former is sufficiently higher than that under the latter.

Lemma 4.3. *Suppose Assumptions 4.1-4.2 hold and* $\alpha = \eta$*, then we have:*

(i) $\theta^B \neq \theta^P$ *and* $u^B \neq u^P$*;*

(ii) $\theta^B > \theta^P$ *and* $u^B < u^P$ *for* $p < \hat{p}$*, and* $\theta^B < \theta^P$*,* $u^B > u^P$ *and* $h^B < h^P$ *for* $p > \hat{p}$*, in which* $\hat{p} \equiv \left[\frac{\beta^{1+\beta}(1-\beta)^{1-\beta}}{(r+\delta)c^{1-\beta}}\right]^{1/\beta} > 0$*;*

(iii) $h^B > h^P$ *if and only if* $\theta^B/\theta^P > \beta^{1/(\beta-1)}$*.*

Proof. See Appendix A. □

As the equilibrium equations determining market tightness are highly nonlinear under both bargaining and posting, we must rely on the assumption $\alpha = \eta$ to guarantee attractability. Intuitively, $\alpha = \eta$

means that a worker's marginal contribution rates to output and matching are equal. Part (i) shows that bargaining and posting diverge on market tightness and search unemployment. Part (ii) provides more specific predictions: if the average cost of human capital investment is smaller than a threshold, then firm entry rate is higher and search unemployment rate is lower under bargaining than under posting; if the average cost is larger than the threshold, then firm entry rate is lower, search unemployment rate is higher and workers make less human capital investments under bargaining than under posting. Part (iii) shows that workers make more human capital investments under bargaining than under posting if and only if firm entry rate under the former is sufficiently higher than that under the latter.

For the three wage-setting mechanisms, the steady-state welfare is a function of three mutually dependent variables determined by three nonlinear equations. Even under Assumptions 4.1-4.2, the three variables cannot be explicitly solved under both bargaining and posting. Thus, achieving direct welfare comparison is not straightforward. Using Assumption 4.1, we can rewrite the welfare function as a function of θ. In addition, the functional form is the same under competitive wage and wage posting.

Lemma 4.4. *Suppose Assumption 4.1 holds and also the existence and uniqueness of decentralized equilibrium are assured under Assumption 4.1, then we have:*

(i) For $\forall j \in \{C, P\}$ and a threshold denoted $\hat{\theta}$: if $\theta \in (0, \hat{\theta}]$, then $\partial \mathcal{W}^j / \partial \theta > 0$; if $\theta \in (\hat{\theta}, \infty)$, then

$$\frac{\partial \mathcal{W}^j}{\partial \theta} \begin{cases} > 0 & \text{for } c < \Theta, \\ = 0 & \text{for } c = \Theta, \\ < 0 & \text{for } c > \Theta, \end{cases}$$

in which

$$\Theta \equiv \frac{\{f'(h(\theta)) - u(\theta)[f'(h(\theta)) + p]\} h'(\theta) - [f(h(\theta)) + ph(\theta)]u'(\theta)}{u(\theta) + u'(\theta)\theta} > 0;$$

(ii) For any given $\theta \in (0, \infty)$ and $\forall j \in \{C, P\}$,

$$\mathcal{W}^j|_{\theta^j=\theta} \begin{cases} > \mathcal{W}^B|_{\theta^B=\theta} & \text{for } p > \tilde{p}, \\ = \mathcal{W}^B|_{\theta^B=\theta} & \text{for } p = \tilde{p}, \\ < \mathcal{W}^B|_{\theta^B=\theta} & \text{for } p < \tilde{p}, \end{cases}$$

in which $\tilde{p} \equiv \frac{\alpha(1-\beta^{1/(1-\alpha)})\delta}{(1-\beta^{\alpha/(1-\alpha)})(r+\delta)} > 0$.

Proof. See Appendix A. □

Part (i) identifies the conditions under which welfare function is monotone with respect to θ, enabling us to compare social welfare between competitive wage and wage posting via comparing their θs (see also Figure 1). Controlling for θ and identifying a threshold for the average cost of human capital investment,

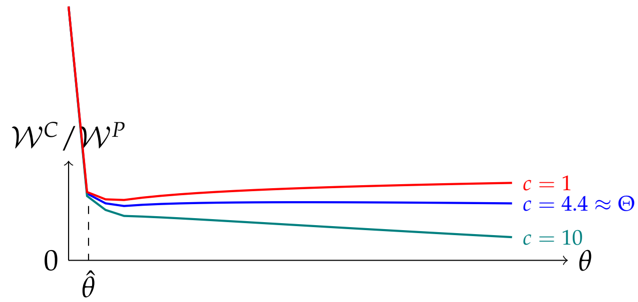

Figure 1: Result (i) of Lemma 4.4: $\alpha = 0.24$, $\eta = 0.58$, $r = 0.004$, $\delta = 0.036$, $p = 1$ and $\hat{\theta} \approx 0.44$.

part (ii) realizes direct welfare comparison between bargaining and the other two wage-setting mechanisms. Therefore, part (i) combined with part (ii) allows for mutual welfare comparison between the three wage-setting mechanisms.

We now state the welfare-comparison predictions in three propositions.

Proposition 4.2. *Suppose Assumption 4.1 holds and also $\alpha \neq \eta$, then we have:*

(i) If $\theta^C \leq \hat{\theta}$, then $\mathcal{W}^C > \mathcal{W}^P$;

(ii) If $\theta^P \geq \hat{\theta}$, then $\mathcal{W}^C > \mathcal{W}^P$ for $c < \Theta$ while $\mathcal{W}^C < \mathcal{W}^P$ for $c > \Theta$.

Proof. An application of Lemmas 4.4 and 4.1. □

Compared to wage posting, Lemma 4.1 confirms the relative advantage of competitive wage in search unemployment and aggregate output as well as its relative disadvantage in the cost of human capital investment and job creation. Proposition 4.2 shows that the relative advantage dominates the relative disadvantage when the cost of creating a job vacancy is bounded above, otherwise the relative advantage is dominated by the relative disadvantage when the cost of creating a job vacancy is bounded below.

Proposition 4.3. *Suppose Assumptions 4.1-4.2 hold, then we have:*

(i) If $p \geq \tilde{p}$ and $c < \Theta$, then for $\mathcal{W}^C > \mathcal{W}^B$ it is necessary that either $1 - \alpha < (1 - \beta)\beta^{\alpha/(1-\alpha)}$ or $1 - \alpha > 1 - \beta$ holds;

(ii) If $p \geq \tilde{p}$ and $c > \Theta$, then for $\mathcal{W}^C > \mathcal{W}^B$ it is necessary that $\alpha(1 - \beta)\beta^{\alpha/(1-\alpha)} < 1 - \alpha < 1 - \beta$ holds;

(iii) If $p \leq \tilde{p}$ and $c > \Theta$, then for $\mathcal{W}^C < \mathcal{W}^B$ it is necessary that either $1 - \alpha < (1 - \beta)\beta^{\alpha/(1-\alpha)}$ or $1 - \alpha > 1 - \beta$ holds;

(iv) If $p \leq \tilde{p}$ and $c < \Theta$, then for $\mathcal{W}^C < \mathcal{W}^B$ it is necessary that $\alpha(1 - \beta)\beta^{\alpha/(1-\alpha)} < 1 - \alpha < 1 - \beta$ holds.

Proof. An application of Lemmas 4.4 and 4.2. □

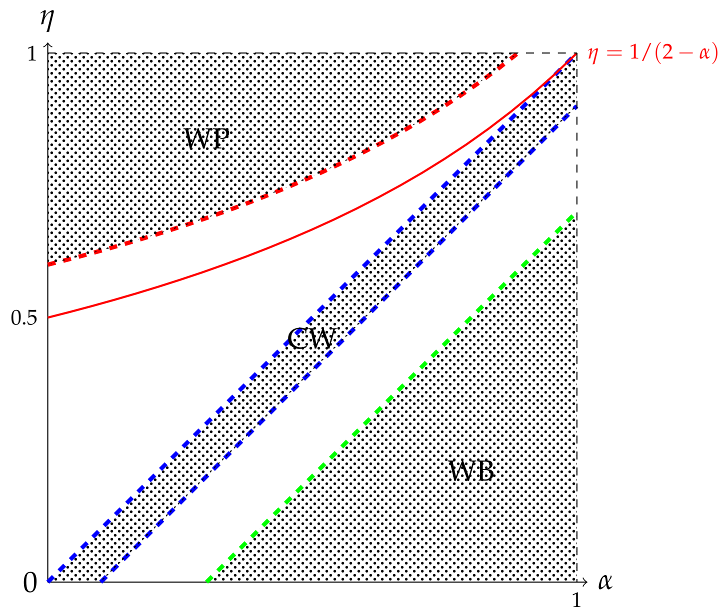

Figure 2: Relative efficiency: $\eta = \beta$, $r + \delta = 1$, $p = 1$ (normalization) and $c > 1$.

We can just establish necessary conditions based on Lemma 4.2. If the average cost of human capital investment and the marginal cost of job creation are bounded, either from above or below, to establish a welfare ranking between competitive wage and wage bargaining it is necessary that either the output share of a firm under competitive wage is smaller than its bargaining share or at least one side of a matched worker-firm pair has an output share greater than the corresponding bargaining share.

Proposition 4.4. *Suppose Assumptions 4.1-4.2 hold and also $\alpha = \eta$, then we have:*

(i) If either $c < \Theta$ and $p > \max\{\hat{p}, \tilde{p}\}$ or $\Theta < c < \Lambda$ and $\tilde{p} \leq p < \hat{p}$ hold, then $\mathcal{W}^P > \mathcal{W}^B$;

(ii) If either $c < \Theta$ and $p < \min\{\hat{p}, \tilde{p}\}$ or $c > \max\{\Theta, \Lambda\}$ and $\hat{p} < p \leq \tilde{p}$ hold, then $\mathcal{W}^P < \mathcal{W}^B$, in which $\Lambda \equiv \left[\frac{\beta(1-\beta)^{1-\beta}(1-\beta^{\beta/(1-\beta)})^{\beta}}{(1-\beta^{1/(1-\beta)})^{\beta}\delta^{\beta}(r+\delta)^{1-\beta}}\right]^{1/(1-\beta)} > 0$.

Proof. An application of Lemmas 4.4 and 4.3. □

Proposition 4.4 provides sufficient conditions consisting of restrictions placed on the average cost of human capital investment and the marginal cost of job creation such that either posting strictly dominates bargaining or bargaining strictly dominates posting. First, even if bargaining induces a higher firm entry rate and a lower search unemployment rate than does posting, it may still be dominated by posting in welfare. Second, even if posting induces a higher firm entry rate, a lower search unemployment rate and more human capital investments, it may still be dominated by bargaining in welfare. Therefore, the cost structures of human capital investment and job vacancy creation are relevant factors for the current comparative analysis with respect to steady-state welfare.

Moreover, as roughly illustrated by Figure 2[8], we can find at least one region of the parameter space

[8] As a caveat, we leave some blank areas and use dotted lines because it is unlikely to explicitly identify the partitioning

generated by α and η such that a typical mechanism prevails within the region. For the specified parameter values, we obtain the following nontrivial observations. If workers' matching contribution measured by η is sufficiently larger than their output contribution measured by α, then wage posting (WP) prevails. If, in contrast, their output contribution is sufficiently larger than their matching contribution, then wage bargaining (WB) prevails. If their output and matching contributions are sufficiently close to each other, then competitive wage (CW) prevails. Our findings reveal that the three wage-setting mechanisms do exhibit different relative advantages and also can be interpreted as being consistent with the survey evidence reported by Hall and Krueger (2012): bargaining is common by minority workers with professional degrees while posting is particularly common among union members, those who took government jobs and non-high-school graduates.

5 Numerical Examples

We provide two numerical examples by calibrating the model to the US labor market to demonstrate how our theoretical framework might work. These numerical experiments quantitatively show how far equilibrium allocations under alternative wage-setting mechanisms are from efficiency.

5.1 Functional Forms, Parameter Values and Calibration Target

Table 1: Commonly-used Parameter Values

Parameter	Value	Description
r	0.004	Interest rate
δ	0.036	Job separation probability

We use the production function $f(h) = h^{\alpha}$ for $\alpha \in (0, 1)$. As suggested by Petrongolo and Pissarides (2001), the matching function is assumed to be Cobb-Douglas $M(u, v) = \bar{m}u^{\eta}v^{1-\eta}$. By using the Job Openings and Labor Turnover Survey (JOLTS) data since December 2000 and the Help-Wanted Index (HWI) adjusted to the JOLTS unites of measurement before then, Pissarides (2009) derived the sample mean for θ during 1960-2006 to be 0.72. Using monthly transitions data, Shimer (2012) derived a mean value for 1960-2004 of 0.594 for the job finding probability and 0.036 for the job separation probability. Applying $\delta = 0.036$ to $u = \delta/[\delta + \theta q(\theta)]$, the implied unemployment rate is 5.7%, which is close to the actual mean. As calculated by Pissarides (2009), we set $r = 0.004$. Using the data on education attainment from UNESCO Institute for Statistics (2013) and the methodology developed by Barro and Lee (2013), Human Development Report Office (HDRO) (2013) estimates the average number of years of education

boundaries. Note that the welfare function is highly nonlinear and endogenous variables cannot be explicitly derived under bargaining and posting, we can just rely on numerical simulations to search for the region within which a mechanism prevails.

received by people ages 25 and older. Based on this data set, we calculate for the U.S. to get the average number of years of schooling as 12.69 between 1980-2013. We summarize the commonly-used parameter values in Table 1, and also the target for calibration is given by $\theta = 0.72$ and $h = 12.69$.

5.2 Steady-State Solutions

We denote by $Y \equiv f(h)(1-u)$ and $\mathcal{W} \equiv f(h)(1-u) - phu - c\theta u$ the aggregate output and social welfare. To make these examples reliable, we calibrate parameters c and p to achieve the target $\theta = 0.72$ and $h = 12.69$ using alternative models: EB (efficiency benchmark), CW, WB and WP. The advantage is that we can eliminate the equilibrium effect resulted from different cost structures, allowing for isolating the equilibrium effect resulted from alternative wage-setting mechanisms.

Example 5.1. $\alpha = 0.24$, $\beta = \eta = 0.54$ and $\bar{m} = 0.69$

Let $\alpha = 0.24$, which is the labor share of the sector[9] including finance, insurance, real estate, rental and leasing in 2012 (see Lawrence, 2015), and $\eta = 0.54$ (see Mortensen and Nagypál, 2007), by which we can get the value of matching function scale as $\bar{m} = 0.69$. Assume that Hosios condition holds and the remaining parameter values are given in Table 1. We obtain the calibrated values of c and p (see Table 2). The steady-state solutions are reported in Tables 3-6.

Table 2: Calibration with Target: $\theta = 0.72$ and $h = 12.69$

Model	Vacancy cost (c)	Marginal cost of investment (p)
EB	9.62	0.58
CW	28.84	0.52
WB	5.64	0.28
WP	8.82	0.52

Table 3: Calibration Based on EB

Model	u	Y	$\mathcal{W}$
EB	5.7%	1.74	0.93
CW	1.8%	2.56	0.29
WB	10.7%	1.05	0.76
WP	6.4%	1.60	0.91

Table 4: Calibration Based on CW

Model	u	Y	$\mathcal{W}$
EB	15.4%	1.13	0.57
CW	5.7%	1.74	0.18
WB	22.8%	0.71	0.47
WP	16.6%	1.05	0.56

We find that workers overinvest under competitive wage while underinvest under bargaining and posting, and that posting strictly dominates competitive wage and bargaining in social welfare. In fact, the efficiency loss under posting can be sufficiently small, and it is even zero in Tables 5-6. Bargaining

[9] In 2012, its GDP contribution to the US economy is around 20%, much bigger than any other sectors.

Table 5: Calibration Based on WB

Model	u	Υ	$\mathcal{W}$
EB	2.6%	2.94	1.58
CW	0.7%	4.35	0.46
WB	5.7%	1.74	1.31
WP	2.9%	2.71	1.58

Table 6: Calibration Based on WP

Model	u	Υ	$\mathcal{W}$
EB	5.1%	1.88	1.00
CW	1.5%	2.79	0.32
WB	9.7%	1.14	0.83
WP	5.7%	1.74	1.00

strictly dominates competitive wage in social welfare. We also perform the robustness exercise via varying parameter β, shown in Table 7, and it turns out that our findings hold for sufficiently various values of β.

Table 7: Robustness Check under Alternative Calibration Models: EB, CW, WP

Model	β	0.95	0.85	0.75	0.65	0.55	0.50	0.45	0.35	0.25	0.15
EB	u	44.2%	23.8%	16.8%	13.2%	10.9%	10.1%	9.3%	8.2%	7.3%	6.6%
	Υ	0.43	0.76	0.92	1.00	1.05	1.06	1.06	1.03	0.98	0.87
	$\mathcal{W}$	0.33	0.58	0.69	0.74	0.76	0.76	0.75	0.71	0.64	0.51
CW	u	67.4%	43.5%	33.2%	27.2%	23.1%	21.6%	20.2%	18.0%	16.3%	15.0%
	Υ	0.19	0.44	0.57	0.66	0.70	0.72	0.72	0.72	0.69	0.61
	$\mathcal{W}$	0.15	0.32	0.41	0.46	0.47	0.47	0.47	0.44	0.38	0.28
WP	u	41.6%	22.0%	15.4%	12.0%	9.9%	9.1%	8.5%	7.5%	6.7%	6.0%
	Υ	0.48	0.84	1.00	1.09	1.13	1.14	1.14	1.11	1.05	0.93
	$\mathcal{W}$	0.37	0.63	0.75	0.81	0.83	0.83	0.82	0.77	0.70	0.56

Example 5.2. $\alpha = 0.24$, $\beta = \eta = 0.50$ and $\bar{m} = 0.70$

Relative to Example 5.1, here we let $\beta = \eta = 0.50$ as in Petrongolo and Pissarides (2001) and Pissarides (2009), then we have $\bar{m} = 0.70$. Let Hosios condition hold and the remaining parameter values be given in Table 1. We obtain new calibrated values of c and p (see Table 8). The steady-state solutions are reported in Tables 9-12.

Table 8: Calibration with Target: $\theta = 0.72$ and $h = 12.69$

Model	Vacancy cost (c)	Marginal cost of investment (p)
EB	11.22	0.58
CW	28.84	0.52
WB	6.27	0.26
WP	10.29	0.52

Table 9: Calibration Based on EB

Model	u	Y	$\mathcal{W}$
EB	5.7%	1.74	0.86
CW	1.7%	2.59	0.36
WB	12.7%	0.95	0.66
WP	6.5%	1.59	0.85

Table 10: Calibration Based on CW

Model	u	Y	$\mathcal{W}$
EB	16.2%	1.10	0.51
CW	5.7%	1.74	0.18
WB	26.3%	0.62	0.40
WP	17.3%	1.03	0.50

Table 11: Calibration Based on WB

Model	u	Y	$\mathcal{W}$
EB	1.9%	3.34	1.69
CW	0.5%	4.98	0.68
WB	5.7%	1.74	1.29
WP	2.3%	3.04	1.67

Table 12: Calibration Based on WP

Model	u	Y	$\mathcal{W}$
EB	4.9%	1.90	0.94
CW	1.4%	2.84	0.39
WB	11.4%	1.03	0.73
WP	5.7%	1.74	0.94

As in Example 5.1, we find that posting strictly dominates competitive wage and bargaining in social welfare. In fact, the efficiency loss under posting can be sufficiently small and no greater than 2%, and it is even zero in Table 12. Similar to Example 5.1, bargaining strictly dominates competitive wage in social welfare. We also perform a robustness check via varying parameter β (see Table 13), which reveals that our finding holds for sufficiently various values of β.

Table 13: Robustness Check under Alternative Calibration Models: EB, CW, WP

Model	β	0.95	0.85	0.75	0.65	0.55	0.50	0.45	0.35	0.25	0.15
EB	u	59.7%	32.4%	22.4%	17.1%	13.9%	12.7%	11.6%	10.0%	8.8%	7.8%
	Y	0.26	0.59	0.77	0.87	0.93	0.95	0.95	0.95	0.90	0.81
	$\mathcal{W}$	0.20	0.44	0.56	0.63	0.66	0.66	0.66	0.63	0.57	0.46
CW	u	81.2%	55.0%	41.8%	33.8%	28.4%	26.3%	24.5%	21.6%	19.3%	17.7%
	Y	0.09	0.30	0.45	0.54	0.60	0.62	0.63	0.64	0.62	0.56
	$\mathcal{W}$	0.07	0.22	0.31	0.37	0.39	0.40	0.39	0.38	0.33	0.25
WP	u	56.6%	29.9%	20.4%	15.5%	12.5%	11.4%	10.5%	9.0%	7.9%	7.0%
	Y	0.30	0.66	0.85	0.96	1.02	1.03	1.04	1.03	0.98	0.88
	$\mathcal{W}$	0.23	0.50	0.62	0.70	0.73	0.73	0.73	0.70	0.63	0.51

6 Conclusion

This paper evaluates the allocative performance of three wage-setting mechanisms with human capital investment and search frictions. We identify explicit conditions under which one mechanism is better than the others in generating a higher level of equilibrium social welfare. Our theoretical results reveal that none of them can unconditionally dominate the other two. To balance between aggregate output and aggregate cost, the cost structures of human capital investment and job vacancy creation turn out to be relevant. We show that there exist reasonable ranges of cost parameters such that each one of the three mechanisms can generate the highest level of equilibrium social welfare. In particular, even though workers tend to be held up and hence underinvest under bargaining, we find that it may still dominate the other two in equilibrium welfare. The intuition is that its relative advantage in saving aggregate cost via inducing low investments and a small number of job openings may outweigh its relative disadvantage in aggregate output, hence dominating the other two in net output of the society.

By calibrating our model to the feature of the US labor market, our numerical examples show the efficiency advantage of wage posting. The insight behind these examples is: competitive wage induces high investments and a high firm entry rate, exhibiting an advantage in aggregate output while a disadvantage in aggregate cost; bargaining induces low investments and a low firm entry rate, exhibiting an advantage in aggregate cost while a disadvantage in aggregate output; posting is in-between and achieves the best balance between aggregate output and aggregate cost under the calibrated parameter values, thereby dominating the other two in equilibrium social welfare.

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Appendix A: Proofs

Proof of Lemma 2.1: We shall complete it in 4 steps.

Step 1. We first derive a group of necessary conditions and then evaluate them at the steady state. Using equations (7) and (8), the Hamiltonian can be written as:

$$\begin{aligned}\mathcal{H}[t,h(t),\theta(t),u(t);\lambda(t)] = e^{-rt}\{f[h(t)][1-u(t)] - ph(t)u(t) - c\theta(t)u(t)\} \\ - \lambda(t)\{\delta[1-u(t)] - \theta(t)q[\theta(t)]u(t)\},\end{aligned}$$

where $\lambda(t) > 0$ denotes a co-state variable. The optimal path of unemployment rate, human capital investment and market tightness satisfies constraint (8) and the following Euler conditions:

$$\mathcal{H}_h = e^{-rt}\left\{f'[h(t)][1-u(t)] - pu(t)\right\} = 0,$$

$$\mathcal{H}_\theta = -e^{-rt}cu(t) + \lambda(t)q[\theta(t)](1-\eta)u(t) = 0, \tag{25}$$

$$\mathcal{H}_u = -e^{-rt}\{f[h(t)] + ph(t) + c\theta(t)\} + \lambda(t)\{\delta + \theta(t)q[\theta(t)]\} = \dot{\lambda}(t). \tag{26}$$

Solving for $\lambda(t)$ from equation (25) and substituting it into equation (26), we can simplify equation (26) as $c(r+\delta) = ((1-\eta)\{f[h(t)] + ph(t)\} - \eta c\theta(t))\, q[\theta(t)]$. Evaluating these equations at the steady state and rearranging the algebra, we hence get equations (9)-(11).

Step 2. We provide a group of sufficient conditions under which equations (9)-(11) determine a unique solution that is also an interior solution. Using equation (9), strict concavity of f implies that $h(\theta) = (f')^{-1}\left[\frac{\delta p}{\theta q(\theta)}\right]$. Plugging this $h(\theta)$ in (10) and rearranging the algebra:

$$\underbrace{c(r+\delta) + \eta c\theta q(\theta)}_{\equiv\varphi(\theta)} = \underbrace{(1-\eta)\left\{f\left((f')^{-1}\left[\frac{\delta p}{\theta q(\theta)}\right]\right) + p(f')^{-1}\left[\frac{\delta p}{\theta q(\theta)}\right]\right\}q(\theta)}_{\equiv\Gamma(\theta)}. \tag{27}$$

By our assumptions, $\varphi'(\theta) > 0$, $\lim_{\theta\downarrow 0}\varphi(\theta) = c(r+\delta)$ and $\lim_{\theta\uparrow\infty}\varphi(\theta) = \infty$. To assure that equation (27) determines a unique solution of θ that is also an interior solution, we can impose restrictions on f and q such that $\Gamma(\theta)$ satisfies: $\Gamma'(\theta) < 0$, $\lim_{\theta\downarrow 0}\Gamma(\theta) > c(r+\delta)$ and $\lim_{\theta\uparrow\infty}\Gamma(\theta) < c(r+\delta)$.

Step 3. We find reasonable functional forms of f and q such that the sufficient conditions established in Step 2 are satisfied. We just need to consider $\Gamma(\theta)$. Let $q(\theta) = \theta^{-\eta}$ and $f(h) = h^\alpha$ for a constant $\alpha \in (0,1)$, then we have:

$$\Gamma(\theta) = (1-\eta)\left[\left(\frac{\alpha}{\delta p}\right)^{\frac{\alpha}{1-\alpha}}\theta^{\frac{\alpha(1-\eta)}{1-\alpha}-\eta} + p\left(\frac{\alpha}{\delta p}\right)^{\frac{1}{1-\alpha}}\theta^{\frac{1-\eta}{1-\alpha}-\eta}\right],$$

based on which we impose the restriction $\eta \geq \frac{1}{2-\alpha}$ so that $\Gamma(\theta)$ satisfies the above requirements.

Step 4. We now prove result (iv). Using $u^* = \delta/[\delta + \theta^* q(\theta^*)]$ and $\delta p = f'(h^*)\theta^* q(\theta^*)$, we thus have $\mathcal{W}^* > 0 \Leftrightarrow [f(h^*) - f'(h^*)h^*]\, q(\theta^*) > \delta c$. It follows from equation (10) that

$$c = \frac{(1-\eta)[f(h^*) + ph^*]q(\theta^*)}{r + \delta + \eta\theta^* q(\theta^*)}. \tag{28}$$

By applying equations (9) and (28) and simplifying the algebra, we get that $[f(h^*) - f'(h^*)h^*] \times q(\theta^*) > \delta c$ is equivalent to $r[f(h^*) - f'(h^*)h^*] + [\delta + \theta^* q(\theta^*)][\eta f(h^*) - f'(h^*)h^*] > 0$. We have $f(h^*) > f'(h^*)h^*$ for $\forall h^* > 0$ by exploiting the strict concavity of f and the assumptions that $f(0) = 0$ and $\lim_{h \downarrow 0} hf'(h) = 0$. Hence, $\mathcal{W}^* > 0$ holds when we put $\eta \geq f'(h^*)h^*/f(h^*)$. Q.E.D.

Proof of Lemma 3.1: We shall complete it in 5 steps.

Step 1. By equation (3), we have $W(h^s) = [w(h^s) + \delta U]/(r+\delta)$, in which h^s denotes the supply of human capital. Plugging it in equation (2) and rearranging the algebra, payoff-maximizing investment/supply of human capital implies that $(r+\delta)p = \theta q(\theta) w'(h^s)$. Correspondingly, we can rewrite equation (5) as $(r+\delta)J(h^d) = f(h^d) - w(h^d)$, where h^d denotes the demand of human capital and we have used the free entry assumption $V = 0$. Thus, maximizing $J(h^d)$ yields that $f'(h^d) = w'(h^d)$. Competitive equilibrium requires that $h^d = h^s$, and also it is determined by equation $(r+\delta)p = \theta q(\theta) f'(h)$.

Step 2. Since we have $J(h) = [f(h) - w(h)]/(r+\delta)$, applying $V = 0$ to equation (4) produces that $c(r+\delta) = [f(h) - w(h)]q(\theta)$, which determines the equilibrium market tightness θ. In the steady state, the flow of workers into unemployment, $\delta(1-u)$, must be equal to the flow of workers out of unemployment, $\theta q(\theta) u$. To conclude, the steady-state equilibrium must simultaneously satisfy (13)-(15). Additionally, reservation wages evaluated at the equilibrium can be written as $rU = \frac{\theta q(\theta)}{r+\delta+\theta q(\theta)}[w(h) - f'(h)h]$, which implies that $w(h) \geq rU$. That is, individual rationality constraint is fulfilled for both workers and firms.

Step 3. We now consider a case where firms offer a constant wage rate, then equilibrium wages can be written as $w(h) = f'(h)h$. Applying Implicit Function Theorem and differentiating both sides of $(r+\delta)p = \theta q(\theta) f'(h)$ with respect to θ, we have

$$\underbrace{-\theta f''(h)}_{>0} \frac{\partial h}{\partial \theta} = \underbrace{(1-\eta) f'(h)}_{>0}, \tag{29}$$

which implies that $\partial h / \partial \theta > 0$. We can hence rewrite $c(r+\delta) = [f(h) - f'(h)h]q(\theta)$ as

$$c(r+\delta) = \underbrace{\{f[h(\theta)] - f'[h(\theta)]h(\theta)\} q(\theta)}_{\equiv \Psi(\theta)},$$

where $h(\theta) = (f')^{-1}\left[\frac{(r+\delta)p}{\theta q(\theta)}\right]$. Noting that $\Psi'(\theta) = \frac{q(\theta)}{\theta} f'[h(\theta)]h(\theta) + f[h(\theta)]q'(\theta)$, where we have used equation (29), then we can get that

$$\Psi'(\theta) > 0 \Leftrightarrow f'[h(\theta)] \frac{h(\theta)}{f[h(\theta)]} > \eta \;\text{ and }\; \Psi'(\theta) < 0 \Leftrightarrow f'[h(\theta)] \frac{h(\theta)}{f[h(\theta)]} < \eta.$$

As a consequence, if condition (16) holds, we have a unique solution of θ that is also an interior solution. It is easy to verify that both equilibrium h and equilibrium u are uniquely determined, and they are interior solutions as well.

Step 4. We show that condition (16) can be satisfied for reasonable functional forms of f and q. Let $q(\theta) = \theta^{-\eta}$ and $f(h) = h^\alpha$ for a constant $\alpha \in (0,1)$, then we obtain $\Psi(\theta) = (1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} \theta^{\frac{\alpha(1-\eta)}{1-\alpha} - \eta}$. Therefore, as long as $\alpha \neq \eta$, equilibrium θ is uniquely determined, and it is an interior solution.

Step 5. We finally verify that the matches with competitive wage are in the mutual interest of workers and firms. In the equilibrium, we have

$$f(h) - rU = \frac{(r+\delta)f(h) + \theta q(\theta)\left[f(h) - w(h) + f'(h)h\right]}{r+\delta+\theta q(\theta)},$$

it is thus immediate that $f(h) \geq w(h) \Rightarrow f(h) \geq rU$ for $\forall h \geq 0$. By using our assumption, group rationality constraint is satisfied in the competitive equilibrium. Q.E.D.

Proof of Lemma 3.2: We shall complete it in 5 steps.

Step 1. Since (ex ante) human capital investment and (ex post) wage bargaining move sequentially in this case, we apply backward induction to derive the search equilibrium. We first derive the Nash wage. The generalized Nash bargaining solution selects a wage contract $w(h)$ to maximize $[W(h) - U]^{\beta}\,[J(h) - V]^{1-\beta}$ for some given $\beta \in (0,1)$. Performing this maximization yields a necessary and sufficient first-order condition:

$$(1-\beta)\left[W(h) - U\right] = \beta\left[J(h) - V\right]. \tag{30}$$

So, the worker receives a fraction β of total match surplus: $W(h) - U = \beta\left[W(h) + J(h) - U - V\right]$. Applying equations (3) and (5) to equation (30) produces that

$$w(h) = \beta f(h) + (1-\beta)rU. \tag{31}$$

The Nash wage is thus a weighted average of the output of the match and the worker's reservation wages.

Step 2. We now establish the three equations determining the steady-state equilibrium. First, making use of equations (2), (3) and (31), we get that

$$rU = \max_{h\geq 0}\left\{\frac{-(r+\delta)ph + \beta\theta q(\theta)f(h)}{r+\delta+\beta\theta q(\theta)}\right\}. \tag{32}$$

Thus, payoff maximization yields a necessary and sufficient first-order condition (17), which determines the equilibrium level of human capital investment. Second, making use of $V = 0$ and equations (4), (5), (31) and (32), we get (18), which determines the equilibrium market tightness. The flow of workers into unemployment, $\delta(1-u)$, must be equal to the flow of workers out of unemployment, $\theta q(\theta)u$, we thus have equation (19).

We next verify that individual rationality constraints facing workers and firms are satisfied. For workers, evaluating equation (32) at the equilibrium outcome and substituting it into equation (31) and simplifying the algebra, we then get the equilibrium Nash wage contract as

$$w(h) = \beta\left\{\frac{(r+\delta)f(h) + \theta q(\theta)[f(h) - f'(h)h] + \beta\theta q(\theta)f'(h)h}{r+\delta+\beta\theta q(\theta)}\right\} \geq 0$$

for $\forall h \geq 0$ and $\forall \theta \geq 0$, where we have used the equilibrium equation (17) as well as the canonical assumptions placed on f. For firms, equations (32) and (31) imply that in equilibrium:

$$f(h) - w(h) = \frac{(1-\beta)(r+\delta)[f(h) + ph]}{r+\delta+\beta\theta q(\theta)} \geq 0$$

for $\forall h \geq 0$. We, accordingly, claim that the individual rationality constraints facing workers and firms are fulfilled under wage bargaining.

Step 3. Here we provide a group of sufficient conditions guaranteeing that we have a unique equilibrium that is also an interior equilibrium. First, by exploiting equation (17) and the strict concavity of f, we get $h(\theta) = (f')^{-1}\left[\frac{(r+\delta)p}{\beta\theta q(\theta)}\right]$. Plugging this $h(\theta)$ in (18) and rearranging the algebra, we have

$$\underbrace{c(r+\delta) + \beta c\theta q(\theta)}_{\equiv \phi(\theta)} = \underbrace{(1-\beta)\left\{f\left((f')^{-1}\left[\frac{(r+\delta)p}{\beta\theta q(\theta)}\right]\right) + p(f')^{-1}\left[\frac{(r+\delta)p}{\beta\theta q(\theta)}\right]\right\}q(\theta).}_{\equiv \Phi(\theta)} \tag{33}$$

By our assumptions, we have $\phi'(\theta) > 0$, $\lim_{\theta\downarrow 0}\phi(\theta) = c(r+\delta)$ and $\lim_{\theta\uparrow\infty}\phi(\theta) = \infty$. Therefore, to make equation (33) have a unique solution of θ that is also an interior solution, we can impose restrictions on f and q such that $\Phi(\theta)$ satisfies: $\Phi'(\theta) < 0$, $\lim_{\theta\downarrow 0}\Phi(\theta) > c(r+\delta)$ and $\lim_{\theta\uparrow\infty}\Phi(\theta) < c(r+\delta)$.

Step 4. We now confirm that we can find reasonable functional forms of f and q so that the sufficient conditions derived in Step 3 are satisfied. Let $q(\theta) = \theta^{-\eta}$ and $f(h) = h^{\alpha}$ for a constant $\alpha \in (0,1)$, then we have

$$\Phi(\theta) = (1-\beta)\left\{\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\theta^{\frac{\alpha(1-\eta)}{1-\alpha}-\eta} + p\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\theta^{\frac{1-\eta}{1-\alpha}-\eta}\right\}.$$

If $\eta \geq \frac{1}{2-\alpha}$, then $\Phi(\theta)$ satisfies the requirements proposed in Step 3. That is, equation (33) indeed determines a unique θ that belongs to $(0,\infty)$.

Step 5. We now verify that the matches with wage bargaining are in the mutual interest of workers and firms. By equation (32), we have in equilibrium: $f(h) \geq rU \Leftrightarrow (r+\delta)[f(h)+ph] \geq 0$, which thus shows that inequality (6) always holds true. Q.E.D.

Proof of Lemma 3.3: We shall complete it in 4 steps.

Step 1. Using equations (2) and (3), we have

$$rU = \max_{h\geq 0}\left\{\frac{-(r+\delta)ph + \theta q(\theta)w(h)}{r+\delta+\theta q(\theta)}\right\}. \tag{34}$$

Using equations (4) and (5), we have

$$rV = \frac{-c(r+\delta) + q(\theta)[f(h) - w(h)]}{r+\delta+q(\theta)}.$$

Similar to Lemma 1 of Acemoglu and Shimer (1999), we can characterize the steady-state equilibrium under wage posting as a solution to the constrained maximization problem:

$$\max_{h,w,\theta}\frac{-(r+\delta)ph + \theta q(\theta)w}{r+\delta+\theta q(\theta)}$$

subject to $rV \geq 0$, i.e., $q(\theta)[f(h) - w] \geq c(r+\delta)$. That is, competitive search equilibrium should select the posted wage contract (h,w) and the market tightness that maximize workers' payoff and simultaneously assure that firms are willing to create job vacancies (i.e., the profits earning from job vacancy creation should be non-negative). The Lagrangian can be written as

$$\mathcal{L}(h, w, \theta; \mu) = \frac{-(r+\delta)ph + \theta q(\theta) w}{r+\delta+\theta q(\theta)} + \mu \left\{ q(\theta)[f(h) - w] - c(r+\delta) \right\},$$

for a Lagrangian multiplier $\mu \geq 0$. We hence obtain a group of necessary first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial h} = \frac{-(r+\delta)p}{r+\delta+\theta q(\theta)} + \mu q(\theta) f'(h) = 0, \tag{35}$$

$$\frac{\partial \mathcal{L}}{\partial w} = \frac{\theta q(\theta)}{r+\delta+\theta q(\theta)} - \mu q(\theta) = 0, \tag{36}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{(1-\eta) q(\theta) w}{r+\delta+\theta q(\theta)} - \frac{(1-\eta)\left[-(r+\delta)ph + \theta q(\theta) w\right] q(\theta)}{\left[r+\delta+\theta q(\theta)\right]^2} + \mu [f(h) - w] q'(\theta) = 0. \tag{37}$$

It follows from equation (35) or equation (36) that $\mu > 0$, and hence

$$q(\theta)[f(h) - w] = c(r+\delta) \tag{38}$$

by using the complementary slackness condition. Equations (35)-(36) imply equilibrium equation (21). Using equations (36) and (38), we can simplify equation (37) and obtain equilibrium equation (22). In addition, the flow of workers into unemployment, $\delta(1-u)$, must be equal to the flow of workers out of unemployment, $\theta q(\theta) u$, we hence have equilibrium equation (23).

In addition, we need to verify whether or not the individual rationality constraints are satisfied in equilibrium. Equation (38) immediately implies that $f(h) - w \geq 0$ for $\forall h \geq 0$ and $\theta \geq 0$. So, the individual rationality constraint facing firms is satisfied. Using equations (38) and (22), we have $w + ph = \eta[f(h) + ph + c\theta] \geq 0$ for $\forall h \geq 0$ and $\theta \geq 0$. So, the individual rationality constraint facing workers is satisfied.

Step 2. Here we provide a group of sufficient conditions guaranteeing that we have a unique equilibrium that is also an interior equilibrium. First, by exploiting equation (21) and the strict concavity of f, we get $h(\theta) = (f')^{-1}\left[\frac{(r+\delta)p}{\theta q(\theta)}\right]$. Then, plugging this $h(\theta)$ in (22) and rearranging the algebra, we have

$$\underbrace{c(r+\delta) + \eta c \theta q(\theta)}_{\equiv \chi(\theta)} = \underbrace{(1-\eta)\left\{ f\left((f')^{-1}\left[\frac{(r+\delta)p}{\theta q(\theta)}\right]\right) + p (f')^{-1}\left[\frac{(r+\delta)p}{\theta q(\theta)}\right]\right\} q(\theta)}_{\equiv \Xi(\theta)}. \tag{39}$$

By our assumptions, we have $\chi'(\theta) > 0$, $\lim_{\theta \downarrow 0} \chi(\theta) = c(r+\delta)$ and $\lim_{\theta \uparrow \infty} \chi(\theta) = \infty$. Therefore, to make equation (39) have a unique solution of θ that is also an interior solution, we can impose restrictions on f and q such that $\Xi(\theta)$ satisfies: $\Xi'(\theta) < 0$, $\lim_{\theta \downarrow 0} \Xi(\theta) > c(r+\delta)$ and $\lim_{\theta \uparrow \infty} \Xi(\theta) < c(r+\delta)$.

Step 3. We now confirm that we can find reasonable functional forms of f and q so that the sufficient conditions derived in Step 2 are satisfied. Let $q(\theta) = \theta^{-\eta}$ and $f(h) = h^{\alpha}$ for a constant $\alpha \in (0,1)$, then we have

$$\Xi(\theta) = (1-\eta)\left\{ \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} \theta^{\frac{\alpha(1-\eta)}{1-\alpha} - \eta} + p \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \theta^{\frac{1-\eta}{1-\alpha} - \eta} \right\}.$$

If $\eta \geq \frac{1}{2-\alpha}$, then $\Xi(\theta)$ satisfies the requirements proposed in Step 2. That is, equation (39) indeed determines a unique θ that belongs to $(0, \infty)$.

Step 4. We now verify that the competitive search equilibrium fulfills that matches with wage posting are in the mutual interest of firms and workers. Note that $f(h) \geq rU \Leftrightarrow f(h) + ph + c\theta \geq 0$ by applying equations (34) and (38), thus inequality (6) holds true in equilibrium. Q.E.D.

Proof of Lemma 4.1: We shall complete it in 4 steps.

Step 1. Applying Assumption 4.1 to (13)-(14) produces:

$$(1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{C}\right)^{\frac{\alpha-\eta}{1-\alpha}} = c(r+\delta) \Leftrightarrow \theta^{C} = \left\{\left[\frac{1-\alpha}{c(r+\delta)}\right]\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}} \tag{40}$$

for $\alpha \neq \eta$. Similarly, applying Assumption 4.1 to (21)-(22) produces:

$$(1-\eta)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{P}\right)^{\frac{\alpha-\eta}{1-\alpha}} + (1-\eta)p\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\left(\theta^{P}\right)^{\frac{1-\eta}{1-\alpha}-\eta} - \eta c\left(\theta^{P}\right)^{1-\eta} = c(r+\delta). \tag{41}$$

Step 2. We first prove that $\theta^{C} \neq \theta^{P}$, and we prove it by means of contradiction. Suppose $\theta^{C} = \theta^{P}$, then we get from (40)-(41) that

$$\theta^{P} < \left\{\left(\frac{1-\eta}{\eta}\right)\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}}, \tag{42}$$

for either $\alpha > \eta$ or $\alpha < \eta$. If $\alpha > \eta$, then combining (40) with (42) shows that

$$\left[\frac{1-\alpha}{c(r+\delta)}\right]\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} > \left(\frac{1-\eta}{\eta}\right)\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \Leftrightarrow \alpha < \eta,$$

a contradiction. If $\alpha < \eta$, then combining (40) with (42) shows that

$$\left[\frac{1-\alpha}{c(r+\delta)}\right]\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} < \left(\frac{1-\eta}{\eta}\right)\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \Leftrightarrow \alpha > \eta,$$

a contradiction. Thus, we should have $\theta^{C} \neq \theta^{P}$.

Step 3. If we assume in (40) and (41) that

$$(1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{C}\right)^{\frac{\alpha-\eta}{1-\alpha}} = (1-\eta)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{P}\right)^{\frac{\alpha-\eta}{1-\alpha}},$$

then we have

$$\frac{\theta^{P}}{\theta^{C}} = \left(\frac{1-\alpha}{1-\eta}\right)^{\frac{1-\alpha}{\alpha-\eta}} \tag{43}$$

as well as

$$\theta^{P} = \left\{\left(\frac{1-\eta}{\eta}\right)\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}} \tag{44}$$

by using (40) and (41) again. Combining (40) with (44) shows that

$$\frac{\theta^{P}}{\theta^{C}} = \left[\frac{(1-\eta)\alpha}{\eta(1-\alpha)}\right]^{\frac{1-\alpha}{\eta-\alpha}},$$

plugging which in (43) results in $\alpha = \eta$, a contradiction. Thus, we should have

$$(1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{C}\right)^{\frac{\alpha-\eta}{1-\alpha}} \neq (1-\eta)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{P}\right)^{\frac{\alpha-\eta}{1-\alpha}}.$$

Step 4. We then prove that $\theta^C < \theta^P$ cannot hold, and we prove it by means of contradiction. Suppose, instead, that $\theta^C < \theta^P$ holds true. We need to consider two cases. First, suppose

$$(1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{C}\right)^{\frac{\alpha-\eta}{1-\alpha}} > (1-\eta)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{P}\right)^{\frac{\alpha-\eta}{1-\alpha}}, \tag{45}$$

then we have $\left(\frac{\theta^P}{\theta^C}\right)^{\frac{\alpha-\eta}{1-\alpha}} < \frac{1-\alpha}{1-\eta}$. If $\alpha > \eta$, then $\frac{\theta^P}{\theta^C} < \left(\frac{1-\alpha}{1-\eta}\right)^{\frac{1-\alpha}{\alpha-\eta}} < 1$, which however contradicts with the assumption that $\theta^C < \theta^P$. So, we assume that $\alpha < \eta$. Also, applying (45) to (40)-(41) shows that (42) is satisfied. Thus $\theta^C < \theta^P$ combined with (40) yields that

$$\left[\frac{1-\alpha}{c(r+\delta)}\right]\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} < \left(\frac{1-\eta}{\eta}\right)\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \Leftrightarrow \alpha > \eta,$$

a contradiction. Thus, we consider the second case:

$$(1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{C}\right)^{\frac{\alpha-\eta}{1-\alpha}} < (1-\eta)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{P}\right)^{\frac{\alpha-\eta}{1-\alpha}}, \tag{46}$$

which implies $\left(\frac{\theta^P}{\theta^C}\right)^{\frac{\alpha-\eta}{1-\alpha}} > \frac{1-\alpha}{1-\eta}$. If $\alpha < \eta$, then $\frac{\theta^C}{\theta^P} > \left(\frac{1-\alpha}{1-\eta}\right)^{\frac{1-\alpha}{\eta-\alpha}} > 1$, which however contradicts with the assumption that $\theta^C < \theta^P$. So, we assume that $\alpha > \eta$. Also, applying (46) to (40)-(41) shows that (42) is satisfied. Thus $\theta^C < \theta^P$ combined with (40) yields that

$$\left[\frac{1-\alpha}{c(r+\delta)}\right]\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} > \left(\frac{1-\eta}{\eta}\right)\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \Leftrightarrow \alpha < \eta,$$

a contradiction. Thus we should have $\theta^C > \theta^P$ other than $\theta^C < \theta^P$. Finally, note that $h^j = \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}(\theta^j)^{\frac{1-\eta}{1-\alpha}}$ and $u^j = \delta/\left[\delta + \left(\theta^j\right)^{1-\eta}\right]$ for $\forall j \in \{C, P\}$ under Assumption 4.1, thus the proof is complete. Q.E.D.

Proof of Lemma 4.2: We shall complete it in 6 steps.

Step 1. Applying Assumption 4.1 to (17)-(18) produces:

$$(1-\beta)\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{B}\right)^{\frac{\alpha-\eta}{1-\alpha}} + (1-\beta)p\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\left(\theta^{B}\right)^{\frac{1-\eta}{1-\alpha}-\eta} - \beta c\left(\theta^{B}\right)^{1-\eta} = c(r+\delta),$$

which combined with (40) shows that

$$\begin{aligned}(1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{C}\right)^{\frac{\alpha-\eta}{1-\alpha}} &= (1-\beta)\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^{B}\right)^{\frac{\alpha-\eta}{1-\alpha}} \\ &+ (1-\beta)p\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\left(\theta^{B}\right)^{\frac{1-\eta}{1-\alpha}-\eta} - \beta c\left(\theta^{B}\right)^{1-\eta}.\end{aligned} \tag{47}$$

Step 2. We prove part (i) by means of contradiction, namely we assume that $\theta^C = \theta^B$. First, if $1-\alpha = (1-\beta)\beta^{\frac{\alpha}{1-\alpha}}$, then it follows from (47) that

$$\theta^B = \left\{ (1-\beta)\beta^{\frac{\alpha}{1-\alpha}} \left(\frac{p}{c}\right) \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \right\}^{\frac{1-\alpha}{\eta-\alpha}}. \tag{48}$$

So, $\theta^C = \theta^B$ implies that

$$\left[\frac{1-\alpha}{c(r+\delta)}\right] \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} = (1-\beta)\beta^{\frac{\alpha}{1-\alpha}} \left(\frac{p}{c}\right) \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \Leftrightarrow 1-\alpha = \alpha(1-\beta)\beta^{\frac{\alpha}{1-\alpha}},$$

where we have used (40). But it contradicts with the assumption $1-\alpha = (1-\beta)\beta^{\frac{\alpha}{1-\alpha}}$. Second, if we assume $1-\alpha > (1-\beta)\beta^{\frac{\alpha}{1-\alpha}}$, then we get from (47) that

$$\left(\theta^B\right)^{\frac{\eta-\alpha}{1-\alpha}} < (1-\beta)\beta^{\frac{\alpha}{1-\alpha}} \left(\frac{p}{c}\right) \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}. \tag{49}$$

Next, if $\alpha > \eta$, then (49) combined with $\theta^C = \theta^B$ and (40) yields

$$\left[\frac{1-\alpha}{c(r+\delta)}\right] \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} < (1-\beta)\beta^{\frac{\alpha}{1-\alpha}} \left(\frac{p}{c}\right) \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \Leftrightarrow 1-\alpha < \alpha(1-\beta)\beta^{\frac{\alpha}{1-\alpha}} < (1-\beta)\beta^{\frac{\alpha}{1-\alpha}},$$

a contradiction. If $\alpha < \eta$, then we can show that (49) combined with $\theta^C = \theta^B$ and (40) yields the same contradiction. Therefore, the required assertion in part (i) follows.

Step 3. We now show that

$$(1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} \left(\theta^C\right)^{\frac{\alpha-\eta}{1-\alpha}} \neq (1-\beta)\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} \left(\theta^B\right)^{\frac{\alpha-\eta}{1-\alpha}},$$

and we prove this by means of contradiction. Suppose, instead, that

$$(1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} \left(\theta^C\right)^{\frac{\alpha-\eta}{1-\alpha}} = (1-\beta)\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} \left(\theta^B\right)^{\frac{\alpha-\eta}{1-\alpha}}, \tag{50}$$

which yields

$$\frac{\theta^B}{\theta^C} = \left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\alpha-\eta}}. \tag{51}$$

Applying (50) to (47) gives rise to (48), which combined with (40) shows that

$$\frac{\theta^B}{\theta^C} = \left[\frac{1-\alpha}{\alpha(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\alpha-\eta}},$$

which combined with (51) implies that $\alpha = 1$, a contradiction.

Step 4. We now show that

$$(1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} \left(\theta^C\right)^{\frac{\alpha-\eta}{1-\alpha}} > (1-\beta)\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} \left(\theta^B\right)^{\frac{\alpha-\eta}{1-\alpha}}$$

does not hold. We also prove this by means of contradiction. Suppose it does hold, then we have $\left(\theta^B/\theta^C\right)^{\frac{\alpha-\eta}{1-\alpha}} < (1-\alpha)/\left[(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\right]$. If $\alpha > \eta$, then we have

$$\theta^B < \left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\alpha-\eta}} \theta^C. \tag{52}$$

Also, note from (47) that in this case we can have

$$\theta^B > \left\{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}}. \tag{53}$$

Making use of (40), (52) and (53), we obtain

$$\left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\alpha-\eta}} \theta^C > \left\{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}} \Leftrightarrow \alpha > 1,$$

a contradiction. If $\alpha < \eta$, then we similarly have

$$\theta^C < \left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\eta-\alpha}} \theta^B \text{ and } \theta^B < \left\{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}}. \tag{54}$$

Thus, making use of (40) and (54), we obtain

$$\theta^C < \left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\eta-\alpha}} \left\{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}} \Leftrightarrow \alpha > 1,$$

also a contradiction.

Step 5. We now prove part (ii) and suppose $\theta^C < \theta^B$. We just need to consider the following case:

$$(1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^C\right)^{\frac{\alpha-\eta}{1-\alpha}} < (1-\beta)\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^B\right)^{\frac{\alpha-\eta}{1-\alpha}},$$

which implies that

$$\left(\frac{\theta^B}{\theta^C}\right)^{\frac{\alpha-\eta}{1-\alpha}} > \frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}. \tag{55}$$

If $\alpha > \eta$, then we have (54) as before. First, using $\theta^C < \theta^B$, (40) and (54), we have

$$\theta^C < \left\{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}} \Leftrightarrow 1-\alpha > \alpha(1-\beta)\beta^{\frac{\alpha}{1-\alpha}},$$

as desired. Second, using (40) and (54), we have

$$\theta^C < \left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\eta-\alpha}} \left\{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}} \Leftrightarrow \alpha < 1,$$

as desired. If $\alpha < \eta$, then we get from (55) that

$$1 > \frac{\theta^C}{\theta^B} > \left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\eta-\alpha}} \Rightarrow 1-\alpha < (1-\beta)\beta^{\frac{\alpha}{1-\alpha}},$$

as desired. Also, note that (53) holds in this case. We hence get

$$\frac{\theta^C}{\theta^B} > \left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\eta-\alpha}} \Leftrightarrow 1 > \alpha,$$

as desired. Note that $\alpha > \eta \Leftrightarrow 1-\alpha < 1-\eta$ and $\alpha < \eta \Leftrightarrow 1-\alpha > 1-\eta$, thus the proof of part (ii) is complete.

Step 6. We now prove part (iii) and suppose $\theta^C > \theta^B$. We just need to consider the following case:

$$(1-\alpha)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^C\right)^{\frac{\alpha-\eta}{1-\alpha}} < (1-\beta)\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^B\right)^{\frac{\alpha-\eta}{1-\alpha}},$$

for either $\alpha > \eta$ or $\alpha < \eta$. First, if $\alpha > \eta$, then we get from this condition and (47) that

$$1 > \frac{\theta^B}{\theta^C} > \left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\alpha-\eta}} \text{ and } \theta^B < \left\{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}},$$

by which we immediately obtain $1-\alpha < \min\left\{1-\eta, (1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\right\}$ and also

$$\left\{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}} > \left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\alpha-\eta}}\theta^C \Leftrightarrow \alpha < 1$$

by using (40) again, hence the desired assertion follows. Second, if $\alpha < \eta$, then we similarly get

$$\frac{\theta^C}{\theta^B} > \max\left\{1, \left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\eta-\alpha}}\right\} \text{ and } \theta^B > \left\{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}},$$

using which we are led to

$$\theta^C > \left[\frac{1-\alpha}{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\eta-\alpha}}\left\{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}} \Leftrightarrow 1 > \alpha$$

and

$$\theta^C > \left\{(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left(\frac{p}{c}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\right\}^{\frac{1-\alpha}{\eta-\alpha}} \Leftrightarrow 1-\alpha > \alpha(1-\beta)\beta^{\frac{\alpha}{1-\alpha}},$$

in which we have used (40) again.

Finally, note that $h^C = \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\left(\theta^C\right)^{\frac{1-\eta}{1-\alpha}}$, $h^B = \left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\left(\theta^B\right)^{\frac{1-\eta}{1-\alpha}}$ and $u^j = \delta/\left[\delta + \left(\theta^j\right)^{1-\eta}\right]$ for $\forall j \in \{C, B\}$ under Assumption 4.1 and also the Hosios condition $\eta = \beta$ is satisfied under Assumption 4.2, the desired assertion immediately follows. Q.E.D.

Proof of Lemma 4.3: We shall complete it in 4 steps.

Step 1. By using (40), (41) and (47), we have

$$\begin{aligned}&(1-\beta)\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^B\right)^{\frac{\alpha-\eta}{1-\alpha}} + (1-\beta)p\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\left(\theta^B\right)^{\frac{1-\eta}{1-\alpha}-\eta} - \beta c\left(\theta^B\right)^{1-\eta} \\ &= (1-\eta)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\theta^P\right)^{\frac{\alpha-\eta}{1-\alpha}} + (1-\eta)p\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\left(\theta^P\right)^{\frac{1-\eta}{1-\alpha}-\eta} - \eta c\left(\theta^P\right)^{1-\eta}.\end{aligned} \tag{56}$$

Step 2. We now prove part (i) by means of contradiction, and hence we assume that $\theta^B = \theta^P$. Then it follows from (56) that

$$\begin{aligned} 0 \;=\; & [(1-\beta)-(1-\eta)]\, c \left(\theta^P\right)^{1-\eta} \\ & + \left[(1-\beta)\beta^{\frac{\alpha}{1-\alpha}} - (1-\eta)\right] \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} \left(\theta^P\right)^{\frac{\alpha-\eta}{1-\alpha}} \\ & + \left[(1-\beta)\beta^{\frac{1}{1-\alpha}} - (1-\eta)\right] p \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \left(\theta^P\right)^{\frac{1-\eta}{1-\alpha}-\eta} \equiv \text{RHS}. \end{aligned} \tag{57}$$

It is easy to show that $1-\beta > (1-\beta)\beta^{\frac{\alpha}{1-\alpha}} > (1-\beta)\beta^{\frac{1}{1-\alpha}}$, hence $1-\beta \le 1-\eta$ implies RHS < 0 and $(1-\beta)\beta^{\frac{1}{1-\alpha}} \ge 1-\eta$ implies RHS > 0, both violating (57). So, the desired assertion in part (i) follows.

Step 3. In what follows, we assume that $\alpha = \eta$, which greatly simplifies (56) and leads us towards

$$\begin{aligned} & (1-\beta)\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} + \left\{(1-\beta)p\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} - \beta c\right\}\left(\theta^B\right)^{1-\eta} \\ & = (1-\eta)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} + \left\{(1-\eta)p\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} - \eta c\right\}\left(\theta^P\right)^{1-\eta}. \end{aligned} \tag{58}$$

Step 4. It follows from Assumption 4.2 that $1-\eta > (1-\beta)\beta^{\frac{\alpha}{1-\alpha}}$, then we get from (58) that

$$\left\{(1-\eta)p\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} - \eta c\right\}\left(\theta^P\right)^{1-\eta} < \left\{(1-\beta)p\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} - \beta c\right\}\left(\theta^B\right)^{1-\eta}. \tag{59}$$

Here we consider three cases. First, if $(1-\beta)p\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} > \beta c$, then (59) implies that $\theta^B > \theta^P$ for

$$\underbrace{\left[(1-\eta)-(1-\beta)\beta^{\frac{1}{1-\alpha}}\right]}_{>0} p\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \ge [(1-\beta)-(1-\eta)]\,c = 0,$$

in which we have used the assumption $1-\eta > (1-\beta)\beta^{\frac{\alpha}{1-\alpha}}$ and Assumption 4.2. As a consequence, we get the corresponding result in part (ii). Second, if $(1-\beta)p\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} < \beta c$, then (59) implies that $\theta^B < \theta^P$ for

$$\underbrace{\left[(1-\eta)-(1-\beta)\beta^{\frac{1}{1-\alpha}}\right]}_{>0} p\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \ge [(1-\beta)-(1-\eta)]\,c = 0,$$

in which we have used the assumption $1-\eta > (1-\beta)\beta^{\frac{\alpha}{1-\alpha}}$ and Assumption 4.2. As a consequence, we get the corresponding result in part (ii). Third, if $(1-\beta)p\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} = \beta c$, then we have from (58) and (41) that $(1-\beta)\beta^{\frac{\alpha}{1-\alpha}}\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} = c(r+\delta)$, which however combined with the assumption $(1-\beta)p\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} = \beta c$ gives rise to $\alpha = 1$, a contradiction. Finally, note that

$$\frac{p}{c}\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} < \frac{\beta}{(1-\beta)\beta^{\frac{1}{1-\alpha}}} \;\Leftrightarrow\; p > \left[\frac{\beta^{1+\beta}(1-\beta)^{1-\beta}}{(r+\delta)c^{1-\beta}}\right]^{\frac{1}{\beta}} \equiv \hat{p},$$

$h^P = \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \left(\theta^P\right)^{\frac{1-\eta}{1-\alpha}}$, $h^B = \left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} \left(\theta^B\right)^{\frac{1-\eta}{1-\alpha}}$ and $u^j = \delta / \left[\delta + \left(\theta^j\right)^{1-\eta}\right]$ for $\forall j \in \{P, B\}$ under Assumption 4.1, the proof is therefore complete. Q.E.D.

Proof of Lemma 4.4: We shall complete it in 4 steps.

Step 1. Applying Assumption 4.1 to Lemmas 3.1-3.3 gives rise to:

$$\begin{aligned} \mathcal{W}^j &= f(h^j) - [f(h^j) + ph^j + c\theta^j]u^j \\ &= \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} (\theta^j)^{\frac{\alpha(1-\eta)}{1-\alpha}} - \left\{ \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} (\theta^j)^{\frac{\alpha(1-\eta)}{1-\alpha}} + p\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} (\theta^j)^{\frac{1-\eta}{1-\alpha}} + c\theta^j \right\} \\ &\quad \times \left[\frac{\delta}{\delta + (\theta^j)^{1-\eta}}\right] \equiv \widetilde{\mathcal{W}}(\theta^j) \end{aligned} \tag{60}$$

for $j \in \{C, P\}$, and

$$\begin{aligned} \mathcal{W}^B &= f(h^B) - [f(h^B) + ph^B + c\theta^B]u^B \\ &= \left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} (\theta^B)^{\frac{\alpha(1-\eta)}{1-\alpha}} - \left\{ \left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}} (\theta^B)^{\frac{\alpha(1-\eta)}{1-\alpha}} + p\left[\frac{\alpha\beta}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}} (\theta^B)^{\frac{1-\eta}{1-\alpha}} + c\theta^B \right\} \\ &\quad \times \left[\frac{\delta}{\delta + (\theta^B)^{1-\eta}}\right] \equiv \widehat{\mathcal{W}}(\theta^B). \end{aligned} \tag{61}$$

Step 2. We first prove part (i). Given that we can write h and u as functions of θ, it follows from (60) that

$$\frac{\partial\widetilde{\mathcal{W}}}{\partial\theta} > 0 \quad \Leftrightarrow \quad cu\left[1 + \frac{\partial u}{\partial\theta}\left(\frac{\theta}{u}\right)\right] < \left\{f'(h) - u[f'(h) + p]\right\}\frac{\partial h}{\partial\theta} - [f(h) + ph]\frac{\partial u}{\partial\theta}.$$

By (15) and (19), we see that

$$\varepsilon_{u,\theta} \equiv \frac{\partial u}{\partial\theta}\left(\frac{\theta}{u}\right) = -\frac{\theta q(\theta)}{\delta + \theta q(\theta)}(1-\eta) \in (-1, 0),$$

hence

$$\frac{\partial\widetilde{\mathcal{W}}}{\partial\theta} > 0 \quad \Leftrightarrow \quad c < \frac{\left\{f'(h) - u[f'(h) + p]\right\}\frac{\partial h}{\partial\theta} - [f(h) + ph]\frac{\partial u}{\partial\theta}}{u\left[1 + \frac{\partial u}{\partial\theta}\left(\frac{\theta}{u}\right)\right]}.$$

Also, note that

$$f'(h) > u[f'(h) + p] \quad \Leftrightarrow \quad u < \frac{(r+\delta)\theta^{\eta-1}}{(r+\delta)\theta^{\eta-1} + 1} \quad \Leftrightarrow \quad 0 < r\theta^{1-\eta},$$

as desired in part (i).

Step 3. Even if we have established the strict monotonicity of the welfare function with respect to θ, mutual welfare comparison is immediate only only when the existence and uniqueness of equilibrium θ are assured. It follows from Lemmas 3.1-3.3 that we should put $\eta \geq 1/(2-\alpha)$, which hence implies that $\eta > \alpha$ (otherwise $\alpha \geq \eta \Rightarrow (\alpha - 1)^2 \leq 0$, an immediate contradiction).

By Step 2 we have

$$\frac{\partial\widetilde{\mathcal{W}}}{\partial\theta} = f'[h(\theta)][1 - u(\theta)]h'(\theta) - \left\{pu(\theta)h'(\theta) + f[h(\theta)]u'(\theta)\right\} - ph(\theta)u'(\theta) - [u(\theta) + u'(\theta)\theta]c, \tag{62}$$

in which $h(\theta)=\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\theta^{\frac{1-\eta}{1-\alpha}}$ and $u(\theta)=\delta/\left[\delta+\theta^{1-\eta}\right]$. First, it is easy to see that

$$\lim_{\theta\downarrow 0}[u(\theta)+u'(\theta)\theta]c=c. \tag{63}$$

Second, note that $ph(\theta)u'(\theta)=-p\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{1}{1-\alpha}}\left[\frac{\delta(1-\eta)}{(\delta+\theta^{1-\eta})^2}\right]\theta^{\frac{1-2\eta+\alpha\eta}{1-\alpha}}$ and $1-2\eta+\alpha\eta\le 0\Leftrightarrow\eta\ge 1/(2-\alpha)$, thus we have

$$-\infty<\lim_{\theta\downarrow 0}ph(\theta)u'(\theta)<0 \text{ for } \eta=1/(2-\alpha) \quad\text{and}\quad \lim_{\theta\downarrow 0}ph(\theta)u'(\theta)=-\infty \text{ for } \eta>1/(2-\alpha). \tag{64}$$

Third, note that

$$\begin{aligned}-\left\{pu(\theta)h'(\theta)+f[h(\theta)]u'(\theta)\right\} &= (1-\eta)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\times\\ &\left[\frac{(1-\alpha)r+(1-2\alpha)\delta-\alpha\theta^{1-\eta}}{(r+\delta)(1-\alpha)(\delta+\theta^{1-\eta})}\right]\left(\frac{\delta}{\delta+\theta^{1-\eta}}\right)\theta^{(\alpha-\eta)/(1-\alpha)}>0,\end{aligned}$$

thus we have

$$\lim_{\theta\downarrow 0}(-\left\{pu(\theta)h'(\theta)+f[h(\theta)]u'(\theta)\right\})=+\infty$$

for either $\alpha\le 1/2$ or $r\ge(2\alpha-1)\delta/(1-\alpha)$. Finally, note that

$$f'[h(\theta)][1-u(\theta)]h'(\theta)=\left[\frac{\alpha(1-\eta)}{1-\alpha}\right]\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left(\frac{1}{\delta+\theta^{1-\eta}}\right)\theta^{(\alpha-\eta)/(1-\alpha)}$$

and

$$\begin{aligned}&f'[h(\theta)][1-u(\theta)]h'(\theta)-\left\{pu(\theta)h'(\theta)+f[h(\theta)]u'(\theta)\right\}\\ &\qquad=\left(\frac{1-\eta}{1-\alpha}\right)\left[\frac{\alpha}{(r+\delta)p}\right]^{\frac{\alpha}{1-\alpha}}\left[\frac{(1-\alpha)\delta^2+r(\delta+\alpha\theta^{1-\eta})}{(r+\delta)(\delta+\theta^{1-\eta})}\right]\left(\frac{1}{\delta+\theta^{1-\eta}}\right)\theta^{(\alpha-\eta)/(1-\alpha)}>0,\end{aligned}$$

thus we actually always have

$$\lim_{\theta\downarrow 0}f'[h(\theta)][1-u(\theta)]h'(\theta)-\left\{pu(\theta)h'(\theta)+f[h(\theta)]u'(\theta)\right\}=+\infty. \tag{65}$$

As a consequence, by applying equations (63)-(65), we get from equation (62) that

$$\lim_{\theta\downarrow 0}\frac{\partial\widetilde{\mathcal{W}}}{\partial\theta}>0$$

for any $c<+\infty$. Since it is easy to show that $\partial\widetilde{\mathcal{W}}/\partial\theta$ is continuous with respect to θ, we thus can find a critical value of θ which is strictly positive such that $\partial\widetilde{\mathcal{W}}(\theta)/\partial\theta>0$ always holds true for any θ smaller than or equalling to this critical value and also any $c<+\infty$. This hence completes the proof of part (i).

Step 4. For any given $\theta\in(0,\infty)$, we get from (60)-(61) that

$$\begin{aligned}\widetilde{\mathcal{W}}(\theta)-\widehat{\mathcal{W}}(\theta) &= \left(1-\beta^{\frac{\alpha}{1-\alpha}}\right)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\theta^{\frac{\alpha(1-\eta)}{1-\alpha}}\left(\frac{\theta^{1-\eta}}{\delta+\theta^{1-\eta}}\right)p^{\frac{\alpha}{\alpha-1}}\\ &\quad-\left(1-\beta^{\frac{1}{1-\alpha}}\right)\left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}\theta^{\frac{1-\eta}{1-\alpha}}\left(\frac{\delta}{\delta+\theta^{1-\eta}}\right)p^{\frac{1}{\alpha-1}},\end{aligned}$$

simplifying which produces the required assertion in part (ii). Q.E.D.

Appendix B: Proof of Proposition 4.1 (Not For Publication)

We complete the proof by proving three claims. Using Lemmas 2.1 and 3.1, we get the first claim:

Claim 6.1. *Let $\eta^* \equiv \frac{w(h^*)+ph^*}{f(h^*)+ph^*+c\theta^*}$, then we have: (i) If $\eta \in (0, \eta^*]$, then either $h^C < h^*$, $\theta^C < \theta^*$ and $u^C > u^*$ or $h^C > h^*$, $\theta^C > \theta^*$ and $u^C < u^*$; (ii) If $\eta \in (\eta^*, 1)$, then either $h^C < h^*$, or $\theta^C > \theta^*$ and $u^C < u^*$, or both.*

Proof. We shall complete it in 4 steps. Step 1. For any $r > 0$, equations (9) and (13) show that

$$f'(h^*)\left[\frac{\theta^* q(\theta^*)}{\delta}\right] = p < f'\left(h^C\right)\left[\frac{\theta^C q\left(\theta^C\right)}{\delta}\right].$$

Then we get either $f'(h^*) < f'\left(h^C\right)$, or $\theta^* q(\theta^*) < \theta^C q\left(\theta^C\right)$, or both. By exploiting strict concavity of f and strict monotonicity of $\theta q(\theta)$, we have either $h^* > h^C$, or $\theta^* < \theta^C$, or both.

Step 2. Using equation (10), $(1-\eta)\left[f(h^*) + ph^*\right] - \eta c\theta^* = f(h^*) - w(h^*)$ implies that

$$\eta = \frac{w(h^*) + ph^*}{f(h^*) + ph^* + c\theta^*} \equiv \eta^* \in (0,1).$$

If $\eta = \eta^*$, then comparing equation (10) to equation (14) reveals that

$$\left[f(h^*) - w(h^*)\right] q(\theta^*) = \left[f\left(h^C\right) - w\left(h^C\right)\right] q\left(\theta^C\right), \tag{66}$$

which implies that $h^* = h^C \Leftrightarrow \theta^* = \theta^C$. Noting that we have shown in Step 1 that either $h^* > h^C$ or $\theta^* < \theta^C$, thus $h^* \neq h^C$ and $\theta^* \neq \theta^C$. On the one hand, if $h^* > h^C$, then $f(h^*) - w(h^*) > f\left(h^C\right) - w\left(h^C\right)$ by using the strict monotonicity of profits, so equation (66) implies that $q(\theta^*) < q\left(\theta^C\right)$. That is, $h^* > h^C \Rightarrow \theta^* > \theta^C$ by using the strict monotonicity of $q(\theta)$. On the other hand, if $\theta^* < \theta^C$, then $q(\theta^*) > q\left(\theta^C\right)$, which combined with equation (66) implies that $f(h^*) - w(h^*) < f\left(h^C\right) - w\left(h^C\right)$. That is, $\theta^* < \theta^C \Rightarrow h^* < h^C$. Therefore, we have either

$$\begin{cases} h^* < h^C \\ \theta^* < \theta^C \end{cases} \text{or} \quad \begin{cases} h^* > h^C \\ \theta^* > \theta^C \end{cases}$$

for $\eta = \eta^*$.

Step 3. If $\eta > \eta^*$, then comparing equation (10) to equation (14) gives

$$\left[f\left(h^C\right) - w\left(h^C\right)\right] q\left(\theta^C\right) = c(r+\delta) < \left[f(h^*) - w(h^*)\right] q(\theta^*),$$

which implies that either $h^* > h^C$, or $\theta^* < \theta^C$, or both. Thus, combining this observation with the claim established in Step 1 reveals that either $h^* > h^C$, or $\theta^* < \theta^C$, or both, for $\forall \eta \in (\eta^*, 1)$.

Step 4. If $\eta < \eta^*$, then comparing equation (10) to equation (14) gives

$$\left[f\left(h^C\right) - w\left(h^C\right)\right] q\left(\theta^C\right) = c(r+\delta) > \left[f(h^*) - w(h^*)\right] q(\theta^*),$$

which implies that either $h^* < h^C$, or $\theta^* > \theta^C$, or both. First, if $h^* < h^C$, then comparing equation (9) to equation (13) shows that

$$\frac{(r+\delta)p}{\theta^C q\left(\theta^C\right)} = f'\left(h^C\right) < f'(h^*) = \frac{\delta p}{\theta^* q(\theta^*)},$$

by which we have $\frac{\theta^* q(\theta^*)}{\theta^C q(\theta^C)} < \frac{\delta}{r+\delta} < 1$. That is, $h^* < h^C \Rightarrow \theta^* < \theta^C$. Second, if $\theta^* > \theta^C$, then comparing equation (9) to equation (13) shows that

$$\frac{f'\left(h^C\right)}{r+\delta} = \frac{p}{\theta^C q\left(\theta^C\right)} > \frac{p}{\theta^* q\left(\theta^*\right)} = \frac{f'\left(h^*\right)}{\delta},$$

by which we have $\frac{f'(h^*)}{f'(h^C)} < \frac{\delta}{r+\delta} < 1$. That is, $\theta^* > \theta^C \Rightarrow h^* > h^C$. To summarize, we have either

$$\begin{cases} h^* < h^C \\ \theta^* < \theta^C \end{cases} \text{or} \quad \begin{cases} h^* > h^C \\ \theta^* > \theta^C \end{cases}$$

for all $\eta \in (0, \eta^*)$. □

Using Lemmas 2.1 and 3.2, we establish the second claim:

Claim 6.2. *(i) If $\beta \in [\eta, 1)$, then either $h^B < h^*$, $\theta^B < \theta^*$ and $u^B > u^*$ or $h^B > h^*$, $\theta^B > \theta^*$ and $u^B < u^*$. (ii) If $\beta \in (0, \eta)$, then either $h^B < h^*$, or $\theta^B > \theta^*$ and $u^B < u^*$, or both.*

Proof. We shall complete it in 4 steps. Step 1. For any $\beta \in (0, 1)$ and $r > 0$, equations (9) and (17) reveal that

$$f'\left(h^*\right)\left[\frac{\theta^* q\left(\theta^*\right)}{\delta}\right] = p < f'\left(h^B\right)\left[\frac{\theta^B q\left(\theta^B\right)}{\delta}\right]. \tag{67}$$

We thus have either $f'\left(h^*\right) < f'\left(h^B\right)$, or $\theta^* q\left(\theta^*\right) < \theta^B q\left(\theta^B\right)$, or both. Exploiting the strict concavity of f and the strict monotonicity of $\theta q(\theta)$, we have either $h^* > h^B$, or $\theta^* < \theta^B$, or both.

Step 2. Let's rewrite equations (10) and (18) as

$$\frac{c\left[r+\delta+\eta\theta^* q\left(\theta^*\right)\right]}{(1-\eta)q\left(\theta^*\right)}\left[\frac{1}{f\left(h^*\right)+ph^*}\right] = 1$$

and

$$\frac{c\left[r+\delta+\beta\theta^B q\left(\theta^B\right)\right]}{(1-\beta)q\left(\theta^B\right)}\left[\frac{1}{f\left(h^B\right)+ph^B}\right] = 1,$$

respectively. Then we have

$$\frac{c\left[r+\delta+\eta\theta^* q\left(\theta^*\right)\right]}{(1-\eta)q\left(\theta^*\right)}\left[\frac{1}{f\left(h^*\right)+ph^*}\right] = \frac{c\left[r+\delta+\beta\theta^B q\left(\theta^B\right)\right]}{(1-\beta)q\left(\theta^B\right)}\left[\frac{1}{f\left(h^B\right)+ph^B}\right]. \tag{68}$$

If $\eta = \beta$, then it's immediate that $h^* = h^B \Leftrightarrow \theta^* = \theta^B$. Since we have shown in Step 1 that either $h^* > h^B$ or $\theta^* < \theta^B$, we conclude that $h^* \neq h^B$ and $\theta^* \neq \theta^B$ for $\eta = \beta$. On the one hand, if $h^* > h^B$, then equation (68) implies that

$$\frac{c\left[r+\delta+\eta\theta^* q\left(\theta^*\right)\right]}{(1-\eta)q\left(\theta^*\right)} > \frac{c\left[r+\delta+\beta\theta^B q\left(\theta^B\right)\right]}{(1-\beta)q\left(\theta^B\right)}.$$

That is, $h^* > h^B$ implies $\theta^* > \theta^B$ for $\eta = \beta$. On the other hand, if $\theta^* < \theta^B$ and $\eta = \beta$, then equation (68) implies that $f\left(h^B\right) + ph^B > f\left(h^*\right) + ph^*$. That is, $\theta^* < \theta^B$ implies $h^* < h^B$ for $\eta = \beta$. To conclude, we have either $h^* > h^B$ and $\theta^* > \theta^B$ or $h^* < h^B$ and $\theta^* < \theta^B$ for $\eta = \beta$.

Step 3. Since the right-hand side of equation (68) is increasing in β, assuming $\eta < \beta$ and substituting for β with η imply that

$$\frac{c\left[r+\delta+\eta\theta^{*}q\left(\theta^{*}\right)\right]}{(1-\eta)q\left(\theta^{*}\right)}\left[\frac{1}{f\left(h^{*}\right)+ph^{*}}\right] > \frac{c\left[r+\delta+\eta\theta^{B}q\left(\theta^{B}\right)\right]}{(1-\eta)q\left(\theta^{B}\right)}\left[\frac{1}{f\left(h^{B}\right)+ph^{B}}\right].$$

Hence either $f\left(h^{*}\right)+ph^{*} < f\left(h^{B}\right)+ph^{B}$, or

$$\frac{c\left[r+\delta+\eta\theta^{*}q\left(\theta^{*}\right)\right]}{(1-\eta)q\left(\theta^{*}\right)} > \frac{c\left[r+\delta+\eta\theta^{B}q\left(\theta^{B}\right)\right]}{(1-\eta)q\left(\theta^{B}\right)},$$

or both. That is, either $h^{*} < h^{B}$, or $\theta^{*} > \theta^{B}$, or both. If $h^{*} < h^{B}$, then equation (67) implies that $\theta^{*} < \theta^{B}$. If $\theta^{*} > \theta^{B}$, then equation (67) implies that $h^{*} > h^{B}$. To conclude, we have either $h^{*} < h^{B}$ and $\theta^{*} < \theta^{B}$ or $h^{*} > h^{B}$ and $\theta^{*} > \theta^{B}$ for $\forall \beta \in (\eta, 1)$.

Step 4. In equation (68), assuming $\eta > \beta$ and substituting for β with η imply that

$$\frac{c\left[r+\delta+\eta\theta^{*}q\left(\theta^{*}\right)\right]}{(1-\eta)q\left(\theta^{*}\right)}\left[\frac{1}{f\left(h^{*}\right)+ph^{*}}\right] < \frac{c\left[r+\delta+\eta\theta^{B}q\left(\theta^{B}\right)\right]}{(1-\eta)q\left(\theta^{B}\right)}\left[\frac{1}{f\left(h^{B}\right)+ph^{B}}\right].$$

Hence either $f\left(h^{*}\right)+ph^{*} > f\left(h^{B}\right)+ph^{B}$, or

$$\frac{c\left[r+\delta+\eta\theta^{*}q\left(\theta^{*}\right)\right]}{(1-\eta)q\left(\theta^{*}\right)} < \frac{c\left[r+\delta+\eta\theta^{B}q\left(\theta^{B}\right)\right]}{(1-\eta)q\left(\theta^{B}\right)},$$

or both. That is, either $h^{*} > h^{B}$, or $\theta^{*} < \theta^{B}$, or both. To summarize, we have either $h^{*} > h^{B}$, or $\theta^{*} < \theta^{B}$, or both, for $\forall \beta \in (0, \eta)$. □

Using Lemmas 2.1 and 3.3, we establish the third claim:

Claim 6.3. *Either $h^{P} < h^{*}$, $\theta^{P} < \theta^{*}$ and $u^{P} > u^{*}$ or $h^{P} > h^{*}$, $\theta^{P} > \theta^{*}$ and $u^{P} < u^{*}$.*

Proof. We shall complete it in 2 steps. Step 1. For any $r > 0$, equations (9) and (21) imply that

$$f'\left(h^{*}\right)\left[\frac{\theta^{*}q\left(\theta^{*}\right)}{\delta}\right] = p < f'\left(h^{P}\right)\left[\frac{\theta^{P}q\left(\theta^{P}\right)}{\delta}\right].$$

We thus have either $f'\left(h^{*}\right) < f'\left(h^{P}\right)$, or $\theta^{*}q\left(\theta^{*}\right) < \theta^{P}q\left(\theta^{P}\right)$, or both. Exploiting the strict concavity of f and the strict monotonicity of $\theta q(\theta)$, we have either $h^{*} > h^{P}$, or $\theta^{*} < \theta^{P}$, or both.

Step 2. Using equations (10) and (22), we have

$$\frac{r+\delta+\eta\theta^{*}q\left(\theta^{*}\right)}{q\left(\theta^{*}\right)}\left[\frac{1}{f\left(h^{*}\right)+ph^{*}}\right] = \frac{r+\delta+\eta\theta^{P}q\left(\theta^{P}\right)}{q\left(\theta^{P}\right)}\left[\frac{1}{f\left(h^{P}\right)+ph^{P}}\right]. \tag{69}$$

It's immediate that $h^{*} = h^{P} \Leftrightarrow \theta^{*} = \theta^{P}$, which however contradicts with the result shown in Step 1. Thus $h^{*} \neq h^{P}$ and $\theta^{*} \neq \theta^{P}$. If $h^{*} > h^{P}$, then $\theta^{*} > \theta^{P}$ by equation (69); if $\theta^{*} < \theta^{P}$, then $h^{*} < h^{P}$ by equation (69). □