GAMES AND ECONOMIC BEHAVIOR **16,** 135–151 (1996)
ARTICLE NO. 0079

Continuous and Feasible Implementation of Rational-Expectations Lindahl Allocations

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Received July 18, 1994

In this paper we consider Bayesian implementation of rational-expectations Lindahl allocations for economic environments with public goods when agents are incompletely informed about environments. We construct a continuous and feasible mechanism whose Bayesian equilibrium allocations coincide with rational-expectations Lindahl allocations. We not only allow the types of individuals to be unknown but also allow both the preferences and the initial endowments to be unknown to the designer. In addition, we allow some types of boundary equilibrium allocations and a continuous information structure. *Journal of Economic Literature* Classification Number: C72, D71, D82, H41. © 1996 Academic Press, Inc.

1. INTRODUCTION

The incentives problem is a basic problem that a social organization, and particularly an economic organization, must consider. Since agents have private information, they may find it advantageous to distort the information they reveal, and thus they may use such information strategically to advance their own interests. Therefore, in many contexts, direct-control (centralized) decision making in such an incomplete information situation is impossible or at least inappropriate and thus indirect control (decentralized) decision making is highly preferable. This implies that mechanism design under incomplete information must provide individuals with appropriate incentives so that individuals' interests are consistent with the goals of the organization. The mechanism design

* The author thanks an associate editor and two anonymous referees for very useful comments and suggestions. Financial support from the Private Enterprise Research Center and Texas A&M University's Program to Enhance Scholarly and Creative Activities is gratefully acknowledged. E-mail address: gqt7885@acs.tamu.edu.

0899-8256/96 $18.00

(implementation) theory, which regards mechanisms as unknown, deals with precisely this problem by designing a game form such that a prespecified welfare criterion is guaranteed to be achieved by the game across a large domain of possible environments. An organization or social choice rule which has this consistency property is called incentive compatibility. This abstract formulation embraces a variety of specific applications. As such, the implementation problem is fundamental to economics and its related social science disciplines. It enables us to compare and study incentive and information aspects of various economic organizations and institutions, including private ownership, state ownership and mixed ownership.

The implementation literature has taken two primary directions since Hurwicz (1972) formalized a general model to deal with the incentives problems. One direction is to characterize what various institutions can be achieved by using incentive compatible mechanisms with various solution concepts of individual behavior. Such solution concepts as dominant strategy equilibrium, Nash equilibrium, Bayesian equilibrium, and maximin equilibrium have been used in the literature. The goal is to design an incentive compatible mechanism such that the set of equilibrium outcomes of the mechanism coincides with the set of socially desirable alternatives for all environments under consideration. A pioneering work in this direction was Maskin (1977) for Nash implementation. The complete proof and further generalizations on full or virtual Nash implementation were provided by Dasgupta, Hammond, and Maskin (1979), Repullo (1987), Saijo (1988), Moore and Repullo (1988, 1990), Matsushima (1988), Abreu and Sen (1990, 1991), and others. Characterization results for full or virtual Bayesian implementation were given by Postlewaite and Schmeidler (1986), Palfrey and Srivastava (1989a,b, 1991), Abreu and Matsushima (1990), Mookherjee and Reichelstein (1990), Jackson (1991), Duggan (1993), Matsushima (1993), and Tian (1994b), among many others. However, due to the general nature of the social choice rules under consideration in this body of work, the implementing mechanisms turn out to be quite complex. Characterization results show what is possible for the implementation of a social choice rule (correspondence), but not what is realistic. Usually, the designer is not required to know individual preferences, and the knowledge of initial endowments, as is well known by now, can be relaxed. Furthermore, these mechanisms are (i) highly discontinuous (so they are not robust with respect to some misspecifications); (ii) have large message spaces; and (iii) they assume the information structure is discrete.

The second direction is toward “better” mechanism design, i.e., designing mechanisms which implement some specific social choice rules such as efficient allocations, individually rational allocations, (rational-expectations) Walrasian allocations, Lindahl allocations, etc., and which have desirable properties such as continuity, feasibility, and lower dimensionality. A seminal work on “better” mechanism design for Nash implementation was given by Groves and Ledyard (1977). This was followed by Hurwicz (1979), Schmeidler (1980), Walker

(1981), Hurwicz, Maskin, and Postlewaite (1984), Postlewaite and Wettstein (1989), Tian (1989, 1990, 1991, 1993) and many others. While the literature is rich in characterization results for Bayesian implementation in the case of incomplete information environments, until recently the only results obtained on designing better mechanisms in the case of incomplete information were given by Wettstein (1990, 1992) for exchange economies with private goods. Wettstein (1990) gave an almost continuous mechanism which Bayesian implements rational-expectations constrained Walrasian allocations, and in Wettstein (1992) he extended his result to a general social choice rule.

On the other hand, Palfrey and Srivastava (1987) introduced the notion of a rational-expectations Lindahl equilibrium which is a close counterpart of the Lindahl equilibrium in economies with complete information. The existence proof of rational-expectations Lindahl equilibria under certain conditions was given by Diamantaras (1992). Palfrey and Srivastava (1987) have shown that rational-expectations Lindahl allocations are implementable in Bayesian equilibrium when restricted to a subset of economies where all equilibrium allocations are interior and the information structure is discrete. Since their results characterize Bayesian implementability for a general correspondence, their approach shares the same drawbacks mentioned in regard to the general mechanism, namely, the designer is required to know individual preferences and endowments and the mechanism is discontinuous. This leaves the question of whether this particular correspondence can be Bayesian implemented by a mechanism with desirable properties.

In this paper we consider Bayesian implementation of rational-expectations Lindahl allocations for economies with public goods in incomplete information environments by giving a mechanism with some nice properties. We construct a continuous and feasible mechanism[1] whose Bayesian equilibrium allocations coincide with rational-expectations Lindahl allocations. In contrast to the existing results on characterizing Bayesian implementation of a social choice correspondence, we not only allow the types of individuals to be unknown to the designer but also allow both the preferences and initial endowments to be unknown to the designer and incorporate unreported endowments which are withheld but not destroyed. In addition, we allow boundary equilibrium allocations and a continuous information structure. The results are shown to be true if (i) there are at least three agents; (ii) utility functions are risk-averse (strictly concave), strictly monotone increasing in the private good, and satisfy the condition of indispensability of the private good; (iii) information is nonexclusive (NEI); (iv) individual endowments are state independent; and (v) other technical conditions.

This paper is organized as follows. Section 2 states some notation and definitions about the framework of analysis. Section 3 presents a mechanism with

[1] That is, the resulting allocations are in the consumption space, (weakly) balanced, and continuous for all messages and types.

the desirable properties. The main results and their proofs are given in Section 4. Finally, the concluding remarks are offered in Section 5.

2. THE MODEL AND DEFINITIONS

2.1. *Model*

In this paper we consider economies in which there are $n \geq 3$ agents (groups, players, or voters) who consume one private good and K public goods, x being private (as a numeraire) and y public. The single private good x can be thought of as a Hicksian composite commodity or money, and public goods y can be thought of as K public projects. Denote by $N = \{1, 2, \ldots, n\}$ the set of agents.

Let T_i be the set of possible types of agent i, which summarizes the preferences and information of the agent. Assume that T_i is a set in $\mathbb{R}^l$.[2] A type profile is denoted by $t = (t_1, \ldots, t_n)$, and $T = \prod_{i \in N} T_i$ denotes the set of all type profiles. Let G be a probability distribution (probability measure) defined on T. Denote by J the support of G, i.e., J is the smallest closed set in T with probability measure 1. We assume that J is compact and convex when T has a continuous information structure (i.e., T is a set with an infinite number of types).[3] Let $\pi_{i_1,\ldots,i_m}(J)$ be the projection of J onto the $(i_1, \ldots, i_m)$ hyperplane. Since J is compact and convex, the projection $\pi_{i_1,\ldots,i_m}(J)$ is also compact and convex.

For each i, denote by $T_{-i} = \prod_{j \neq i} T_j$ the set of possible profiles of the types of all agents other than i. We summarize i's beliefs about the other agents by means of a collection of conditional distribution functions on T_{-i} which can be derived from the common joint distribution G. For each i, let $G_i(t_{-i} \mid t_i)$ denote agent i's conditional probability measure that other agents receive the profile of types t_{-i} when agent i receives the type t_i.

Given a profile of types t and a consumption bundle (x_i, y), the (ex-post) utility function of agent i is given by $u_i(x_i, y, t)$ which is assumed to be strictly increasing in x_i, risk averse (strictly concave) in (x_i, y), and measurable in t. Each individual i in the economy is endowed with $\overset{\circ}{w}_i > 0$ units of the private good in each state, which is assumed to be independent of the types of agent. We assume there are no initial endowments of public goods, but the public goods can be produced from the private good under constant returns to scale. That is, the production function f_k is given by $y^k = f^k(x) = (1/\beta^k)x$ for each

[2] T_i can be a set in a Banach space.

[3] This assumption ensures that all projections are carried out continuously. (When T is finite, it trivially renders continuity with regard to the information structure T.) This assumption can be dropped if continuity of the outcome function is not required. In this case, the projection operation turns out to be a correspondence, but by Lemma 1 in Hildenbrand (1974, p. 55), the correspondence has a measurable selection.

$k = 1, \ldots, K$. Thus each unit of public good y^k requires β^k units of private good. Hence the feasibility constraint becomes

$$\sum_{i=1}^{n} x_i + \beta \cdot y \leq \sum_{i=1}^{n} \overset{\circ}{w}_i, \tag{1}$$

where $\beta = (\beta^1, \ldots, \beta^K) \in \mathbb{R}^K_{++}$.

Let A denote the set of all such feasible allocations,

$$A = \left\{ (x, y) \in R^{K+n}_{+} \colon \sum_{i=1}^{n} x_i + \beta \cdot y \leq \sum_{i=1}^{n} \overset{\circ}{w}_i \right\}.$$

An allocation rule is a measurable function $f \colon T \to A$. Let

$$X = \{ f \colon T \to A \}$$

be the set of feasible environment-contingent allocation rules. A social choice set is a subset $F \subset X$.[4]

Given an allocation rule, $f \colon T \to A$, the interim (conditional expected) utility of f to agent i's type t_i is

$$V_i(f_i, t_i) = \int_{T_{-i}} u_i(f_i(t), t) dG_i(t_{-i} \mid t_i).$$

Here $f_i = (x_i, y) \colon T \to R^{1+K}$ is agent i's consumption bundle for the private good and the public goods determined by the allocation rule f.

The tuple $e = \langle N, A, T, G, \{u_i\} \rangle$ is called an economic environment. We assume, as is standard, that the structure of e is common knowledge to all agents and that each agent i knows his own type. Denote by E all such environments.

The designer is assumed to know T and G, but does not know the true types of agents, the true utility functions, or the true initial endowments. To achieve a social goal, the designer needs to construct an informationally decentralized mechanism $\langle M, g \rangle$, where $M = \prod_{i \in N} M_i$ with an element $m = (m_1, \ldots, m_n)$. The set M_i is the set of possible messages agent i can use and M is the message space. $g \colon M \to A$ is an outcome function or, more explicitly, $g(m) =$

[4] A social choice set is sometimes called a social choice correspondence in the literature. However, this may cause confusion between a social choice correspondence setup for complete information and a social choice correspondence setup for incomplete information. A social choice correspondence in the complete information setup is usually defined as a mapping which assigns to each environment a set of desirable allocations. In contrast, a social choice set in incomplete information environments is defined as a set of social choice rules, each of which is a mapping that assigns to each type ("state") in the class of environments a desirable allocation. Therefore, to avoid confusion, the incomplete information correspondence is called a social choice set. We will define a social choice correspondence for incomplete information below.

$(g_1(m), \ldots, g_n(m))$. Here $g_i \equiv (x_i, y)$ is agent i's consumption bundle for the private good and the public goods determined by the outcome function g_i. For each $m \in M$, $g(m)$ yields an allocation in A. Given a mechanism $\langle M, g\rangle$, each agent i chooses messages m_i as a function of his types. We call a mapping $m_i\colon T_i \to M_i$ a strategy for agent i. The notations m_{-i} and M_{-i} are similarly defined. Given a strategy (message) profile $m = (m_1, \ldots, m_n)$, the interim utility to i when of type t_i is given by

$$W_i(m; t_i) = \int_{T_{-i}} u_i(g_i(m(t)), t)dG_i(t_{-i} \mid t_i).$$

A mechanism $\langle M, g\rangle$ defined on the domain E is *feasible* if $g_i(m) \in \mathbb{R}^{1+K}_+$ and the resulting allocations satisfy the global material constraint (1) for all $i \in N$ and all $m \in M$.

DEFINITION 1. For an economic environment e, the message $m*$ is a *Bayesian equilibrium* of a mechanism $\langle M, g\rangle$ defined on E if

$$W_i(m^*; t_i) \geq W_i(m_i, m^*_{-i}; t_i).$$

for all $i \in N$ and all $m_i \in M_i$. When m^* is a Bayesian equilibrium, $g(m^*)$ is called a Bayesian equilibrium allocation. Denote by $B_{\langle M,g\rangle}(e)$ the set of all such allocations for environment e.

DEFINITION 2. For an economic environment e, a mechanism $\langle M, g\rangle$ is said to *Bayesian-implement* a social choice set F if

(i) For any $f \in F$, there is a Bayesian equilibrium m^* to $\langle M, g\rangle$ such that $g(m^*(t)) = f(t)$ for all $t \in T$.

(ii) If m^* is a Bayesian equilibrium of the mechanism $\langle M, g\rangle$, then $g(m^*) \in F$.

If there is a mechanism $\langle M, g\rangle$ which Bayesian-implements F, then F is said to be *Bayesian-implementable*.

One can also consider global Bayesian-implementation. Define a social choice correspondence $\mathcal{F}\colon E \to 2^X$ on the domain E as a set-valued function which assigns to every economy $e \in E$ a social choice set $\mathcal{F}(e) \subset X$.

DEFINITION 3. A social choice correspondence $\mathcal{F}\colon E \to 2^X$ is said to be *globally Bayesian-implementable* relative to E if, for all $e \in E$, $\mathcal{F}(e)$ is Bayesian-implementable.

2.2. *Rational-Expectations Lindahl Allocations*

In this paper we will use the rational-expectations Lindahl correspondence as a social choice correspondence and will present a continuous and feasible mechanism which globally implements rational-expectations Lindahl allocations in

Bayesian equilibrium. The concept of rational-expectations Lindahl equilibrium allows the possibility that prices, as public information, partially reveal the state of the world. We now define the notion of a rational-expectations Lindahl equilibrium.

We normalize the price of the private good equal to one and denote by $P_i(t)$ the personalized price function of agent i when the vector of types is t. Given a price function $P_i\colon T \to R_+^K$, let

$$E_i(t_i, p_i; P_i) = \{(t_i, t'_{-i}) \in T\colon P_i(t_i, t'_{-i}) = p_i\}$$

be the set of types profiles which i cannot distinguish based either on his own type t_i or on the personalized price vector p_i. Let $G_i(i_{-i} \mid t_i, p_i, P_i)$ be the distribution function of agent i, conditional on $t \in E_i(t_i, p_i; P_i)$. This distribution function incorporates both the private information of agent i and the public information contained in the personalized price vectors. For instance, if P_i is fully revealing, $P_i(t) \neq P_i(t')$ if $t \neq t'$, then $E_i(t_i, p_i; P_i)$ is either a singleton or empty for each (t_i, p_i) pair, and $G_i(t_{-i} \mid t_i, P_i(t), P_i) = 1$ when t is the true type and 0 otherwise.

An allocation $(x^*, y^*)\colon T \to \mathbb{R}_+^{n+K}$ is a *rational-expectations Lindahl equilibrium allocation* for an economy e if there are measurable personalized price functions $P_i^*\colon T \to \mathbb{R}_+^K$, one for each i, such that

(1) For all $i \in N$ and almost every $t \in T$, (x_i^*, y^*) maximizes the expected utility

$$\int_{E_i(t_i, P_i^*(t); P_i^*)} u_i(x_i, y, t) dG_i(t_{-i} \mid t_i, P_i^*(t), P_i^*)$$

subject to $x_i + P_i^*(t) \cdot y \leq \overset{\circ}{w}_i$;

(2) $\sum_{i=1}^n x_i^*(t) + \beta \cdot y^*(t) = \sum_{i=1}^n \overset{\circ}{w}_i$ for all $t \in T$;

(3) $\sum_{i=1}^n P_i^*(t) = \beta$ for all $t \in T$.

Denote by REL(e) the set of all such allocations and thus $REL\colon E \to 2^X$ is a correspondence from E to X.

To achieve global implementability of rational-expectations Lindahl allocations, we assume that the support J of G satisfies the so-called nonexclusive information (NEI) condition.

ASSUMPTION 1. (NEI). For all $i \in N$, there exists a function $s_i\colon T_{-i} \to T_i$ such that for each $t \in J$, $t_i = s_i(t_{-i})$.

To ensure the continuity of the outcome function defined below, we must require that s_i be continuous on T_{-i} for all $i \in N$. The NEI condition implies that each agent's private information can be precisely inferred from the joint observations of all the remaining agents. In other words, if the information observed by all other agents is perfectly transmitted, the contribution to social knowledge of any one agent's private information is redundant. The NEI condition guarantees that the incentive compatibility condition holds for all social choice sets in an economy.[5] Postlewaite and Schmeidler (1986) showed that the NEI condition, together with the Bayesian monotonicity condition, is a sufficient condition for the implementability of a social choice rule under incomplete information. Further, Blume and Easley (1983, 1990) have shown that the NEI condition is in fact a necessary condition for implementability of rational-expectations Walrasian allocations in exchange economies. Intuitively, when NEI does not hold, some agent has truly private information which he may choose to exploit. His ability to exploit this information may destroy the incentive compatibility of a social choice rule. For detailed discussions about this condition, see Blume and Easley (1990). Since a rational-expectations Walrasian equilibrium is a special case of a rational-expectations Lindahl equilibrium, the NEI condition is also a necessary condition for implementability of rational-expectations Lindahl allocations.[6]

The following condition is also necessary for implementability of REL allocations.

ASSUMPTION 2. (Indispensability of Private Good) For all $i \in N, u_i(x_i, y, t) > u_i(x_i', y', t)$ for all $t \in T$, $x_i \in \mathbb{R}_{++}$, $x_i' \in \partial\mathbb{R}_+$, and $y, y' \in \mathbb{R}_+^K$, where $\partial\mathbb{R}_+^m$ is the boundary of $\mathbb{R}_+^m$.

REMARK 1. Assumption 2 has been called the "indispensability of money" by Mas-Colell (1980). This assumption cannot be dispensed with. Tian (1988) showed that the (constrained) Lindahl allocations violate Maskin's (1977) monotonicity condition even under the strict monotonicity and convexity of utility functions, and thus cannot be Nash implemented by a feasible mechanism. Since Nash implementation is a special case of Bayesian implementation when all individuals in the economy have complete information, it is also necessary to have Bayesian implementability of REL allocations. Note that, under this condition, the consumption of public goods can be zero although the consumption of the private good must be positive at an equilibrium. Therefore, boundary points are possible.

3. MECHANISM

In this section we present a continuous and feasible mechanism which globally Bayesian implements the rational-expectations Lindahl allocations. Before

[5] In fact, with NEI, incentive constraints are never binding in pure exchange private economies and public goods economies with constant returns to scale.

[6] By constructing a similar economy as in Blume and Easley (1990), one can directly show that violation of NEI leads to nonimplementability of rational-expectations Lindahl allocations.

giving the specific mechanism, we will briefly describe the mechanism. The mechanism proceeds as follows: First, construct the personalized price function of each individual using the pricing functions proposed by others. Next, use the projection mappings π_{-i} to form n–profiles in J for each reported type $\hat{t}$, and then value the personalized price functions of individuals at these n–profiles in J. Then, define a feasible choice correspondence $B\colon\ M \to R^K$ for public goods that can be produced with total endowments and can be purchased by all agents. The outcome $Y(m)$ for public goods will be chosen from $B(m)$ so that it is the closest to the sum of the contributions that each agent is willing to pay (i.e., it is the projection of the sum of the contributions onto $B(m)$). The outcome for the private good is then determined so that the budget constraint of each agent is satisfied. This mechanism will be continuous and feasible. Further, under conditions imposed in the paper, we show that the set of Bayesian allocations coincides with the set of rational-expectations Lindahl allocations over E.

We now begin the formal construction of the mechanism. For each agent $i \in N$, his message domain is of the form

$$M_i = Q_i \times T_i \times (0, 1] \times (0, \overset{\circ}{w}_i] \times \mathbb{R}^K \times (0, 1]. \tag{2}$$

Here Q_i is a set of all measurable functions. A generic element of M_i is $m_i = (q_i, \hat{t}_i, \delta_i, w_i, y_i, \eta_i)$. We require that q_i be chosen before any private information is observed, $\hat{t}_i$ be picked after the observation of t_i, and $(\delta_i, w_i, y_i, \eta_i)$ be decided upon after observing $q_i(\hat{t}_i)$. This requirement on the timing of messages captures the idea of prices revealing information to the individuals and market behavior under incomplete information. The components of m_i have the following interpretations. The component $q_i\colon\ T_i \to R^K$ denotes the price function proposed by agent i for use in constructing other agents' personalized price functions. The component $\hat{t}_i\colon\ T_i \to T_i$ is a reported, possibly false, type of agent i. The component δ_i denotes the degree of desirability for the private good. When $\delta_i = 1$, agent i wishes that public goods would not be produced. The designer will use the smallest δ_i of all agents to determine the level of public goods (see Eq. (5) below). The component w_i denotes a profession of agent i's endowment. The inequality $0 < w_i \leq \overset{\circ}{w}_i$ means that the agent cannot overstate his own endowment; on the other hand, the endowment can be understated, but the claimed endowment w_i must be positive. Note that, although the true endowment is the upper bound of the reported endowment, the designer does not need to know this upper bound. This is because whenever an agent claims an endowment of a certain amount, the designer can ask him to *exhibit* it (one may, for instance, imagine that the rules of the game require that the agent "put on the table" the reported amount w_i). Since $w_i \leqq \overset{\circ}{w}_i$ is permitted, the agent is able to withhold a part of the true endowment. The component y_i denotes the proposed level of public goods that agent i is willing to contribute (a negative y_i means the agent wants to receive a subsidy from the society). The component η_i is the penalty index of agent i when the reported type $\hat{t}$ is not in J.

We now turn to the construction of the outcome functions of the mechanism in the following steps. First, for each agent i, define the personalized price function $\bar{P}_i \colon T \to R^K$,

$$\bar{P}_i(\cdot) = \frac{1}{n}\beta + q_{i+1}(\cdot) - q_{i+2}(\cdot), \tag{3}$$

where $n+1$ and $n+2$ are to be read as 1 and 2, respectively. Observe that, by construction, $\sum_{i=1}^n \bar{P}_i(t) = \beta$ for all $t \in T$, and each agent's personalized price function is independent of his own reported pricing function message.

Next, construct n-profiles in J as follows. For each reported type $\hat{t}$, define the ith agent's profile, $(\tilde{t}_1^i, \ldots, \tilde{t}_n^i)$, by

$$\tilde{t}_{-i}^i = \{t_{-i} \in \pi_{-i}(J) \colon \min \left\| t_{-i} - \hat{t}_{-i} \right\|\} \tag{4}$$

which is the closest point to $\hat{t}_{-i}$. Note that $\tilde{t}_{-i}^i$ is a single-valued and continuous function of $\hat{t}_{-i}$.[7] Then define $\tilde{t}_i^i = s_i(\tilde{t}_{-i}^i)$. Thus the profile $\tilde{t}^i = (\tilde{t}_1^i, \ldots, \tilde{t}_n^i)$ is in J for all $i \in N$. Note that all the n-profiles are identical and equal to $\hat{t}$ if $\hat{t}$ is in J, and further, a change in the strategy of agent i will not change the ith profile $\tilde{t}^i$.

Now we have n personalized price functions $\bar{P}_i$ and n profiles $\tilde{t}_i$. At this point, each individual i is informed that his personalized price vector is given by $\bar{P}_i(\tilde{t}^i)$.

Define a correspondence $B \colon M \to 2^{\mathbb{R}_+^K}$ by

$$B(m) = \{y \in \mathbb{R}_+^K \colon (1-\delta)w_i - \bar{P}_i(\tilde{t}^i)\cdot y \geq 0 \, \forall i \in N \text{ and } \sum_{i=1}^n \bar{P}_i(\tilde{t}^i)\cdot y \geq \beta \cdot y\}, \tag{5}$$

which is clearly a continuous correspondence with nonempty compact convex values.[8] Here $\delta = \min\{\delta_i, \ldots, \delta_n\}$.

Next, define the outcome function for public goods $Y \colon M \to B$ by

$$Y(m) = \{y \colon \min_{y \in B(m)} \|y - \tilde{y}\|\}, \tag{6}$$

which is the closest point to $\tilde{y}$. Here $\tilde{y} = \sum_{i=1}^n y_i$. Then $Y(m)$ is single-valued and continuous on M. For each individual i, define the taxing function $H_i \colon M \to \mathbb{R}$ by

$$H_i(m) = \bar{P}_i(\tilde{t}^i) \cdot Y(m). \tag{7}$$

[7] This is because $\tilde{t}_{-i}^i \colon T_{-i} \to \pi_{-i}(j)$ is an upper semicontinuous correspondence by Berge's maximum theorem (see Debreu, 1959, p. 19) and single-valued (see Mas-Colell, 1985, p. 28).

[8] $B(m)$ is shown to be compact by noting that $\beta > 0$ and $\beta \cdot y \leq \sum_{i=1}^n \bar{P}_i(\tilde{t}^i) \cdot y \leq \sum_{i=1}^n w_i$ for all $y \in B(m)$. Note that, in general, $\sum_{i=1}^n \bar{P}_i(\tilde{t}^i) \neq \beta$. However, when $t \in J$, $\tilde{t}^i = t$ for all $i \in N$ and thus $\sum_{i=1}^n \bar{P}_i(\tilde{t}^i) = \sum_{i=1}^n \bar{P}_i(\hat{t}) = \beta$.

Then, by (5),

$$\sum_{i=1}^{n} H_i(m) \geq \beta \cdot Y(m). \tag{8}$$

The outcome function $X(m)$: $M \to \mathbb{R}_+$ is given by

$$X_i(m) = \frac{1}{1 + \eta_i \left\| \hat{t} - \pi_J(\hat{t}) \right\|}[w_i - \bar{P}_i(\tilde{t}^i) \cdot Y(m)]. \tag{9}$$

where $\pi_J(\hat{t})$ is the projection of $\hat{t}$ onto J. Note that $X_i(m) > 0$ by the definition of the constrained set $B(m)$ and the total (final) consumption of agent i for the private good is the sum of $X_i(m)$ and $(\overset{\circ}{w}_i - w_i)$. That is, it is the sum of the amount of private good allocated by the mechanism and the unreported amount of his own endowment. Thus the outcome function of agent i, which is given by

$$g_i(m) = (X_i(m) + \overset{\circ}{w}_i - w_i, Y(m)),$$

is continuous on M. Also, $g_i(m) \in \mathbb{R}_+^{1+K}$, and from Eqs. (8) and (9), we have

$$\sum_{i=1}^{n} [X_i(m) + \overset{\circ}{w}_i - w_i] + \beta \cdot Y(m) \leq \sum_{i=1}^{n} \overset{\circ}{w}_i \tag{10}$$

for all $m \in M$. Therefore, the mechanism is feasible.

From (10), we have

$$\sum_{i=1}^{n} X_i(m) + \beta \cdot Y(m) \leq \sum_{i=1}^{n} w_i, \tag{11}$$

which means the sum of the aggregate consumption of the private good and consumption of public goods allocated by the mechanism are not greater than the aggregate of endowments reported by agents for all $m \in M$.

4. MAIN RESULTS

In the remainder of this paper we prove the global Bayesian implementability of the rational-expectations Lindahl correspondence, which is stated in Theorem 1. The proof consists of two propositions on equivalence between Bayesian equilibrium allocations and rational-expectations Lindahl allocations. Proposition 1 proves that every Bayesian equilibrium allocation is a rational-expectations Lindahl allocation. Proposition 2 proves that every rational-expectations Lindahl allocation is a Bayesian allocation.

THEOREM 1. *For the class of economic environments E with one private and K public goods, if the following assumptions are satisfied*:

(1) $n \geq 3$;
(2) u_i *is strictly increasing in* x_i, *strictly concave in* (x_i, y), *measurable in* t, *and satisfies indispensability of private good*;
(3) J *is compact and convex, and satisfies NEI*,

then there exists a continuous and feasible mechanism which globally Bayesian implements the rational-expectations Lindahl correspondence on E.

The proof of Theorem 1 consists of the following propositions. All we need to show is that the set of Bayesian equilibrium allocations coincides precisely with the set of rational-expectations Lindahl allocations. That is, $B_{\langle M,g\rangle}(e) = \mathrm{REL}(e)$ for all $e \in E$, satisfying Conditions 1–3 in Theorem 1.

PROPOSITION 1. *Under Conditions* 1–3 *of Theorem* 1, *if* (x^*, y^*) *is a rational-expectations Lindahl allocation with the Lindahl price vector function* $P^* = (P_1^*, \ldots, P_n^*)$, *there is a Bayesian equilibrium* m^* *for the mechanism defined above such that* $g_i(m^*(t)) = (x_i^*(t), y^*(t))$ *and* $\bar{P}_i^*(t) = P_i^*(t)$ *for all* $i \in N$ *and* $t \in T$.

Proof. We first note that $x^* \in \mathbb{R}^n_{++}$ by the assumption of indispensability of private good. We need to show that there is a Bayesian equilibrium message m^* such that (x^*, y^*) is a Bayesian equilibrium allocation. Let $q_1^* = 0$, $q_2^* = (1/n)\beta - P_n^*$, $q_i^* = (1/n)\beta + q_{i-1}^* - P_{i-2}^*$ for $i = 3, \ldots, n$. Let $\hat{t}_i^* = t_i$. Let $w_i^* = \mathring{w}_i$. Let δ_i^* be sufficiently small so that $(1 - \delta(m^*))\mathring{w}_i - P_i^* \cdot y^* > 0$. Let $y_i^* = (1/n)y^*$ and let $\eta_i^* = 1$. Then it can be easily verified that $\bar{P}_i^* = P_i^*$, $Y(m^*) = y^*$, and $X_i(m^*) = x_i^*$, for all $i \in N$.

Now we must show that the strategy profile m^* forms a Bayesian equilibrium. Note that each agent i's personalized price function and profile $(\tilde{t}_1^i, \ldots, \tilde{t}_n^i)$ are independent of his own messages. The person cannot change his personalized prices by changing his messages. Since $\hat{t} = t \in J$, we have $\bar{P}_i^*(\tilde{t}^i) = \bar{P}_i^*(t)$ for all $i \in N$. Thus, the personalized price vector of each individual i is the true personalized price vector, $P_i^*(t)$. Thus, $(X_i(m_i, m*_{-i}) + \mathring{w}_i - w_i, Y(m_i, m_{-i}^*)) \in \mathbb{R}_+^{1+K}$ and $[X_i(m_i(t), m_{-i}^*(t)) + \mathring{w}_i - w_i] + P_i^*(t) \cdot Y(m_i(t), m_{-i}^*(t)) \leq \mathring{w}_i$ for all $i \in N$, $m_i \in M_i$, and for all $t \in T$. Note that, by the timing of mechanism, $P_i^*(t)$ is revealed to individual i before choices of $(\delta_i, w_i, y_i, \eta_i)$ are taken and thus $P_i^*(t)$ is independent of the choice of these messages. Therefore, by the definition of rational-expectations Lindahl allocation, we know that, when $t_i \in T_i$ and price signal P_i^* is observed, $(X_i(m^*), Y(m^*)) = (x_i^*, y^*)$ gives agent i the most preferred point for all $(\delta_i, w_i, y_i, \eta_i)$. Thus $(X_i(m^*), Y(m^*))$ forms a Bayesian equilibrium allocation. ∎

PROPOSITION 2. *Under Conditions* 1–3 *of Theorem* 1, *if the mechanism defined above has a Bayesian equilibrium* m^*, *the Bayesian equilibrium allocation* $(X(m^*) + \overset{\circ}{w} - w^*, Y(m^*))$ *is a rational-expectations Lindahl allocation with* $(\bar{P}_1^*(\hat{t}), \ldots, \bar{P}_n^*(\hat{t}))$ *as the Lindahl price vector function.*

Proof. Let m^* be a Bayesian equilibrium. We first note that $\hat{t}^*$ with $\hat{t}_i^*\colon T_i \to T_i$ must be in J so that $(\tilde{t}_1^{*i}, \ldots, \tilde{t}_{*n}^{i}) = \hat{t}^*$ for all $i \in N$; otherwise, agent i can choose a smaller $\eta_i < \eta_i^*$ in $(0, 1]$ so that his consumption of the private good becomes larger. Hence, no choice of η_i could constitute part of a Bayesian equilibrium strategy. Let $P_i^*(t) = \bar{P}_i^*(\hat{t}^i(t))$. Note that, by the imposed timing of messages, these price vectors of functions $P^*({}_it)$ are known by the individuals before choices of $(\delta_i, w_i, y_i, \eta_i)$ are taken and thus $P_i^*(t)$ are independent of the choice of $(\delta_i, w_i, y_i, \eta_i)$.

Now we prove that $(X(m^*) + \overset{\circ}{w} - w^*, Y(m^*))$ is a rational-expectations Lindahl allocation with $(P_1^*(t), \ldots, P_n^*(t))$ as the Lindahl price function. Since the mechanism is feasible, $\sum_{i=1}^n \bar{P}_i^* = \beta$, and $[X_i(m^*(t)) + \overset{\circ}{w}_i - w_i^*] + P_i^*(t) \cdot Y(m^*(t)) = \overset{\circ}{w}_i$ for all $i \in N$, we only need to show that each individual is maximizing expected utility. Suppose, by way of contradiction, that there is some $(x_i, y) \in \mathbb{R}_+^{1+K}$ such that

$$\begin{aligned}&\int_{E_i(t_i, P_i^*(t); P_i^*)} u_i(x_i, y, t) dG_i(t_{-i} \mid t_i, P_i^*(t), P_i^*)\\ &\qquad > \int_{E_i(t_i, P_i^*(t); P_i^*)} u_i(g_i(m^*), t) dG_i(t_{-i} \mid t_i, P_i^*(t), P_i^*)\end{aligned}$$

and $x_i + P_i^*(t) \cdot y \leq \overset{\circ}{w}_i$ for some $t \in T$. Because of monotonicity of the utility function, it will be enough to confine ourselves to the case of $x_i + q_i^*(t) \cdot y = \overset{\circ}{w}_i$. Let

$$\begin{aligned}x_{\lambda i} &= \lambda x_i + (1 - \lambda)[X_i(m^*) + \overset{\circ}{w}_i - w_i^*]\\ y_\lambda &= \lambda y + (1 - \lambda) Y(m^*).\end{aligned}$$

Then by strict concavity of the utility function we have

$$\begin{aligned}&\int_{E_i(t_i, P_i^*(t); P_i^*)} u_i(x_{\lambda i}, y_\lambda, t) G_i(t_{-i} \mid t_i, P_i^*(t), P_I^*)\\ &\qquad > \int_{E_i(t_i, P_i^*(t); P_i^*)} u_i(g_i(m^*), t) G_i(t_{-i} \mid t_i, P_i^*(t), P_i^*)\end{aligned}$$

for any $0 < \lambda < 1$. Also, $(x_{\lambda i}, y_\lambda) \in \mathbb{R}_+^{1+K}$ and $x_{\lambda i} + P_i^*(t) \cdot y_\lambda = \overset{\circ}{w}_i$. Now suppose that player i chooses δ_i so that $\delta_i < \delta^*$, $y_i = y_\lambda - \sum_{j \neq i}^n y_j^*$, and it keeps $\hat{t}_i^*$, w_i^*, η_i^*, and q_i^* unchanged. Then $\delta = \delta_i < \delta^*$ and thus $(1 - \delta) w_j^* - P_j^*(t)$.

$Y(m^*)(t) > (1-\delta(m^*))w_j^* - P_j^*(t)\cdot Y(m^*)(t) \geq 0$ for all $j \in N$ by the construction of the mechanism. Thus, we have $(1-\delta)w_j^* - P_j^*(t)\cdot y_\lambda > 0$ for all $j \in N$ as λ is sufficiently small. Hence $y_\lambda \in B(m_i, m_{-i}^*)$, and therefore, $Y(m_i, m_{-i}^*) = y_\lambda$ and $X_i(m_i(t), m_{-i}^*(t)) = w_i^* - P_i^*(t)\cdot Y(m_i(t), m_{-i}^*(t)) = w_i^* - P_i^*(t)\cdot y_\lambda$. Then, $X_i(m_i, m_{-i}^*) + \overset{\circ}{w}_i - w_i^* = x_{i\lambda}$. From

$$\int_{E_i(t_i, P_i^*(t); P_i^*)} u_i(x_{\lambda i}, y_\lambda, t)dG_i(t_{-i} \mid t_i, P_i^*(t), P_I^*)$$
$$> \int_{E_i(t_i, P_i^*(t); P_i^*)} u_i(g_i(m^*), t)dG_i(t_{-i} \mid t_i, P_i^*(t), P_i^*),$$

we have

$$\int_{E_i(t_i, P_i^*(t); P_i^*)} u_i(g_i(m_i^*, m_{-i}), t)dG_i(t_{-i} \mid t_i, P_i^*(t), P_i^*)$$
$$> \int_{E_i(t_i, P_i^*(t); P_i^*)} u_i(g_i(m^*), t)dG_i(t_{-i} \mid t_i, P_i^*(t), P_i^*).$$

This contradicts the hypothesis that $(X(m^*) + \overset{\circ}{w} - w^*, Y(m^*))$ is a Bayesian equilibrium allocation. ■

From Propositions 1 and 2, we know that $B_{\langle M,g\rangle}(e) = \mathrm{REL}(e)$ and thus the proof of Theorem 1 is completed.

In summary, we conclude that, for one private and K public goods economies E satisfying assumptions imposed in the above theorem, there exists a feasible and continuous mechanism which globally Bayesian implements the rational-expectations Lindahl allocations.

5. CONCLUDING REMARKS

In this paper, we have presented an appealing mechanism which globally Bayesian implements the rational-expectations Lindahl allocations for economic environments with public goods when agents are incompletely informed about environments. We not only allow the types of individuals to be unknown to the designer, we also allow both the preferences and initial endowments to be unknown, and have unreported endowments, which are withheld. The constructed mechanism is well behaved in the sense that it is feasible and continuous. Though this paper only considers Bayesian implementation of rational-expectations Lindahl allocations for public economies with one private good, the mechanism presented here can be modified to implement rational-expectations Lindahl allocations for public goods economies with any number of goods. Such extensions are similar to those given in Tian and Li (1995b) which Nash implement Lindahl allocations for economies with any number of public goods. Also, some of the techniques developed in this paper can be used to generalized some other social choice rules such as (generalized) ratio allocations, linear cost share, and Lindahl-ratio allocations to rational expectations versions; and also to consider Bayesian implementation of these rational-expectations social-choice correspondences by modifying Nash implementation results of Tian (1994a) and Tian and Li (1994, 1995a).

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