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SYMPOSIUM

Detrimental externalities, pollution rights, and the "Coase theorem"

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Abstract This paper, which builds on Chipman (The economist's vision. Essays in modern economic perspectives, 131–162, 1998), analyzes a simple model formulated by Hurwicz (Jpn World Econ 7:49–74, 1995) of two agents—a polluter and a pollutee—and two commodities: "money" (standing for an exchangeable private good desired by both agents) and "pollution" (a public commodity desired by the polluter but undesired by the pollutee). There is also a government that issues legal rights to the two agents to emit a certain amount of pollution, which can be bought and sold with money. It is assumed that both agents act as price-takers in the market for pollution rights, so that competitive equilibrium is possible. The "Coase theorem" (so-called by Stigler (The theory of price, 1966) asserts that the equilibrium amount of pollution is independent of the allocation of pollution rights. A sufficient condition for this was (in another context) obtained by Edgeworth (Giorn Econ 2:233–245, 1891), namely that preferences of the two agents be "parallel" in the money commodity, whose marginal utility is constant. Hurwicz (Jpn World Econ 7:49–74, 1995) argued that this parallelism is also necessary. This paper, which provides an exposition of the problem, raises some questions about this result and provides an alternative necessary and sufficient condition.

This paper is dedicated to the memory of our esteemed respective former colleague and former thesis advisor Leonid Hurwicz. We greatly regret not having been able to discuss the final section with him before his death. Thanks are due to Augustine Mok for his help with the diagrams.

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1 Introduction

This paper reconsiders the question of the conditions for the validity of the "Coase theorem" (Coase 1960, IV, p. 6; Stigler 1966, p. 113) according to which the equilibrium amount of pollution is independent of the assignment of legal rights as between a polluter and the pollutee. Hurwicz (1995) analyzed this problem as a two-person, two-commodity exchange equilibrium between a polluter and a pollutee exchanging "money" and pollution, and characterized the Coasian solution as one in which the set of Pareto optima in the Edgeworth box exhibits a constant level of pollution for positive money holdings of both parties, as depicted in Fig. 2 below. He recognized that a sufficient condition for this outcome is that preferences be "parallel" with respect to the x-commodity (money in this case)—a result that in fact goes back to Edgeworth (1891)—and endeavored to show that this condition is also necessary for the result. In this paper, we show that this result is incorrect, but that a weaker (yet still restrictive) necessary and sufficient condition leads to the Coase result.

In recent years, a very interesting literature has developed in which these concepts have been applied to *countries* as opposed to individuals, with "pollution" taking the form of emission of carbon dioxide and other greenhouse gases; cf. Chichilnisky and Heal (1994, 2000a), Sheeran (2006) and Chichilnisky et al. (2000), also (in this issue) Chichilnisky (2011), Asheim et al. (2011). An important question is then how to implement policies in the context of such detrimental externatities, as discussed in Burniaux and Martins (2011), Dutta and Radner (2011), Figuières and Tidball (2011), Karp and Zhang (2011), Lauwers (2011), Lecocq and Hourcade (2011), Ostrom (2011) and Rezai et al. (2011). One solution (adopted in this paper) is to follow the Coasian approach of clearly defining property rights. However, the problem of climate change differs somewhat from the Coasian one in that CO_2 is caused by breathing on the part of humans, cattle, and other animals, and only beyond a certain level (which may certainly be claimed to have been reached) by fuel combustion; but also in that it requires assumptions on preferences needed to aggregate individuals to countries (for such aggregation conditions see, e.g., Chipman 2006). It turns out that some of these assumptions are consistent with but others are incompatible with those needed to justify the Coase theorem; hence in order to avoid confusion, it is safer to conduct the exposition in terms of the two-individual model. This path will be followed here.

We shall suppose that there are two individuals: individual 1 who likes to engage in an activity (e.g., smoking and blowing leaves) that is annoying to individual 2 because it produces a "detrimental externality" (smoke and noise) which may be characterized as pollution. Individual 1 will be called the polluter and individual 2 the pollutee. The cost of the activity to the polluter (e.g., purchase of cigarettes, fuel for the leaf-blower and time taken to engage in the polluting activity) will be disregarded. This externality may be "internalized" by the introduction of pollution rights or permits which can be traded. Suppose that there is a maximum amount of pollution that could be produced by individual 1 per period of time, indicated by η, and let s denote the actual amount of pollution (smoke, or noise) produced during this period of time. Then of course

$$0 \leqq s \leqq \eta. \tag{1.1}$$

Other things being equal, individual 1 will wish to increase s and individual 2 will wish to reduce it. Pollution is a public commodity in the sense of Samuelson (1954, 1955, 1969)—a public good for individual 1 and a public bad for individual 2.

Suppose a system is developed whereby a quantity η of pollution rights (permits) is made available by the government, initially allocated between the two individuals, according to

$$\eta_1 + \eta_2 = \eta, \tag{1.2}$$

where η_i is the initial allocation of pollution rights to individual i. Then, individual 1 has the legal right (which we assume will be exercised) to emit η_1 units of pollution, while individual 2 has the legal right to emit $\eta_2 = \eta - \eta_1$ units of pollution, which (since we assume it will *not* be exercised) is equivalent to a right to η_2 units of *pollution avoidance*. Suppose further that the two individuals start out with amounts ξ_i of another good which may be called "money",

$$\xi_1 + \xi_2 = \xi. \tag{1.3}$$

Letting p denote the price of a pollution right in terms of money, letting y_i denote the amount of pollution rights held by individual i, and letting x_i denote the amount of money individual i has left over after purchasing or selling pollution rights, individual i is constrained by the budget inequality

$$x_i + py_i \leqq \xi_i + p\eta_i \quad (i = 1, 2). \tag{1.4}$$

Now, since it may be assumed that each individual will desire a larger final holding x_i of money than less, and for the reasons given above will also want to end up with a larger holding y_i of pollution *rights* than less, (1.4) will be an equality

$$x_i + py_i = \xi_i + p\eta_i \quad (i = 1, 2). \tag{1.5}$$

Since this equality is valid for all prices, p, summing (1.5) over the two individuals, we see from (1.2) and (1.3) that[1]

$$x_1 + x_2 = \xi; \quad y_1 + y_2 = \eta. \tag{1.6}$$

[1] This formulation differs from that of Hurwicz (1995) (which it otherwise follows closely), who considers pollution rights $z_1 \equiv y_1$ and $z_2 \equiv \eta - y_2$, which must satisfy $z_1 = z_2$, analogously to the theory of public goods. This corresponds to the second equation of (1.6), since $z_1 = y_1 = \eta - y_2 = z_2$. Thus the difference is largely one of notation. The present formulation, which provides a notation for individual i's final holdings of money and of pollution rights (x_i, y_i), makes it somewhat easier to interpret the diagrams in Figs. 1 and 2 below.

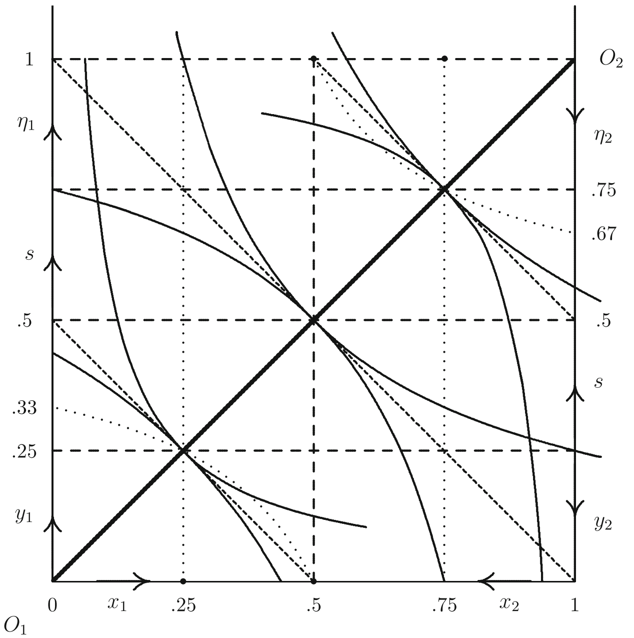

Fig. 1 Edgeworth box in the case of homothetic preferences

We finally must relate the pollution rights (which are just pieces of paper) to the pollution itself. It would not be beneficial for individual 1 (the polluter) to hold onto y_1 pollution rights unless he or she intended to exercise them, i.e., to produce an equal amount, s, of pollution. Consequently, we may assume that $y_1 = s$. Likewise, in order to limit him or herself to an amount s of pollution, individual 2 (the pollutee) will need to limit individual 1′s pollution rights to $y_1 = s$ units and will therefore need to obtain possession of the remaining $y_2 = \eta - y_1 = \eta - s$ rights.[2] Thus, we have

$$y_1 = s \quad \text{and} \quad y_2 = \eta - s. \tag{1.7}$$

Hence the same piece of paper which gives individual 1 the right to emit 1 unit of pollution, if transferred to individual 2, gives individual 2 the right to 1 unit of *pollution avoidance*.

Now, the preferences of the polluter and the pollutee may be represented by (differentiable) utility functions $U_1(x_1, s)$ and $U_2(x_2, s)$, respectively, where $\partial U_i/\partial x_i > 0$ and $\partial U_1/\partial s > 0$ but $\partial U_2/\partial s < 0$ for $(x_i, s) \in (0, \xi) \times (0, \eta)$, $i = 1, 2$.

[2] This shows that the validity of the analysis in this paper is limited to the case of a single pollutee. The introduction of a second pollutee at once introduces a "free-rider" problem.

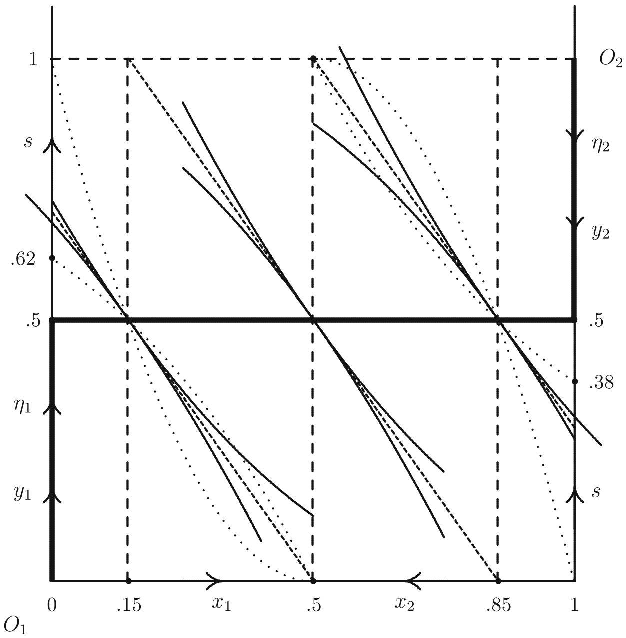

Fig. 2 Edgeworth box in the case of parallel preferences

Then, the necessary first-order interior (tangency) condition for Pareto-optimality takes the form (as in the Lindahl-Samuelson public-goods condition)[3]

$$\frac{\partial U_1}{\partial s} \Big/ \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial s} \Big/ \frac{\partial U_2}{\partial x_2} = 0, \tag{1.8}$$

where $\frac{\partial U_i}{\partial s} \Big/ \frac{\partial U_i}{\partial x_i}$ is agent i's marginal rate of substitution of s for x.

2 Homothetic preferences

Let the utility functions of the two individuals be given by

$$\begin{aligned} U_1(x_1, s) &= x_1 s, \\ U_2(x_2, s) &= x_2(\eta - s). \end{aligned} \tag{2.1}$$

For a given amount x_1 of money, individual 1 will have maximum utility when $s = \eta$ and minimum (zero) utility when $s = 0$. Likewise, for a given amount x_2 of money,

[3] Note that the first-order necessary (Lindahl-Samuelson) condition for the Pareto-optimality for public-goods economies is given by $\frac{\partial U_1}{\partial s} / \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial s} / \frac{\partial U_2}{\partial x_2} = g'(s)$, where $g(s)$ is the x-input requirement for producing s units of the public good s. But in the pollution model there is no x-input requirement to produce s, so $g(s) = 0$ identically. Hence the two marginal rates of substitution add up to zero.

individual 2 will have maximum utility when $s = 0$ and minimum (zero) utility when $s = \eta$. Because of the relations (1.7), the utility functions (2.1) may be expressed in terms of the pollution rights instead of the pollution itself:

$$\begin{aligned} U_1(x_1, y_1) &= x_1 y_1, \\ U_2(x_2, y_2) &= x_2 y_2. \end{aligned} \tag{2.2}$$

Both functions are strictly quasi-concave and increasing in both arguments. We derive the two individuals' demand functions for x_i and y_i as functions of the price, p, of the permits.

Individual i's objective is to maximize $U_i(x_i, y_i) = x_i y_i$ subject to (1.4). Equating the marginal rate of substitution to the price of a right, we have

$$\frac{\partial U_i/\partial y_i}{\partial U_i/\partial x_i} = \frac{x_i}{y_i} = p, \quad \text{hence } x_i = p y_i \quad (i = 1, 2). \tag{2.3}$$

Since (from (1.5)) equality must hold in (1.4), substituting (2.3) into this equality, we obtain

$$\begin{aligned} 2x_i &= \xi_i + p\eta_i \ \text{ hence } \ x_i = \tfrac{\xi_i}{2} + \tfrac{\eta_i}{2} p; \\ 2py_i &= \xi_i + p\eta_i \ \text{ hence } \ y_i = \tfrac{\xi_i}{2p} + \tfrac{\eta_i}{2} \end{aligned} \quad (i = 1, 2). \tag{2.4}$$

Summing the two individuals' demands for money and for pollution rights from (2.4) and using the facts (from (1.6), (1.3), and (1.2)) that $x_1 + x_2 = \xi_1 + \xi_2 = \xi$ and $y_1 + y_2 = \eta_1 + \eta_2 = \eta$, we have

$$\xi = x_1 + x_2 = \frac{\xi}{2} + \frac{\eta}{2} p, \quad \eta = y_1 + y_2 = \frac{\xi}{2p} + \frac{\eta}{2},$$

from either of which it follows that

$$p = \frac{\xi}{\eta}. \tag{2.5}$$

Evaluating the demand functions (2.4) at the equilibrium price (2.5) we obtain

$$x_i = \frac{\xi_i}{2} + \frac{\eta_i}{2}\frac{\xi}{\eta} \quad \text{and} \quad y_i = \frac{\xi_i}{2}\frac{\eta}{\xi} + \frac{\eta_i}{2} \quad (i = 1, 2). \tag{2.6}$$

This is the desired competitive equilibrium.

Now, let us consider two cases, in both of which $\xi = \eta = 1$ and $\xi_1 = \xi_2 = \frac{1}{2}$. Then from (2.5), $p = 1$. In Case (i), the polluter (individual 1) starts out with initial holdings $(\xi_1, \eta_1) = \left(\frac{1}{2}, 0\right)$, i.e., with half the money and no pollution rights, and ends up with final holdings $(x_1, y_1) = \left(\frac{1}{4}, \frac{1}{4}\right)$, i.e., with one quarter of the money (half of his initial amount, the other half of which is used to purchase pollution rights from the pollutee) and the right to pollute only one quarter of the time. On the other hand, in Case (ii), the polluter starts out with $(\xi_1, \eta_1) = \left(\frac{1}{2}, 1\right)$, i.e., with half the money and all the pollution rights, and ends up with final holdings $(x_1, y_1) = \left(\frac{3}{4}, \frac{3}{4}\right)$, i.e., three quarters of the money (one quarter of which is obtained from selling pollution rights to the pollutee) and the right to pollute three quarters of the time. This violates the "Coase theorem" according to which the assignment of rights does not affect the amount of pollution.

The situation is depicted in Fig. 1, a slightly modified Edgeworth box in which individual 1′s initial (ξ_1, η_1) and final (x_1, y_1) holdings of money and pollution rights are measured rightward and upward from the southwest origin $O_1 = (0, 0)$, while individual 2′s initial (ξ_2, η_2) and final (x_2, y_2) holdings of money and pollution rights are measured leftward and downward from the northeast origin $O_2 = (1, 1)$. For both individuals, the amount of pollution itself (desired by individual 1 and undesired by individual 2) is measured upward from the bottom of the box. All the numbers shown in the figure are measured from O_1. With the assumed initial values $\xi = \eta = 1, \xi_1 = \xi_2 = \frac{1}{2}$, and $\eta_1 + \eta_2 = \eta$, we consider Case (i) in which the initial holdings of money and pollution rights are $(\xi_1, \eta_1) = \left(\frac{1}{2}, 0\right)$ and $(\xi_2, \eta_2) = \left(\frac{1}{2}, 1\right)$ [shown by the point (0.5, 0) in the box, measured from O_1] and the final equilibrium holdings of money and pollution rights are $(x_1, y_1) = \left(\frac{1}{4}, \frac{1}{4}\right)$ and $(x_2, y_2) = \left(\frac{3}{4}, \frac{3}{4}\right)$ [shown by the point (0.25, 0.25) in the box, measured from O_1]; and Case (ii) in which the initial holdings are $(\xi_1, \eta_1) = \left(\frac{1}{2}, 1\right)$ and $(\xi_2, \eta_2) = \left(\frac{1}{2}, 0\right)$ [shown by the point (0.5,1) in the box, measured from O_1] and the final (equilibrium) holdings are $(x_1, y_1) = \left(\frac{3}{4}, \frac{3}{4}\right)$ and $(x_2, y_2) = \left(\frac{1}{4}, \frac{1}{4}\right)$ [shown by the point (0.75,0.75) in the box, measured from O_1]. The tangential indifference curves of the two individuals are displayed at these two points, with the budget lines (shown as short-dashed lines) going through the points $(x_1, y_1) = \left(\frac{1}{2}, 0\right)$ and $\left(\frac{1}{4}, \frac{1}{4}\right)$ in Case (i) and through the points $(x_1, y_1) = \left(\frac{1}{2}, 1\right)$ and $\left(\frac{3}{4}, \frac{3}{4}\right)$ (measured from O_1) in Case (ii). (In both cases, $x_1 + x_2 = y_1 + y_2 = 1$.)

Figure 1 also displays (as dotted curves) the two individuals' offer curves. These are obtained from the two demand Eq. (2.4) for individual 1 with $\xi_1 = \frac{1}{2}$ and (in Case (i)) with $\eta_1 = 0$ to obtain

$$x_1 = \frac{1}{4}, \quad y_1 = \frac{1}{4p} \quad \text{(independently of } x_1\text{)}. \tag{2.7}$$

This is shown in the southwest part of Fig. 1 as the vertical straight line at $x_1 = \frac{1}{4}$ for individual 1. For individual 2, we eliminate the price variable from (2.4) to obtain (again in Case (i) for $\eta_1 = 0$ and $\eta_2 = 1$)

$$y_2 = \frac{1}{2} + \frac{1}{8x_2 - 2}. \tag{2.8}$$

This is shown in the southwest part of Fig. 1 as the dotted curve starting at the initial-endowment point $(\xi_2, \eta_2) = (0.5, 1)$ (measured from O_2 and corresponding to $(\xi - \xi_2, \eta - \eta_2) = (0.5, 0)$ measured from O_1), going through the equilibrium point $(x_2, y_2) = (0.75, 0.75)$ (measured from O_2 and corresponding to the equilibrium point $(1 - x_2, 1 - y_2) = (0.25, 0.25)$ measured from O_1) and ending at the point $(x_2, y_2) =$

$\left(1, \frac{2}{3}\right)$ (measured from O_2 and corresponding to the point $(1 - x_2, 1 - y_1) = \left(0, \frac{1}{3}\right)$ measured from O_1).

In Case (ii), with $\eta_1 = 1 (\eta_2 = 0)$, we eliminate p from the demand Eq. (2.4) to obtain, for individual 1,

$$y_1 = \frac{1}{2} + \frac{1}{8x_1 - 1}, \tag{2.9}$$

a curve which is shown in the northeast part of Fig. 1 starting at $(\xi_1, \eta_1) = \left(\frac{1}{2}, 1\right)$ and going through the equilibrium point $(x_1, y_1) = \left(\frac{3}{4}, \frac{3}{4}\right)$ and ending at the point $(x_1, y_1) = \left(1, \frac{2}{3}\right)$. For individual 2, from the Eq. (2.4) for the case $\xi_2 = \frac{1}{2}$ and $\eta_2 = 0$, we obtain

$$x_2 = \frac{1}{4}, \quad y_2 = \frac{1}{4p} \quad \text{(independently of } x_2\text{)}, \tag{2.10}$$

showing that individual $2'$s offer curve is the vertical straight line $x_2 = \frac{1}{4}$ corresponding to $1 - x_2 = \frac{3}{4}$ in the diagram (measured from O_1).

This being a case of pure exchange with identical homothetic preferences, the set of Pareto optima (the "contract curve") is the dark diagonal of the box. The equilibrium amount of pollution is $y = \frac{1}{4}$ when the polluter starts out with no pollution rights, compared with $y = \frac{3}{4}$ when the polluter starts out with all the pollution rights, in contradiction to the "Coase theorem". Since the assumption of identical homothetic preferences is the leading condition making possible the aggregation of individuals into groups (cf. Chipman 1974), this raises problems with the application of Coasian economics to groups of individuals.

A third point $(x_1, y_1) = (0.5, 0.5)$ is shown in the diagram; this corresponds to the case in which not only the initial holdings of money are the same ($\xi_1 = \xi_2 = \frac{1}{2}$) but also the initial holdings of pollution rights are the same ($\eta_1 = \eta_2 = \frac{1}{2}$). This point is also Pareto-optimal, and no trading of pollution rights is needed to attain it.

3 Parallel preferences

Now let the utility functions of the two individuals, expressed in terms of money and pollution, be given by

$$\begin{aligned} U_1(x_1, s) &= x_1 + \sqrt{s}, \\ U_2(x_2, s) &= x_2 + \sqrt{\eta - s}. \end{aligned} \tag{3.1}$$

As before, for a given amount x_1 of money, individual $1'$s utility is maximized when $s = \eta$ and minimized when $s = 0$, whereas for a given amount x_2 of money, individual $2'$s utility is maximized when $s = 0$ and minimized when $s = \eta$. Substituting (1.7), the utility functions (3.1) may be expressed in terms of money and pollution rights:

$$
\begin{aligned}
U_1(x_1, y_1) &= x_1 + \sqrt{y_1},\\
U_2(x_2, y_2) &= x_2 + \sqrt{y_2}.
\end{aligned}
\tag{3.2}
$$

This is a case of identical "parallel" preferences.[4]

We will examine the above two cases (i) and (ii) with these utility functions in place of the utility functions (2.1).

With the utility functions (3.2), (2.3) is replaced by

$$
\frac{\partial U_i/\partial y_i}{\partial U_i/\partial x_i} = \frac{1}{2\sqrt{y_i}} = p \quad \text{hence } y_i = \frac{1}{4p^2} \quad (i = 1, 2). \tag{3.3}
$$

Substituting (3.3) in the budget constraint (1.4) (with equality), we obtain the two demand functions for individual i:

$$
x_i = \xi_i + \eta_i p - \frac{1}{4p} \quad \text{and} \quad y_i = \frac{1}{4p^2}. \tag{3.4}
$$

Thus, each individual's demand y_i for pollution rights is independent of his or her initial money holdings ξ_i or holdings of pollution rights η_i and of the budget constraint (so long as it is consistent with the budget constraint).

Now setting $x_1 + x_2 = \xi$ and $y_1 + y_2 = \eta$, we have from (3.4)

$$
\xi = x_1 + x_2 = \xi + \eta p - \frac{1}{2p}, \quad \eta = y_1 + y_2 = \frac{1}{2p^2}
$$

from both of which, we conclude that

$$
p = \frac{1}{\sqrt{2\eta}}. \tag{3.5}
$$

Substituting this price in the demand functions (3.4), we obtain

$$
x_i = \xi_i + \frac{\eta_i}{\sqrt{2\eta}} - \frac{\sqrt{2\eta}}{4} \quad \text{and} \quad y_i = \frac{\eta}{2} \quad (i = 1, 2). \tag{3.6}
$$

This is the desired competitive equilibrium.

Now let us look as before at the special case $\xi = \eta = 1$ and $\xi_1 = \xi_2 = \frac{1}{2}$ and consider two cases (see Fig. 2). When the utility functions are as in (3.1), in Case (i),

[4] A parallel preference ordering is one that is representable by a *quasi-linear* utility function $U(x, y) = \nu x + \phi(y)$ for $\nu > 0$ and $\phi'(y) > 0$ (cf. Hurwicz 1995, p. 55n). The term "parallel" was introduced by Boulding (1945) and followed by Samuelson (1964), though the concept goes back to Auspitz and Lieben (1889, Appendix II, Sect. 2, pp. 470–483), Edgeworth (1891, p. 237n; 1925, p. 317n) and Berry (1891, p. 550). The concept was also analyzed by Pareto (1892); Samuelson (1942) and by Katzner (1970, pp. 23–26) who describes such preferences as "quasi-linear". See also Chipman and Moore (1976, pp. 86–91, 108–110; 1980, pp. 940–946), Chipman (2006, p. 109).

while the polluter starts out without any pollution rights ($\eta_1 = 0$) and ends up with $\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) = 0.14645$ of the money (less than 30% of his initial amount $\xi_1 = \frac{1}{2}$, the other 70% of which is used to purchase pollution rights from the pollutee), he ends up with the right to emit pollution half the time. In Case (ii), the polluter starts out with all the pollution rights ($\eta_1 = 1$) and ends up with $x_1 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = 0.85355$ of the money (more than 70% of his initial amount, the extra 0.35355 coming from the sale of pollution rights to the pollutee), but the right to emit pollution only half the time. This is in accord with the "Coase theorem" that states that the initial allocation of property rights does not affect the amount of pollution. This follows from a basic property of "parallel" preferences according to which the set of Pareto optima (the "contract curve") is (for $0 < x_i < \xi$) a horizontal straight line, shown as the dark line $y_1 = 0.5$ in Fig. 2. Allowing for zero amounts of money ($x_i = 0$), the entire set of Pareto optima is the half-swastika-shaped dark line shown.[5]

In the case of the utility functions (3.1), in Case (i) when $\xi_i = \frac{1}{2}$ and $\eta_1 = 0 (\eta_2 = 1)$, elimination of p from the demand Eq. (3.4) yields individual 1′s offer function

$$y_1 = 4\left(\frac{1}{2} - x_1\right)^2, \tag{3.7}$$

shown as the dotted convex curve in the left part of Fig. 2 starting at $(x_1, y_1) = \left(\frac{1}{2}, 0\right)$, going through the equilibrium point $(x_1, y_1) = \left((1 - 1/\sqrt{2})/2, 1/2\right) = (0.1464466, \quad 0.5)$, and ending at the point (0,1). Likewise, elimination of p from the demand Eq. (3.4) yields individual 2′s offer function (for $\eta_2 = 1$ in Case (i))

$$x_2 = \frac{1}{2} + \frac{1}{2\sqrt{y_2}} - \frac{\sqrt{y_2}}{2}, \tag{3.8}$$

which may be written as a quadratic equation in $\sqrt{y_2}$:

$$y_2 + (2x_1 - 1)\sqrt{y_2} - 1 = 0.$$

Taking the positive root, this gives

$$y_2 = \frac{1}{4}\left(1 - 2x_2 + \sqrt{(2x_2 - 1)^2 + 4}\right)^2. \tag{3.9}$$

This is shown in the left part of Fig. 2 as the concave dotted offer curve starting at the initial-endowment point $(\xi_2, \eta_2) = \left(\frac{1}{2}, 1\right)$ (measured from O_2 and corresponding to $\left(\frac{1}{2}, 0\right)$ measured from O_1), going through the equilibrium point $(x_2, y_2) = \left((1 + 1/\sqrt{2})/2, \frac{1}{2}\right) = (0.85355339, 0.5)$ (measured from O_2 and corresponding to

[5] The swastika is an ancient Buddhist and Hindu symbol found on temples in central Asia. Hitler adopted it (after rotating it clockwise 45 degrees) as the symbol of his Nazi party. As L. Hurwicz reminded the first author in a seminar presentation, the German word for swastika is *Hakenkreuz* (hook-cross); so the set of Pareto optima in the Edgeworth box with parallel preferences is one of these hooks.

$(1 - x_2, 1 - y_2) = \left((1 - 1/\sqrt{2})/2, \frac{1}{2}\right) = (0.14644661, 0.5)$ measured from O_1), and ending at $(x_2, y_2) = (1, 0.381966)$ (corresponding to (0,0.618034) as measured from O_1).

It remains to consider Case (ii) in which $\eta_1 = 1(\eta_2 = 0)$. Eliminating p from Eq. (3.4), we obtain for individual 1 the offer function

$$x_1 = \frac{1}{2} + \frac{1}{2\sqrt{y_1}} - \frac{\sqrt{y_1}}{2}, \tag{3.10}$$

which may also be expressed as a quadratic equation in $\sqrt{y_1}$:

$$y_1 + (2x_1 - 1)\sqrt{y_1} - 1 = 0.$$

Taking the positive root, this yields

$$y_1 = \frac{1}{4}\left(1 - 2x_1 + \sqrt{(2x_1 - 1)^2 + 4}\right)^2. \tag{3.11}$$

This is shown in the right part of Fig. 2 by the convex dotted offer curve starting at the initial-endowment point $(\xi_1, \eta_1) = (0.5, 1)$, going through the equilibrium point $(x_1, y_1) = ((1 + 1/\sqrt{2})/2, 1/2) = (0.85355339, 0.5)$, and ending at the point $(1, (\sqrt{5} - 1)^2)/4) = (1, 0.381966)$. In the case of individual 2, with $\eta_2 = 0$, elimination of p from Eq. (3.4) yields the offer function for individual 2:

$$y_2 = 4\left(\frac{1}{2} - x_2\right)^2. \tag{3.12}$$

This is shown in the right part of Fig. 2 by the dotted concave curve starting at the initial-endowment point $(\xi_2, \eta_2) = (0.5, 0)$ (measured from O_2 and corresponding to the point (0.5, 1) as measured from O_1), going through the equilibrium point

$$(x_2, y_2) = \left(\frac{1 - 1/\sqrt{2}}{2}, \frac{1}{2}\right) = (0.14644661, 0.5)$$

(corresponding to

$$(1 - x_2, 1 - y_2) = \left(\frac{1 + 1/\sqrt{2}}{2}, \frac{1}{2}\right) = (0.85355339, 0.5),$$

as measured from O_1), and ending at the point $(x_2, y_2) = (0, 1)$ (measured from O_2 and corresponding to $(1 - x_1, 1 - y_2) = (1, 0)$ as measured from O_1).

Because of the "parallel" nature of the preferences, the set of Pareto optima is (for $0 < x_i < \xi$) the horizontal line at $y = 0.5$. In this case, the Coase theorem holds:

the equilibrium amount of pollution is independent of the initial allocation of pollution rights. Note, however, that it does not hold if either the polluter has no money ($x_1 = 0$, $x_2 = \xi$) or the pollutee has no money ($x_1 = \xi$, $x_2 = 0$).

As in the previous case, a third Pareto-optimal point is shown at $(x_1, y_1) = (0.5, 0.5)$, and this is the case in which no trading is required to reach the optimum.

4 The question of the necessity of parallel preferences

The set of Pareto optima (equivalent here to the set of competitive equilibria—Edgeworth's "contract curve" (1881)) may be obtained as in Lange (1942) by maximizing the pollutee's utility $U_2(x_2, y_2)$ subject to that of the polluter $U_1(x_1, y_1)$ being constant at u_1, i.e.,

$$\text{Maximize } U_2(\xi - x_1, \eta - y_1) \text{ subject to } U_1(x_1, y_1) = u_1. \tag{4.1}$$

Setting up the Lagrangean expression

$$\mathcal{L}(x_1, y_1; \lambda) = U_2(\xi - x_1, \eta - y_1) - \lambda[U_1(x_1, y_1) - u_1] \tag{4.2}$$

and differentiating it with respect to x_1 and y_1, one obtains after eliminating λ the well-known mutual tangency condition

$$\frac{\partial U_1}{\partial y_1} \bigg/ \frac{\partial U_1}{\partial x_1} = \frac{\partial U_2}{\partial y_2} \bigg/ \frac{\partial U_2}{\partial x_2}, \quad \text{or} \quad \frac{\partial U_2}{\partial x_2} \cdot \frac{\partial U_1}{\partial y_1} = \frac{\partial U_2}{\partial y_2} \cdot \frac{\partial U_1}{\partial x_1} \tag{4.3}$$

as obtained by Edgeworth (1891, p. 236; 1925, p. 316). Now Edgeworth (1891, p. 237n; 1925, p. 317n) introduced the *sufficient* conditions that the marginal utilities of money of the respective individuals be positive constants $\partial U_i/\partial x_i = \nu_i > 0$ for $x_i > 0$, so that[6]

$$U_i(x_i, y_i) = \nu_i x_i + \phi_i(y_i) \quad (i = 1, 2) \tag{4.4}$$

(for $x_i > 0$), from which (4.3) reduces (together with (1.6)) to

$$\phi_1'(y_1)/\nu_1 = \phi_2'(\eta - y_1)/\nu_2. \tag{4.5}$$

Edgeworth concluded that one can solve this equation for y_1 independently of x_1,—i.e., that $y_1 = \text{constant}$—a conclusion that follows if it is assumed that $\phi_i' > 0$ and $\phi_i'' < 0$, as well as $\phi_2'(\eta)/\nu_2 < \phi_1'(0)/\nu_1$ and $\phi_2'(0)/\nu_2 > \phi_1'(\eta)/\nu_1$. Thus, the contract curve for $0 < x_i < \xi$ is a horizontal line in the (x, y) space.[7]

[6] The symbols x and y need to be interchanged to reconcile the present notation with Edgeworth's.

[7] Edgeworth's interest in this problem stemmed from his inquiry into the conditions under which the bargaining process introduced by Marshall in his Note on Berry (1891, pp. 395–397, 755–756; 1961, I, pp. 844–845; II, pp. 791–798), involving a succession of partial contracts at independently reached prices, with recontracting

It was further shown by Berry (1891, p. 550n)—see also Marshall (1961, II, pp. 793–5)—that the equilibrium price ratio is constant along this horizontal contract curve. His reasoning was that at any point (x_1, y_1) of an indifference curve $\nu_1 x_1 + \phi_1(y_1) = \bar{u}_1$, the slope is $\nu_1 dx_1/dy_1 + \phi_1'(y_1) = 0$; hence, along the horizontal line $y_1 =$ constant, the slope $dx_1/dy_1 = \phi'(y_1)/\nu_1$ is constant.[8]

It was Hurwicz's aim to show the *necessity* of parallel preferences in order to justify Coase's result. His argument will be followed here, except that the present exposition is cast in terms of pollution *rights* rather than pollution itself; hence, it applies generally to a two-agent, two-commodity model in which the marginal utilities of both commodities are nonnegative. We may characterize as *Coase's condition* the condition that the set of Pareto optima (the contract curve) in the (x, y) space for $x_i > 0$ is a horizontal line $y =$ constant. It was shown above that a sufficient condition for this (given by Edgeworth) is the cardinal condition $\partial U_i/\partial x_i =$ constant for $i = 1, 2$. In order to investigate the necessity, we must obtain a corresponding ordinal condition.

It was shown by Hurwicz (1995, p. 67) that preferences that are parallel with respect to the x-commodity, i.e., representable by a differentiable utility function $U(x, y) = f(\nu x + \phi(y))$, where $\nu > 0$, $\phi' > 0$, and $f' > 0$, are characterized by the condition

$$\frac{\partial^2 U}{\partial x^2} \cdot \frac{\partial U}{\partial y} = \frac{\partial^2 U}{\partial y \partial x} \cdot \frac{\partial U}{\partial x}, \quad \text{or} \quad \frac{\partial^2 U}{\partial x^2} - \frac{\partial U/\partial x}{\partial U/\partial y} \cdot \frac{\partial^2 U}{\partial y \partial x} = 0. \tag{4.6}$$

This is verified immediately by performing the computations. The second condition of (4.6) is the ordinal counterpart of Edgeworth's cardinal condition $\partial^2 U/\partial x^2 = 0$.

Now we assume that the contract curve for $x_i > 0$ is a horizontal line $y_i = \bar{y}$. Accordingly, following Hurwicz (1995, p. 67), we differentiate the competitive equilibrium condition (4.3) with respect to x_1 subject to $x_2 = \xi - x_1$ and $y_i = \bar{y}$ for $i = 1, 2$. This gives

$$\frac{\partial^2 U_1}{\partial y_1 \partial x_1} \cdot \frac{\partial U_2}{\partial x_2} + \frac{\partial U_1}{\partial y_1} \cdot \frac{\partial^2 U_2}{\partial x_2^2} = \frac{\partial^2 U_1}{\partial x_1^2} \cdot \frac{\partial U_2}{\partial y_2} + \frac{\partial U_1}{\partial x_1} \cdot \frac{\partial^2 U_2}{\partial y_2 \partial x_2}. \tag{4.7}$$

Now dividing (4.7) through by $\partial U_1/\partial y_1 \cdot \partial U_2/\partial y_2$ and employing the tangency condition (4.3), we obtain Hurwicz's important formula (A.3):

$$\frac{1}{\partial U_1/\partial y_1}\left[\frac{\partial^2 U_1}{\partial x_1^2} - \frac{\partial U_1/\partial x_1}{\partial U_1/\partial y_1}\frac{\partial^2 U_1}{\partial y_1 \partial x_1}\right] = \frac{1}{\partial U_2/\partial y_2}\left[\frac{\partial^2 U_2}{\partial x_2^2} - \frac{\partial U_2/\partial x_2}{\partial U_2/\partial y_2}\frac{\partial^2 U_2}{\partial y_2 \partial x_2}\right]. \tag{4.8}$$

Footnote 7 continued
ruled out, would lead to a competitive equilibrium with in fact a uniform price, and thus a "determinate" solution.

[8] See Fig. 2 above, also Fig. 1 in Hurwicz (1995, p. 57). But this condition is violated in Fig. 2 of Samuelson (1969, p. 113).

In view of the second formula of (4.6), Hurwicz's formula (4.8) shows that, assuming (as we do) that $\partial U_i/\partial y_i > 0$ (pollution rights are positively desired by both agents), if individual $2'$s preferences are parallel (the bracketed term on the right is zero), so must be individual $1'$s, and *vice versa*. But of course this does not imply the desired converse of Edgeworth's proposition, namely that the horizontality of the contract curve implies that both individuals' preferences must be parallel.

In his *Sketch of proof* of the desired proposition, Hurwicz (1995, p. 71) assumed that individual 2 has a "linear preference", i.e., one representable by a utility function $U_2(x_2, y_2) = ax_2 + by_2$. But this is a special case of parallel preferences. This oversight appears to have escaped his attention. The problem of obtaining necessary conditions for the "Coase conjecture" was therefore left open. In effect, the proof consisted in showing how, starting from individual $2'$s linear preference (which is varied throughout the proof), one can infer that individual $1'$s must be parallel in order for the contract curve to be a horizontal line. But this already followed from (4.8).[9]

In fact, there are two problems in Hurwicz (1995). First, the equation in (4.8) alone cannot be used to fully characterize competitive equilibrium with $0 < x_i < \xi$ and $y_1 = y_2 = \bar{y}$. The following example shows that although the equation in (4.8) is satisfied, it cannot guarantee that the contract curve is horizontal so that the set of Pareto optima for the utility functions need not be y = constant.

Example 4.1 Suppose the initial endowments for money $\xi = 1$ and pollution rights $\eta = 1$. Let $U_i = x_i - x_i^2/2 + y_i$ which is not quasilinear in x_i. But U_i is monotonically increasing for $(x_i, y_i) \in [0, 1] \times [0, 1]$ and concave. Moreover, it can be easily checked that (4.8) is satisfied for all $(x_i, y_i) \in [0, 1] \times [0, 1]$.

In fact, if we let

$$U_i = x_i - x_i^2/2 + \phi(y_i), \tag{4.9}$$

where ϕ is any concave and monotonically increasing function, then (4.8) is also satisfied for all $x_i, \in [0, 1]$ and $y_1 = y_2$. Thus, for the class of utility functions given by (4.9), although (4.8) is satisfied, the set of Pareto optima is not a horizontal line $y =$ constant.

To make the equation in (4.8) fully characterize competitive equilibrium with $0 < x_i < \xi$ and $y_1 = y_2 = \bar{y}$, we need to assume that the mutual tangency (first-order) condition (4.3) is also satisfied for all $x_i \in [0, \xi]$ and $y_1 = y_2 = \bar{y}$. Note that for the class of utility functions given by (4.9), (4.3) is satisfied only for $x_i = 1/2$. This is why, even if (4.3) is satisfied for all $x_i \in [0, \xi]$ and $y_1 = y_2 = \bar{y}$, the set of Pareto optima is not a horizontal line y = constant.

[9] Hurwicz mentioned (p. 68) that he had found an example (unfortunately not displayed) of cubic utility functions generating horizontal contract curves, but he dismissed this on the ground that "it must be possible to choose the two utility functions independently, while in the cubic case 'the choice of u^2 is limited by the choice of u^1.'" But formula (4.8) above, as well as the mutual tangency (4.3), shows that assuming the horizontality of the contract curve to be true, the two terms in brackets cannot be entirely unrelated. This does not imply any *psychological* dependence between the utility functions, but simply that there must be some kind of relationship, e.g. as in Edgeworth's formula (4.5), in order for the horizontality of the contract curve to be true.

Secondly, Hurwicz's argument on the necessity of parallel preferences for "Coase's conjecture" is also problematic. To see this, without loss of generality,[10] we study the allocation of pollution s rather than the individuals' pollution rights y_i and consider the following class of utility functions $U_i(x_i, s)$ that have the functional form:

$$U_i(x_i, s) = x_i e^{-s} + \phi_i(s), \qquad i = 1, 2 \tag{4.10}$$

where

$$\phi_i(s) = \int e^{-s} b_i(s) ds. \tag{4.11}$$

$U_i(x_i, s)$ is then clearly not quasi-linear in x_i. It is further assumed that for all $s \in (0, \eta]$, $b_1(s) > \xi$, $b_2(s) < 0$, $b_i'(s) < 0 (i = 1, 2)$, $b_1(0) + b_2(0) \geq \xi$, and $b_1(\eta) + b_2(\eta) \leq \xi$.

We then have

$$\begin{aligned} \partial U_i/\partial x_i &= e^{-s} > 0, \quad i = 1, 2, \\ \partial U_1/\partial s &= -x_1 e^{-s} + b_1(s) e^{-s} > e^{-s}[\xi - x_1] \geq 0, \\ \partial U_2/\partial s &= -x_2 e^{-s} + b_2(s) e^{-s} < 0 \end{aligned}$$

for $(x_i, s) \in (0, \xi) \times (0, \eta)$, $i = 1, 2$. Thus, by (1.8), we have

$$\begin{aligned} 0 = \frac{\partial U_1}{\partial s} \Bigg/ \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial s} \Bigg/ \frac{\partial U_2}{\partial x_2} &= -x_1 - x_2 + b_1(s) + b_2(s) \\ = b_1(s) + b_2(s) - \xi, \end{aligned} \tag{4.12}$$

which is independent of x_i. Hence, if (x_1, x_2, s) is Pareto-optimal, so is (x_1', x_2', s) provided $x_1 + x_2 = x_1' + x_2' = \xi$. Also, note that $b_i'(s) < 0 (i = 1, 2)$, $b_1(0) + b_2(0) \geq \xi$, and $b_1(\eta) + b_2(\eta) \leq \xi$. Then, $b_1(s) + b_2(s)$ is strictly monotone, and thus, there is a unique $s \in [0, \eta]$, satisfying (4.12). Thus, the contract curve is horizontal even though individuals' preferences need not be parallel.

Example 4.2 Suppose $b_1(s) = (1 + s)^{\alpha} \eta^{\eta} + \xi$ with $\alpha < 0$ and $b_2(s) = -s^{\eta}$. Then, for all $s \in (0, \eta]$, $b_1(s) > \xi$, $b_2(s) < 0$, $b_i'(s) < 0 (i = 1, 2)$, $b_1(0) + b_2(0) > \xi$, and $b_1(\eta) + b_2(\eta) < \xi$. Thus, $\phi_i(s) = \int e^{-s} b_i(s) ds$ is concave, and $U_i(x_i, s) = x_i e^{-s} + \int e^{-s} b_i(s) ds$ is quasi-concave, $\partial U_i/\partial x_i > 0$ and $\partial U_1/\partial s > 0$, and $\partial U_2/\partial s < 0$ for $(x_i, s) \in (0, \xi) \times (0, \eta)$, $i = 1, 2$, but it is not quasi-linear in x_i.

[10] Since Eq. (1.7) defines a continuous one-to-one mapping between the individuals' pollution rights y_i with $y_1 + y_2 = \eta$ and allocation of pollution s, the two constrained optimization problems are equivalent. Note that through such a monotonic transformation, one may transform an original problem into a concave optimization problem, in which the object function is (quasi)concave and the constraint sets are convex. Since this technique has been widely used in the literature such as in the moral hazard model in the Principal-Agent Theory (cf. Laffont and Martimort (2002, pp. 158–159)).

Now, we investigate the necessity for the "Coase conjecture" that the level of pollution is independent of the assignments of property rights. This reduces to developing the necessary and sufficient conditions that guarantee that the contract curve is horizontal so that the set of Pareto optima for the utility functions is s-constant. This in turn reduces to finding the class of utility functions such that the mutual tangency (first-order) condition (4.3) does not contain x_i, and consequently, it is a function, denoted by $g(s)$, of s only:

$$\frac{\partial U_1}{\partial s} \Big/ \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial s} \Big/ \frac{\partial U_2}{\partial x_2} = g(s) = 0. \tag{4.13}$$

Let $F_i(x_i, s) = \frac{\partial U_i}{\partial s} / \frac{\partial U_i}{\partial x_i} (i = 1, 2)$, which can be generally expressed as

$$F_i(x_i, s) = x_i h_i(s) + f_i(x_i, s) + b_i(s),$$

where the $f_i(x_i, s)$ are nonseparable and nonlinear in x_i. $h_i(s)$, $b_i(s)$, and $f_i(x_i, s)$ will be further specified below.

Let $F(x, s) = F_1(x, s) + F_2(\xi - x, s)$. Then, (1.8) can be rewritten as

$$F(x, s) = 0. \tag{4.14}$$

Thus, the contract curve, i.e., the locus of Pareto-optimal allocations, can be expressed by a function $s = f(x)$ that is implicitly defined by (4.14).

Then, the Coase Neutrality Theorem, which is characterized by the condition that the set of Pareto optima (the contract curve) in the (x, s) space for $x_i > 0$ is a horizontal line $s =$ constant, implies that

$$s = f(x) = \bar{s}$$

with $\bar{s}$ constant, and thus, we have

$$\frac{ds}{dx} = -\frac{F_x}{F_s} = 0$$

for all $x \in [0, \xi]$ and $F_s \neq 0$, which means that the function $F(x, s)$ is independent of x. Then, for all $x \in [0, \xi]$,

$$\begin{aligned} F(x, s) = x h_1(s) + (\xi - x) h_2(s) + f_1(x, s) + f_2(\xi - x, s) \\ + b_1(s) + b_2(s) \equiv g(s). \end{aligned} \tag{4.15}$$

Since the utility functions U_1 and U_2 are functionally independent and x disappears in (4.15), we must have $h_1(s) = h_2(s) \equiv h(s)$, and $f_1(x, s) = -f_2(\xi - x, s) = 0$ for all $x \in [0, \xi]$. Therefore,

$$F(x, s) = \xi h(s) + b_1(s) + b_2(s) \equiv g(s), \tag{4.16}$$

and

$$\frac{\partial U_i}{\partial s} \Big/ \frac{\partial U_i}{\partial x_i} = F_i(x_i, s) = x_i h(s) + b_i(s) \tag{4.17}$$

which is a first-order linear partial differential equation. Then, from Polyanin et al. (2002),[11] we know that the principal integral $U_i(x_i, s)$ of (4.17) is given by

$$U_i(x_i, s) = x_i e^{\int h(s)} + \phi_i(s), \qquad i = 1, 2 \tag{4.18}$$

with

$$\phi_i(s) = \int e^{\int h(s)} b_i(s) ds. \tag{4.19}$$

The general solution of (4.17) is then given by $\bar{U}_i(x, y) = \psi(U_i)$, where ψ is an arbitrary function. Since a monotonic transformation preserves orderings of preferences, we can regard the principal solution $U_i(x_i, s)$ as a general functional form of utility functions that are fully characterized by (4.17).

Note that (4.18) is a general utility function that contains quasi-linear utility in x_i and the utility function given in (4.10) as special cases. Indeed, it represents parallel preferences when $h(s) \equiv 0$ and also reduces to the utility function given by (4.10) when $h(s) = -1$.

To make the mutual tangency (first-order) condition (4.13) be also sufficient for the contract curve to be horizontal in a pollution economy, we assume that for all $s \in (0, \eta], x_1 h(s) + b_1(s) > 0, x_2 h(s) + b_2(s) < 0, h'(s) \leq 0, b_i'(s) < 0 (i = 1, 2), \xi h(0) + b_1(0) + b_2(0) \geq 0$, and $\xi h(\eta) + b_1(\eta) + b_2(\eta) \leq 0$.

We then have for $(x_i, s) \in (0, \xi) \times (0, \eta), i = 1, 2,$

$$\begin{aligned} \partial U_i / \partial x_i &= e^{\int h(s)} > 0, \quad i = 1, 2, \\ \partial U_1 / \partial s &= e^{\int h(s)} [x_1 h(s) + b_1(s)] > 0, \\ \partial U_2 / \partial s &= e^{\int h(s)} [x_2 h(s) + b_2(s)] < 0, \end{aligned}$$

and thus

$$\begin{aligned} 0 = \frac{\partial U_1}{\partial s} \Big/ \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial s} \Big/ \frac{\partial U_2}{\partial x_2} &= (x_1 + x_2) h(s) + b_1(s) + b_2(s) \\ &= \xi h(s) + b_1(s) + b_2(s), \end{aligned} \tag{4.20}$$

which does not contain x_i. Hence, if (x_1, x_2, s) is Pareto-optimal, so is (x_1', x_2', s) provided $x_1 + x_2 = x_1' + x_2' = \xi$. Also, note that $h'(s) \leq 0, b_i'(s) < 0 (i = 1, 2), \xi h(0) + b_1(0) + b_2(0) \geq 0$, and $\xi h(\eta) + b_1(\eta) + b_2(\eta) \leq 0$. Then $\xi h(s) + b_1(s) + b_2(s)$ is strictly monotone, and thus, there is a unique $s \in [0, \eta]$ that satisfies (4.20). Thus, the contract curve is horizontal even though individuals' preferences need not be parallel.

In summary, we have the following proposition.

[11] It can be also seen from http://eqworld.ipmnet.ru/en/solutions/fpde/fpde1104.pdf.

Proposition 4.1 (Coase Neutrality Theorem) *In a pollution economy considered in this paper, suppose that the transaction cost equals zero and that the utility functions $U_i(x_i, s)$ are differentiable and such that $\partial U_i/\partial x_i > 0$, and $\partial U_1/\partial s > 0$ but $\partial U_2/\partial s < 0$ for $(x_i, s) \in (0, \xi) \times (0, \eta)$, $i = 1, 2$. Then, the level of pollution is independent of the assignment of property rights if and only if the utility functions $U_i(x, y)$, up to a monotonic transformation, have a functional form given by*

$$U_i(x_i, s) = x_i e^{\int h(s)} + \int e^{\int h(s)} b_i(s) ds, \tag{4.21}$$

where h and b_i are arbitrary functions such that the $U_i(x_i, s)$ are differentiable, $\partial U_i/\partial x_i > 0$, and $\partial U_1/\partial s > 0$ but $\partial U_2/\partial s < 0$ for $(x_i, s) \in (0, \xi) \times (0, \eta)$, $i = 1, 2$.

Although the above Coase neutrality theorem covers a much wider class of preferences, it still puts a significant restriction on the domain of its validity due to the special functional forms of the utility functions.

In this paper, we only consider the economy in which one individual is the polluter and the other is the pollutee. By using the first-order conditions for Pareto-optimality in an economy with negative externalities, which is developed in Tian and Yang (2009), we can also study "Coase's conjecture" in an economy with more than two individuals in which the individuals are polluters and pollute each other.

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