ECMT 660

Mathematical Economics
(Lecture Notes)

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I. THE NATURE OF MATHEMATICL ECONOMICS

Mathematical economics is an approach to economic analysis, in which the economists make use of mathematical symbols in the statement of the problem and also draw upon known mathematical theorems to aid in reasoning.

The purpose of this course is to introduce the most fundamental aspects of the mathematical methods such as those matrix algebra, mathematical analysis, and optimization theory.

1.1 Mathematical Versus Nonmathematical Economics

Since mathematical economics is merely an approach to economic analysis, it should not and does not differ from the nonmathematical approach to economic analysis in any fundamental way. The difference between these two approaches is that in the former, the assumptions and conclusions are stated in mathematical symbols rather than words and in equations rather than sentences.

Advantages of the Mathematical Approaches

(1) the analysis is more rigorous;
(2) it allows us to treat the general n-variable case; and
(3) the "language" used is more concise and precise.

III. EQUILIBRIUM ANALYSIS IN ECONOMICS

3.1 The Meaning of Equilibrium

Like any economic term, equilibrium can be defined in various ways. One definition here is that an equilibrium is a constellation of selected interrelated variables so adjusted to one another that inherent tendency to change prevails in the model which they constitute.

In essence, an equilibrium for a specific model is a situation that is characterized by a lack of tendency to change. It is for this reason that the analysis of equilibrium is referred to as statics. The fact that an equilibrium implies no tendency to change may tempt one to conclude that an equilibrium necessarily constitutes a desirable or ideal state of affairs.

This chapter provides two examples of equilibrium. One is the equilibrium attained by a market under given demand and supply conditions. The other is the equilibrium of national income under given conditions of consumption and investment patterns.

3.2 Partial Market Equilibrium - A Linear Model

In a static-equilibrium model, the standard problem is that of finding the set of values of the endogenous variables which will satisfy the equilibrium conditions of the model.

Partial-Equilibrium Market Model--a model of price determination in an isolated market.

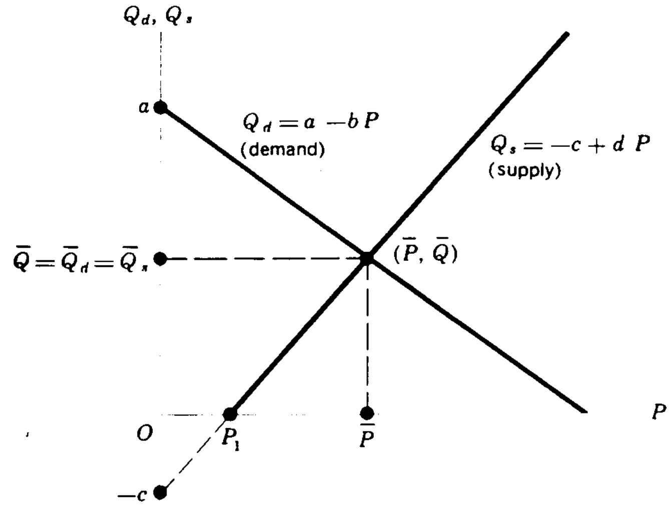
Three variables

Q_d = the quantity demanded of the commodity;
Q_s = the quantity supplied of the commodity;
P = the price of the commodity.

The Equilibrium Condition: $Q_d = Q_s$.

The model is

$$Q_d = Q_s$$
$$Q_d = a - bp \qquad (a, b > 0)$$
$$Q_s = -c + dp \qquad (c, d > 0)$$



The slope of Q_d = -b, the vertical intercept = a.
The slope of Q_s = d, the vertical intercept = -c.
Note that, contrary to the usual practice, quantity rather than price has been plotted vertically in the Figure.

One way of finding the equilibrium is by successive elimination of variables and equations through substitution.

From $Q_s = Q_d$, we have

$$a - bp = -c + dp$$

and thus

$$(b + d)p = a + c.$$

Since $b + d \neq 0$, then the equilibrium price is

$$\bar{p} = \frac{a+c}{b+d}$$

The equilibrium quantity can be obtained by substituting $\bar{p}$ into either Q_s or Q_d:

$$\bar{Q} = \frac{ad - bc}{b + d}$$

Since the denominator (b+d) is positive, the positivity of $\bar{Q}$ requires that the numerator (ad-bc) > 0. Thus, to be economically meanful, the model should contain the additional restriction that ad > bc.

3.3 Partial Market Equilibrium - A Nonlinear Model

The partial market model can be nonlinear. Suppose a model is given by

$$Q_d = Q_d$$
$$Q_d = 4 - p^2$$
$$Q_s = 4p - 1$$

As previously stated, this system of three equations can be reduced to a single equation by substitution.

$$4 - p^2 = 4p - 1$$

or

$$p^2 + 4p - 5 = 0$$

which is a quadratic equation. In general, given a quadratic equation in the form

$$ax^2 + bx + c = 0 \qquad (a \neq 0),$$

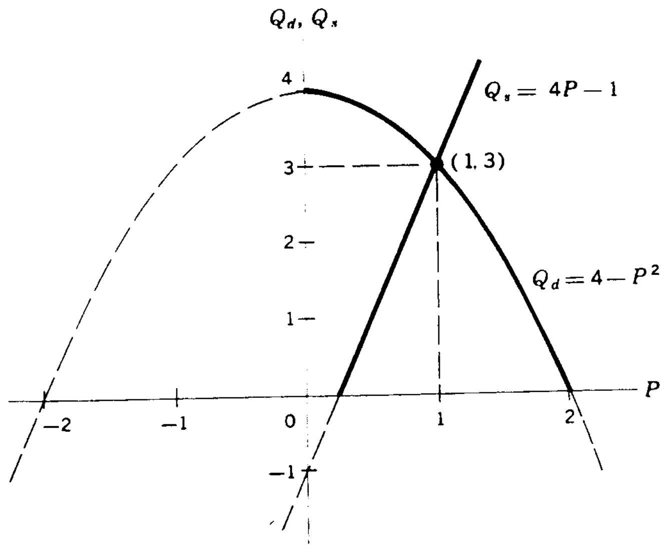
its two roots can be obtained from the quadratic formula:

$$\bar{x}_1, \bar{x}_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the "+" part of the "±" sign yields $\bar{x}_1$ and the "-" part yields $\bar{x}_2$.

Thus, by applying the quadratic formulas to $p^2 + 4p - 5 = 0$, we have $\bar{p}_1 = 1$ and $\bar{p}_2 = -5$, but only the first is economically admissible, as negative prices are ruled out.

The Graphical Solution



3.4 General Market Equilibrium

In the above, we have discussed methods of an isolated market, wherein the Q_d and Q_s of a commodity are functions of the price of that commodity alone. In the real world, there would normally exist many substitutes and complementary goods. Thus a more realistic model for the demand and supply functions of a commodity should take into account the effects not only of the price of the commodity itself but also of the prices of other commodities. As a result, the price and quantity variables of multiple commodities must enter endogenously into the model. Thus, when several interdependent commodities are simultaneously considered, equilibrium would require the absence of excess demand, which is the difference between demand and supply, for each and every commodity included in the model. Consequently, the equilibrium condition of an n-commodity market model will involve n equations, one for each commodity, in the form

$$E_i \equiv Q_{di} - Q_{si} = 0 \qquad (i=1,2,\ldots,n)$$

where $Q_{di} = Q_{di}(P_1,P_2,\ldots,P_n)$ and $Q_{si} = Q_{si}(P_1,\ldots,P_n)$ are the demand and supply functions of commodity i, and $(P_1,P_2,\ldots,P_n)$ are prices of commodities.

Thus solving n equations for P:

$$E_i(P_1,P_2,\ldots,P_n) = 0$$

we obtain the n equilibrium prices $\bar{P}_i$--if a solution does indeed exist. And then the $\bar{Q}_i$ may be derived from the demand or supply functions.

Two-Commodity Market Model

To illustrate the problem, let us consider a two-commodity market model with linear demand and supply functions. In parametric terms, such a model can be written as

$$
\begin{aligned}
&Q_{d1} - Q_{s1} = 0 \\
&Q_{d1} = a_0 + a_1P_1 + a_2P_2 \\
&Q_{s1} = b_0 + b_1P_1 + b_2P_2 \\
&Q_{d2} - Q_{s2} = 0 \\
&Q_{d2} = \alpha_0 + \alpha_1P_1 + \alpha_2P_2 \\
&Q_{s2} = \beta_0 + \beta_1P_1 + \beta_2P_2
\end{aligned}
$$

By substituting the second and third equations into the first and the fifth and sixth equations into the fourth, the model is reduced to two equations in two variables:

$$(a_0 - b_0) + (a_1 - b_1)P_1 + (a_2 - b_2)P_2 = 0$$

$$(\alpha_0 - \beta_0) + (\alpha_1 - \beta_1)P_1 + (\alpha_2 - \beta_2)P_2 = 0$$

If we let

$$
\left.
\begin{aligned}
c_i &= a_i - b_i \\
\gamma_i &= \alpha_i - \beta_i
\end{aligned}
\right\} \quad (i=0,1,2),
$$

the above two linear equations can be written as

$$c_1P_1 + c_2P_2 = -c_0$$

$$\gamma_1P_1 + \gamma_2P_2 = -\gamma_0$$

which can be solved by further elimination of variables.

The solutions are

$$\bar{P}_1 = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1}$$

$$\bar{P}_2 = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1}$$

For these two values to make sense, certain restrictions should be imposed on the model. First, we require the common denominator $c_1\gamma_2 - c_2\gamma_1 \neq 0$. Second, to assure positivity, the numerator must have the same sign as the denominator.

Numerical Example

Suppose that the demand and supply functions are numerically as follows:

$$Q_{d1} = 10 - 2P_1 + P_2$$
$$Q_{s1} = -2 + 3P_1$$
$$Q_{d2} = 15 + P_1 - P_2$$
$$Q_{s2} = -1 + 2P_2$$

By substitution, we have

$$5P_1 - P_2 = 12$$
$$-P_1 + 3P_2 = 16$$

which are two linear equations. The solutions for the equilibrium prices and quantities are $\bar{P}_1 = 52/14$, $\bar{P}_2 = 92/14$, $\bar{Q}_1 = 64/7$, and $\bar{Q}_2 = 85/7$.

Similarly, for the n-commodity market model, when demand and supply functions are linear in prices, we can have n linear equations. In the above, we assume that an equal number of equations and unknowns has a unique solution. However, some very simple examples should convince us that an equal number of equations and unknowns does not necessarily guarantee the existence of a unique solution.

For the two linear equations,

$$\begin{cases} x + y = 8 \\ x + y = 9 \end{cases}$$

we can easily see there is no solution.

The second example shows a system has an infinite number of solutions:

$$2x + y = 12$$
$$4x + 2y = 24$$

These two equations are functionally dependent, which means that one can be derived from the other. Consequently, one equation is redundant and may be dropped from the system. Any pair $(\bar{x},\bar{y})$ is the solution as long as $(\bar{x},\bar{y})$ satisfies $y = 12 - x$.

Now consider the case of more equations than unknowns. In general, there is no solution. But, when the number of unknowns equals the number of functional independent equations, the solution exists and is unique. The following example shows this fact.

$$2x + 3y = 58$$

$$y = 18$$

$$x + y = 20$$

Thus for simultaneous-equation model, we need systematic methods of testing the existence of a unique (or determinate) solution. These are our tasks in the following chapters.

3.5 Equilibrium in National-Income Analysis

The equilibrium analysis can be also applied to other areas of economics. As a simple example, we may cite the familiar Keynesian national-income model,

$$Y = C + I_0 + G_0 \qquad \text{(equilibrium condition)}$$

$$C = a + bY \qquad \text{(the consumption function)}$$

where Y and C stand for the endogenous variables national income and consumption expenditure, respectively, and I_0 and G_0 represent the exogenously determined investment and government expenditures.

Solving these two linear equations, we obtain the equilibrium national income and consumption expenditure:

$$\bar{Y} = \frac{a + I_0 + G_0}{1 - b}$$

$$\bar{C} = \frac{a + b(I_0 + G_0)}{1 - b}$$

IV. LINEAR MODELS AND MATRIX ALGEBRA

From the last chapter we have seen that for the one-commodity model, the solutions for $\bar{P}$ and $\bar{Q}$ are relatively simple, even though a number of parameters are involved. As more and more commodities are incorporated into the model, such solution formulas quickly become cumbersome and unwieldy. We need to have new methods suitable for handling a large system of simultaneous equations. Such a method is found in matrix algebra.

Matrix algebra can enable us to do many things. (1) It provides a compact way of writing an equation system, even an extremely large one. (2) It leads to a way of testing the existence of a solution by evaluation of a determinant--a concept closely related to that of a matrix. (3) It gives a method of finding that solution if it exists.

4.1 Matrix and Vectors

In general, a system of m linear equations in n variables $(x_1, x_2, \ldots, x_n)$ can be arranged into such a formula.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= d_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= d_m. \end{aligned} \tag{4.1}$$

where the doubled-subscripted symbol a_{ij} represents the coefficient appearing in the ith equation and attached to the jth variable x_j and d_j represents the constant term in the jth equation.

Example: The two-commodity linear market model can be written--after eliminating the quantity variables--as a system of two linear equations.

$$\begin{aligned} c_1P_1 + c_2P_2 &= -c_0 \\ \gamma_1P_1 + \gamma_2P_2 &= -\gamma_0 \end{aligned}$$

Matrix as Arrays

There are essentially three types of ingredients in the equation system (4.1). The first is the set of coefficients a_{ij}; the second is the set of variables $x_1, x_2, \ldots, x_n$; and the last is the set of constant terms $d_1, \ldots, d_m$. If we arrange the three sets as three rectangular arrays and label them, respectively, as, A, x, and d, then we have

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix} \tag{4.2}$$

Example: Given the linear-equation system

$$6x_1 + 3x_2 + x_3 = 22$$
$$x_1 + 4x_2 - 2x_3 = 12$$
$$4x_1 - x_2 + 5x_3 = 10$$

we can write

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$$

Each of three arrays in (4.2) constitutes a matrix.

A matrix is defined as a rectangular array of numbers, parameters, or variables. As a shorthand device, the array in matrix A can be written more simply as

$$A = [a_{ij}] \quad \begin{pmatrix} i = 1,2,\ldots,n \\ j = 1,2,\ldots,m \end{pmatrix}$$

Vectors as Special Matrices

The number of rows and the number of columns in a matrix together define the dimension of the matrix. For instance, A is said to be of dimension $m \times n$. In the special case where $m=n$, the matrix is called a square matrix.

If a matrix contains only one column (row), it is called a column (row) vector. For notation purposes, a row vector is often distinguished from a column vector by the use of a primed symbol.

$$x' = [x_1, x_2, \ldots, x_n]$$

Remark. A vector is merely an ordered n-triple and as such it may be interpreted as a point in an n-dimensional space.

With the matrices defined in (4.2), we can express the equation system (4.1) simply as

$$Ax = d.$$

However, the equation $Ax = d$ prompts at least two questions. How do we multiply two matrices A and x? What is meant by the equality of Ax and d? Since matrices involve whole blocks of numbers, the familiar algebraic operations defined for single numbers are not divertly applicable, and there is need for new set of operation rules.

4.2 Matrix Operations

The Equality of Two Matrices.

$A = B$ if and only if $a_{ij} = b_{ij}$ for all $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

Addition and Subtraction of Matrices.

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

i.e. the addition of A and B is defined as the addition of each pair of corresponding elements.

Remark. Two matrices can be added (equal) if and only if they have the same dimension.

Example:

$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}$$

Example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

<u>The Subtraction of Matrices</u>

A - B is defined by

$$[a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$$

Example:

$$\begin{bmatrix} 19 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 13 & -5 \\ 1 & -3 \end{bmatrix}$$

<u>Scalar Multiplication</u>

$$\lambda A = \lambda[a_{ij}] = [\lambda a_{ij}]$$

i.e. to multiply a matrix by a number is to multiply every element of that matrix by the given scalar.

Example:

$$7 \begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 21 & -7 \\ 0 & 35 \end{bmatrix}$$

Example:

$$-1 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{21} & -a_{31} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$$

Multiplication of Matrices

Given two matrices A_{mxn} and B_{pxq}, the conformability condition for multiplication AB is that the column dimension of A must be equal to be the row dimension of B, i.e., the matrix product AB will be defined if and only if n = p. If defined, the product AB will have the dimension $m \times q$.

The product AB is defined by

$$AB = C$$

with $$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{i\in}b_{nj} = \sum_{\ell=1}^{n} a_{i\ell}b_{\ell j}$$

Example:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$$

Example:

$$A_{2x2} = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \qquad B_{2x2} = \begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} -3+20 & 35 \\ -4+24 & 42 \end{bmatrix} = \begin{bmatrix} 17 & 35 \\ 20 & 42 \end{bmatrix}$$

Example:

$$u' = [u_1, u_2, \ldots, u_n] \text{ and } v' = [v_1, v_2, \ldots, v_n]$$

$$u'v = u_1v_1 + u_2v_2 + \ldots + u_nv_n = \sum_{i=1}^{n} u_iv_i$$

This can also be described by using the concept of the <u>inner product</u> of two vectors u and v.

$$u \cdot v = u_1v_1 + u_2v_2 + \ldots + u_nv_n = u'v.$$

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n \\ a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n \end{bmatrix}$$

Example: Given $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v' = [\,1 \;\; 4 \;\; 5\,]$

$$uv' = \begin{bmatrix} 3\times1 & 3\times4 & 3\times5 \\ 2\times1 & 2\times8 & 2\times5 \end{bmatrix} = \begin{bmatrix} 3 & 12 & 15 \\ 2 & 8 & 10 \end{bmatrix}$$

It is important to distinguish the meanings of uv' (a matrix larger than 1×1) and u'v (a 1×1 matrix, or a scalar).

4.3 Linear Dependence of Vectors

A set of vectors $v_1,\ldots,v_n$ is said to be linearly dependent if and only if any one of them can be expressed as a linear combination of the remaining vectors; otherwise they are linearly independent.

Example: The three vectors

$$v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

are linearly dependent since v_3 is a linear combination of v_1 and v_2:

$$3v_1 - 2v_2 = \begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = v_3.$$

or $3v_1 - 2v_2 - v_3 = 0$

where $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ represents a zero vector.

An equivalence definition of linear dependence is: a set of m-vectors $v_1,v_2,\ldots,v_n$ is linearly dependent if and only if there exists a set of scalars $k_1,k_2,\ldots,k_n$ (not all zero) such that

$$\sum_{i=1}^{n} k_i v_i = 0$$

If this equation can be satisfied only when $k_i = 0$ for all i, these vectors are linearly independent.

4.4 Commutative, Associative, and Distributive Laws

The commutative and associative laws of matrix can be stated as follows:

Commutative Law $\qquad A + B = B + A$

Proof: $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}] = B + A$

Associative Law $(A + B) + C = A + (B + C)$

Matrix Multiplication

Matrix multiplication is not commutative, that is

$$AB \neq BA$$

Even when AB is defined, BA may not be; but even if both products are defined, AB = BA may not still hold.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$ then $AB = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}$, but

$$BA = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}.$$

The scalar multiplication of a matrix does obey. The commutative law:

$$kA = Ak$$

if k is a scalar.

Associative Law $(AB)C = A(BC)$

if A is mxn, B is nxp, and C is pxq.

Distribution Law

$A(B + C) = AB + AC$ [premultiplication by A]
$(B + C)A = BA + CA$ [postmultiplication by A]

4.5 Identity Matrices and Null Matrices

Identity Matrix is a square matrix with ones in its principal diagonal and zeros everywhere else. It is denoted by I or I_n, in which n indicates its dimension.

Fact 1 Given an mxn matrix A, we have

$$I_m A = A I_n = A$$

Fact 2

$$A_{mxn} I_n B_{nxp} = (AI)B = AB$$

Fact 3 $(I_n)^k = I_n$

Null Matrices. A null--or zero matrix--denoted by 0, plays the role of the number 0. A null matrix is simply a matrix whose elements are all zero. Unlike I, the zero matrix is not restricted to being square. Null matrices obey the following rules of operation.

$$A_{mxn} + 0_{mxn} = A_{mxn}$$
$$A_{mxn}\, 0_{nxp} = 0_{mxp}$$
$$0_{qxm}\, A_{mxn} = 0_{qxn}$$

Remark. CD = CE does not imply D = E. For instance,

$$C = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

we can see that

$$CD = CE = \begin{bmatrix} 5 & 8 \\ 15 & 24 \end{bmatrix}$$

even though D $\neq$ E.

4.6 Transposes and Inverses

The transpose of a matrix A is a matrix which is obtained by interchanging the rows and columns of the matrix A. It is denoted by A′ or A^T.

Example: For $A = \begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix}$

$$A' = \begin{bmatrix} 3 & 1 \\ 8 & 0 \\ -9 & 4 \end{bmatrix}$$

Thus, by definition, if a matrix A is mxn, then its transpose A′ must be nxm.

Example:

$$D = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$$

Its transpose $D' = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix} = D$

A matrix A is said to be symmetric if A′ = A.

Properties of Transposes

a) (A′)′ = A
b) (A + B)′ = A′ + B′
c) (AB)′ = B′A′

The property C states that the transpose of a product is the product of the transposes in reverse order.

Inverses and Their Properties

For a given matrix A, A′ is always derivable. On the other hand, its inverse matrix may or may not exist. The inverse of A, denoted by A^{-1}, is defined only if A is a square matrix, in which case the inverse is the matrix that satisfies the condition.

$$AA^{-1} = A^{-1}A = I$$

Remarks.

1. Not every square matrix has an inverse--squareness is a necessary but not sufficient condition for the existence of an inverse. If a square matrix A has an inverse, A is said to be nonsingular, if A possesses no inverse, it is said to be a singular matrix.

2. If A is nonsingular, then A and A^{-1} are inverses of each other, i.e., $(A^{-1})^{-1} = A$.

3. If A is nxn, then A^{-1} is also nxn.

4. The inverse of A is unique.
   Proof. Let B and C both be inverses of A. Then

   $$B = BI = BAC = IC = C.$$

5. $AA^{-1} = I$ implies $A^{-1}A = I$.
   Proof. We need to show that if $AA^{-1} = I$, and if there is a matrix B such that $BA = I$, then $B = A^{-1}$. Postmultiplying both sides of $BA = I$ by A^{-1}, we have $BAA^{-1} = A^{-1}$ and thus $B = A^{-1}$.

Example: let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$

Then $AB = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \frac{1}{6} = \begin{bmatrix} 6 & \\ & 6 \end{bmatrix} \frac{1}{6} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$

So B is the inverse of A. Suppose A and B ar nonsingular matrices with dimension nxn.

6. (a) $(AB)^{-1} = B^{-1}A^{-1}$

   (b) $(A')^{-1} = (A^{-1})'$

Inverse Matrix and Solution of Linear-Equation System

The application of the concept of inverse matrix to the solution of a simultaneous-equation is immediate and direct. Consider

$$Ax = d$$

If A is a nonsingular matrix, then premultiplying both sides of $Ax = d$, we have

$$A^{-1}Ax = A^{-1}d$$

So, $x = A^{-1}d$ is the solution of $Ax = d$ and the solution is unique since A^{-1} is unique. Methods of testing the existence of the inverse and of its calculation will be discussed in the next chapter.

V. LINEAR MODELS AND MATRIX ALGEBRA (CONTINUED)

In Chapter 4, it was shown that a linear-equation system can be written in a compact notation. Furthermore, such an equation system can be solved by finding the inverse of the coefficient matrix, provided the inverse exists. This chapter studies how to test for the existence of the inverse and how to find that inverse.

5.1 Conditions for Nonsingularity of a Matrix

As was pointed out earlier, the squareness condition is necessary but not sufficient for the existence of the inverse A^{-1} of a matrix A.

Conditions for Nonsingularity

When the squareness condition is already met, a sufficient condition for the nonsingularity of a matrix is that its rows (or equivalently, its columns) are linearly independent. In fact, the necessary and sufficient conditions for nonsingularity are that the matrix satisfies the squareness and linear independence conditions.

An nxn coefficient matrix A can be considered an ordered set of row vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix}$$

where $v_i' = [a_{i1}, a_{i2}, \ldots, a_{in}]$, $i = 1, 2, \ldots, n$. For the rows to be linearly independent, for any set of scalars k_i, $\sum_{i=1}^{n} k_i v_i = 0$ if and only if $k_i = 0$ for all i.

Example: For a given matrix,

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 8 & 10 \end{bmatrix},$$

since $v'_3 = 2v'_1 + 0v'_2$, so the matrix is singular.

Rank of a Matrix

Even though the concept of row independence has been discussed only with regard to square matrices, it is equally applicable to any mxn rectangular matrix. If the maximum number of linearly independent rows that can be found in such a matrix is γ, the matrix is said to be of rank γ. The rank also tells us the maximum number of linearly independent columns in the said matrix. The rank of an mxn matrix can be at most m or n, whichever is smaller.

By definition, an $n \times n$ nonsingular matrix A has n linearly independent rows (or columns); consequently it must be of rank n. Conversely, an $n \times n$ matrix having rank n must be nonsingular.

5.2 Test of Nonsingularity by Use of Determinant

To determine whether a square matrix is nonsingular, we can make use of the concept of determinant.

Determinants and Nonsingularity

The determinant of a square matrix A, denoted by $|A|$, is a uniquely defined scalar associated with that matrix. Determinants are defined only for square matrices. For a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{22} & a_{22} \end{bmatrix}, \quad \text{its determinant is defined as follows:}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

In view of the dimension of matrix A, $|A|$ as defined in the above is called a second-order determinant.

Example: Given $A = \begin{bmatrix} 10 & 4 \\ 8 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}$, then

$$|A| = \begin{vmatrix} 10 & 4 \\ 8 & 5 \end{vmatrix} = 50 - 32 = 18$$

$$|B| = \begin{vmatrix} 3 & 5 \\ 0 & -1 \end{vmatrix} = -3 - 5x0 = -3$$

Example: $A = \begin{bmatrix} 2 & 6 \\ 8 & 24 \end{bmatrix}$

Then its determinant

$$|A| = \begin{vmatrix} 2 & 6 \\ 8 & 24 \end{vmatrix} = 2x24 - 6x8 = 48 - 48 = 0$$

This example shows that the determinant is equal to zero if its rows are linearly dependent. As will be seen, the value of a determinant $|A|$ can serve not only as a criterion for testing the linear independence of the rows (hence nonsingularity) of matrix A but also as an input in the calculation of the inverse A^{-1}, if it exists.

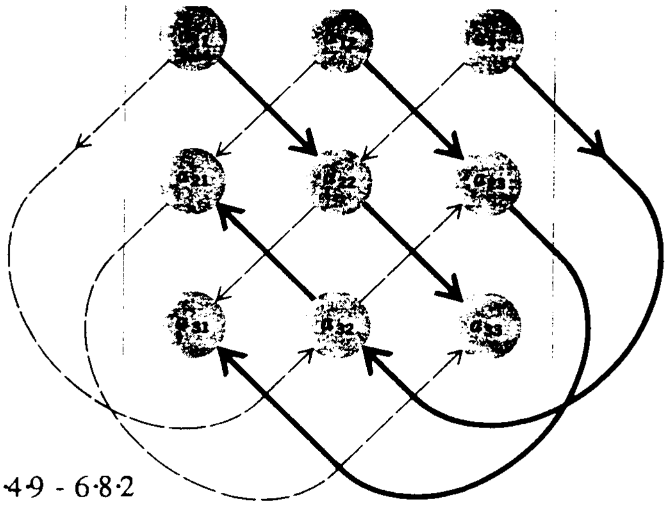
Evaluating a Third-Order Determinant

For a 3×3 matrix A, its determinants have the value

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$- a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

We can use the following diagram to calculate the third-order determinant.



Example:

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 2 \cdot 5 \cdot 9 + 1 \cdot 6 \cdot 7 + 4 \cdot 8 \cdot 3 - 3 \cdot 5 \cdot 7 - 1 \cdot 4 \cdot 9 - 6 \cdot 8 \cdot 2$$

$$= 90 + 42 + 96 - 105 - 36 - 96 = -9$$

This method of cross-diagonal multiplication provides a handy way of evaluating a third-order determinant, but unfortunately it is not applicable to determinants of orders higher than 3. For the latter, we must resort to the so-called "Laplace expansion" of the determinant.

Evaluating an nth-Order Determinant by Laplace Expansion

The minor of the element a_{ij} of a determinant $|A|$, denoted by $|M_{ij}|$ can be obtained by deleting the ith row and jth column of the determinant $|A|$. Since a minor is itself a determinant, it has a value. For example, for a 3x3 determinant $|A|$, the minors of a_{11}, a_{12}, and a_{13} are

$$|M_{11}| - \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad |M_{12}| - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad |M_{13}| - \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

A concept closely related to the minor is that of the cofactor. A cofactor, denoted by $|C_{ij}|$, is a minor with a prescribed algebraic sign attached to it. Formerly, it is defined by

$|C_{ij}| = (-1)^{i+j} |M_{ij}|$

Thus, if the sum of the two subscripts i and j in $|M_{ij}|$ is even, then $|C_{ij}| = |M_{ij}|$. If it is odd, then $|C_{ij}| = -|M_{ij}|$. Using these new concepts, we can express a third-order determinant as

$$
\begin{aligned}
|A| &= a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| \\
&= a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}|
\end{aligned}
$$

The Laplace expansion of a third-order determinant serves to reduce the evaluation problem to one of evaluating only certain second-order determinants. In general, the Laplace expansion of an nth-order determinant will reduce the problem to one of evaluating n cofactors, each of which is of the (n-1)st order, and the repeated application of the process will methodically lead to lower and lower orders of determinants, eventually culminating in the basic second-order determinants. Then the value of the original determinant can be easily calculated. Formerly, the value of a determinant $|A|$ of order n can be found by the Laplace expansion of <u>any row or any column</u> as follows:

$$|A| = \sum_{j=1}^{n} a_{ij}|C_{ij}| \qquad \text{[expansion by the ith row]}$$

$$= \sum_{i=1}^{n} a_{ij}|C_{ij}| \qquad \text{[expansion by the jth column].}$$

Even though one can expand $|A|$ by any row or any column, as the numerical calculation is concerned, a row or column with largest number of 0s or 1s is always preferable for this purpose, because a 0 times its cofactor is simply 0.

Example: For the $|A| = \begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{vmatrix}$,

the easiest way to expand the determinant is by the third column, which consists of the elements 1, 0, and 0. Thus,

$$|A| = 1(-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} = -6 - 21 = -27$$

5.3 Basic Properties of Determinants

Property I. The determinant of a matrix A has the same value as that of its transpose A′, i.e.,

$$|A| = |A'|.$$

Example:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$|A'| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - cb = |A|$$

Property II. The interchange of any two rows (or any two columns) will alter the sign, but not the numerical value of the determinant.

Example: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$ but the interchange of the two rows yields

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ab - cd).$$

Property III. The multiplication of any one row (or one column) by a scalar k will change the value of the determinant k-fold, i.e. for $|A|$,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k|A|$$

In contrast, the factoring of a matrix requires the presence of a common divisor for <u>all</u> its elements, as in

$$k \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

Property IV. The addition (subtraction) of a multiple of any row (or column) to (from) another row (or column) will leave the value of the determinant unaltered.

Example:

$$\begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = a(d+kb) - b(c+ka)$$

$$= ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Property V. If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Example:

$$\begin{vmatrix} ka & kb \\ a & b \end{vmatrix} = kab - kab = 0$$

Remark. Property V is a logic consequence of Property IV.

The basic properties just discussed are useful in several ways. For one thing, they can be of great help in simplifying the task of evaluating determinants. By subtracting multipliers of one row (or column) from another, for instance, the elements of the determinant may be reduced to much simpler and simpler numbers. If we can indeed apply these properties to transform some row or column into a form containing mostly 0s or 1s, Laplace expansion of the determinant will become a much more manageable task.

Determinantal Criterion for Nonsingularity

Our present concern is primarily to link the linear dependence of rows with the vanishing of a determinant. By Property I, we can easily see that row independence is equivalent to column independence.

Given a linear-equation system $Ax = d$, where A is an nxn coefficient matrix, we have

$$
\begin{aligned}
|A| \neq 0 &\Leftrightarrow A \text{ is row (or column) independent} \\
&\Leftrightarrow A \text{ is nonsingular} \\
&\Leftrightarrow A^{-1} \text{ exists} \\
&\Leftrightarrow \text{a unique solution } \tilde{x} = A^{-1}d \text{ exists.}
\end{aligned}
$$

Thus the value of the determinant of A provides a convenient criterion for testing the nonsingularity of matrix A and the existence of a unique solution to the equation system $Ax = d$.

Rank of a Matrix Redefined

The rank of a matrix A was earlier defined to be the maximum number of linearly independent rows in A. In view of the link between row independence and the nonvanishing of the determinant, we can redefine the rank of an mxn matrix as the maximum order of a nonvanishing determinant that can be constructed from the rows and columns of that matrix. The rank of any matrix is a unique number.

Obviously, the rank can at most be m or n for a mxn matrix A, whichever is smaller, because a determinant is defined only for a square matrix. Symbolically, this fact can be expressed as follows:

$$\gamma(A) \le \min\{m,n\}.$$

The rank of an nxn nonsingular matrix A must be n; in that case, we may write $\gamma(A) = n$. For the product of two matrix,

$$\gamma(AB) \le \min\{\gamma(A), \gamma(B)\}.$$

5.4 Finding the Inverse Matrix

If the matrix A in the linear-equation system $Ax = d$ is nonsingular, then A^{-1} exists, and the solution of the system will be $\bar{x} = A^{-1}d$. We have learned to test the nonsingularity of A by the criterion $|A| \neq 0$. The next question is how we can find the inverse A^{-1} if A does pass that test.

Expansion of a Determinant by Alien Cofactors

We have known that the value of a determinant $|A|$ of order n can be found by the Laplace expansion of any row or any column as follows:

$$|A| = \sum_{j=1}^{n} a_{ij}|C_{ij}| \qquad \text{[expansion by the ith row]}$$

$$= \sum_{i=1}^{n} a_{ij}|C_{ij}| \qquad \text{[expansion by the jth column]}$$

Now what happens if we replace a_{ij} by $a_{i'j}$ for $i \neq i'$ or by $a_{ij'}$, for $j \neq j'$. Then we have the following important property of determinants.

Property VI. The expansion of a determinant by alien cofactors (the cofactors of a "wrong" row or column) always yields a value of zero. That is, we have

$$\sum_{j=1}^{n} a_{i'j}|C_{ij}| = 0 \qquad (i \neq i') \quad \text{[expansion by i'th row and use of cofactors of ith row]} \qquad (5.10)$$

$$\sum_{i=1}^{n} a_{ij'} |C_{ij}| = 0 \qquad (j \neq j')$$ [expansion by j'th column and use of cofactors of jth column]

The reason for this outcome lies in the factor that the above formula can be considered as the results of the regular expansion by the ith row (jth column) of another determinant, which differs from $|A|$ only in its i'th row (j'th column) and its ith row (jth column) and i'th row (j'th column) are identical.

Example: For the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

consider another determinant

$$|A^*| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

If we expand $|A^*|$ by the second row, then we have

$$0 = |A^*| = a_{11}|C_{21}| + a_{12}|C_{22}| + a_{13}|C_{23}|$$
$$= \sum_{j=1}^{3} a_{1j} |C_{2j}|$$

Matrix Inversion

Property VI is of finding the inverse of a matrix. For a $n \times n$ matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

since each element of A has a cofactor $|C_{ij}|$, we can form a matrix of cofactors by replacing each element a_{ij} with its cofactor $|C_{ij}|$. Such a cofactor matrix $C = [\,|C_{ij}|\,]$ is also $n \times n$. For our present purpose, however, the transpose of C is of more interest. This transpose C′ is commonly referred to as the adjoint of A and is denoted by adj A. Thus:

$$C' \equiv adj\ A \equiv \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \vdots & \vdots & \vdots & \vdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix}$$

By utilizing the formula for the Laplace expansion and Property VI, we have

$$AC' = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}|C_{1j}| & \sum_{j=1}^{n} a_{1j}|C_{2j}| & \cdots & \sum_{j=1}^{n} a_{1j}|C_{nj}| \\ \sum_{j=1}^{n} a_{2j}|C_{1j}| & \sum_{j=1}^{n} a_{2j}|C_{2j}| & \cdots & \sum_{j=1}^{n} a_{2j}|C_{nj}| \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^{n} a_{nj}|C_{1j}| & \sum_{j=1}^{n} a_{nj}|C_{2j}| & \cdots & \sum_{j=1}^{n} a_{nj}|C_{nj}| \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= |A| \, I_n$$

Since $|A| \neq 0$, then

$$A \frac{C'}{|A|} = I.$$

Thus, by the uniqueness of A^{-1} of A, we know

$$A^{-1} = \frac{C'}{|A|} = \frac{\text{adj } A}{|A|}$$

Now we have found a way to invert the matrix A. The general procedures for finding the inverse of a square A are: (1) find $|A|$; (2) find the cofactors of all elements of A and form $C = [\,|C_{ij}|\,]$; (3) form C' to get adj A; and (4) determine A^{-1} by $\frac{\text{adj } A}{|A|}$.

In particular, for a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have the following formula:

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is a very useful formula.

Example: $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

The inverse of A is given by

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Example: Find the inverse of $B = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$.

Since $|B| = 99 \neq 0$, B^{-1} exists. The cofactor matrix is:

$$\begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 21 & 6 & 9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}$$

Then,

$$adj\ B = \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ 9 & 3 & 12 \end{bmatrix}$$

Hence,

$$B^{-1} = \frac{1}{99}\begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ 9 & 3 & 12 \end{bmatrix}$$

5.5 Cramer's Rule

The method of matrix inversion just discussed enables us to derive a convenient way of solving a linear-equation system, known as Cramer's rule.

<u>Derivation of the Rule</u>

Given an equation system Ax = d, the solution can be written as

$$\bar{x} = A^{-1}d = \frac{1}{|A|}(\text{adj } A)d$$

provided A is nonsingular. Thus,

$$\bar{x} = \frac{1}{|A|}\begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \vdots & \vdots & \vdots & \vdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} \sum_{i=1}^{n} d_i|C_{i1}| \\ \sum_{i=1}^{n} d_i|C_{i2}| \\ \vdots \\ \sum_{i=1}^{n} d_i|C_{in}| \end{bmatrix}$$

That is, the $\bar{x}_i$ is given by

$$\bar{x}_j = \frac{1}{|A|}\sum_{i=1}^{n} d_i |C_{ij}|$$

$$= \frac{1}{|A|}\begin{bmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{bmatrix}$$

$$\equiv \frac{1}{|A|}|A_j|$$

where $|A_j|$ is obtained by replacing the jth column of $|A|$ with the constant terms $d_1,\ldots,d_n$. This result is the statement of Cramer's rule.

Example:

$5x_1 + 3x_2 = 30$
$6x_1 - 2x_2 = 8$

$$|A| = \begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix} = -28$$

$$|A_1| = \begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix} = -84$$

$$|A_2| = \begin{vmatrix} 5 & 30 \\ 6 & 8 \end{vmatrix} = -140$$

Therefore, by Cramer's rule, we have

$$\bar{x}_1 = \frac{|A_1|}{|A|} = \frac{-84}{-28} = 3 \text{ and } \bar{x}_2 = \frac{|A_2|}{|A|} = \frac{-140}{-28} = 5.$$

Note on Homogeneous - Equation System

A linear-equation system $Ax = d$ is said to be a homogeneous-equation system if $d=0$, i.e. if $Ax = 0$. If $|A| \neq 0$, $\bar{x}=0$ is a unique solution of $Ax = 0$ since $\bar{x} = A^{-1}0 = 0$. This is a "trivial solution." Thus, the only way to get a nontrivial solution from the homogeneous-equation system is to have $|A| = 0$, i.e. A is singular. In this case, Cramer's rule is not applicable. Of course, this does not mean that we cannot obtain solutions; it means only that we cannot get a unique solution. In fact, it has an infinite number of solutions.

Example:

$$a_{11}x_1 + a_{12}x_2 = 0$$
$$a_{21}x_2 + a_{22}x_2 = 0$$

If $|A| = 0$, then its rows are linearly dependent. As a result, one of two equations is redundant. By deleting, say, the second equation, we end up with one equation with two variables. The solutions are

$$\bar{x}_1 = -\frac{a_{12}}{a_{11}} x_2 \qquad \text{if } a_{11} \neq 0$$

Solution Outcomes for $Ax = d$.

<table>
<tr><th colspan="2">d / |A|</th><th>$d \neq 0$</th><th>$d = 0$</th></tr>
<tr><td colspan="2">$|A| \neq 0$</td><td>The solution is unique and $\bar{x} \neq 0$</td><td>The solution is unique and $x = 0$</td></tr>
<tr><td rowspan="2">$|A| = 0$</td><td>Equations dependent</td><td>An infinite number of solutions and $\bar{x} \neq 0$</td><td>There is an infinite number of solutions</td></tr>
<tr><td>Equations inconsistent</td><td>No solution exists</td><td>[Not applicable]</td></tr>
</table>

5.6 Application to Market and National-Income Models

Market Model

The two-commodity model described in Chapter 3 can be written as follows:

$$c_1P_1 + c_2P_2 = -C_0$$
$$\gamma_1P_1 + \gamma_2P_2 = -\gamma_2.$$

Thus $|A| = \begin{vmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{vmatrix} = c_1\gamma_2 - c_2\gamma_1$

$$|A_1| = \begin{vmatrix} -c_0 & c_2 \\ -\gamma_0 & \gamma_2 \end{vmatrix} = -c_0\gamma_2 + c_2\gamma_0$$

$$|A_2| = \begin{vmatrix} c_1 & -c_0 \\ \gamma_1 & -\gamma_0 \end{vmatrix} = c_1\gamma_0 + c_0\gamma_1$$

Thus the equilibrium is given by

$$\bar{P}_1 = \frac{|A_1|}{|A|} = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1} \text{ and } \bar{P}_2 = \frac{|A_2|}{|A|} = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1}$$

National-Income Model

$$Y = c + I_0 + G_0$$
$$c = a + bY \qquad (a>0,\ 0<b<1)$$

These can be rearranged into the form

$$Y - c = I_0 + G_0$$
$$-bY + c = a$$

While we can solve $\bar{Y}$ and $\bar{c}$ by Cramer's rule, here we solve this model by inverting the coefficient matrix.

$$\text{Since } A = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}, \quad \text{then } A^{-1} = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}$$

Hence

$$\begin{bmatrix} \bar{Y} \\ \bar{c} \end{bmatrix} = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}\begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$
$$= \frac{1}{1-b}\begin{bmatrix} I_0 + G_0 + a \\ b(I_0 + G_0) + a \end{bmatrix}$$

VI. COMPARATIVE STATICS AND THE CONCEPT OF DERIVATIVE

6.1 The Nature of Comparative Statics

Comparative statics is concerned with the comparison of different equilibrium states that are associated with different sets of values of parameters and exogeneous variables. When the value of some parameter or exogeneous variable that is associated with an initial equilibrium changes, we can get a new equilibrium. Then the question posed in the comparative-static analysis is: How would the new equilibrium compare with the old?

It should be noted that in the comparative-statics analysis we don't concern with the process of adjustment of the variables; we merely compare the initial (prechange) equilibrium state with the final (postchange) equilibrium state. We also preclude the possibility of instability of equilibrium for we assume the equilibrium to be attainable.

It should be clear that the problem under consideration is essentially one of finding a rate of change: the rate of change of the equilibrium value of an endogenous variable with respect to the change in a particular parameter or exogeneous variable. For this reason, the mathematical concept of derivative takes on preponderant significance in comparative statics.

6.2 Rate of Change and the Derivative

We want to study the rate of change of any variable y in response to a change in another variable x, where the two variables are related to each other by the function

$$y = f(x)$$

Applied in the comparative static context, the variable y will represent the equilibrium value of an endogeneous variable, and x will be some parameter.

<u>The Difference Quotient</u>

We use the symbol Δ to denote the change from one point, say x_0, to another point, say x_1. Thus $\Delta x = x_1 - x_0$. When x changes from an initial value x_0 to a new value $(x_0 + \Delta x)$, the value of the function $y=f(x)$ changes from $f(x_0)$ to $f(x_0 + \Delta x)$. The change in y per unit of change in x can be represented by the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Example: $y = f(x) = 3x^2 - 4$

Then $f(x_0) = 3x_0^2 - 4$ $\quad f(x_0 + \Delta x) = 3(x_0 + \Delta x)^2 - 4$
and thus,

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$= \frac{3(x_0 + \Delta x)^2 - 4 - (3x_0^2 - 4)}{\Delta x} = \frac{6x_0\Delta x + 3(\Delta x)^2}{\Delta x}$$

$$= 6x_0 + 3\Delta x$$

The Derivative

Frequently, we are interested in the rate of change of y when Δx is very small. In particular, we want to know the rate of $\Delta y/\Delta x$ when Δx approaches to zero. If, as $\Delta x \rightarrow 0$, the limit of the difference quotient $\Delta y/\Delta x$ exists, that limit is called the derivative of the function $y=f(x)$, and the derivative is denoted by

$$\frac{dy}{dx} \equiv y' \equiv f'(x) \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Remark. Several points should be noted about the derivative: (1) a derivative is a function. Whereas the difference quotient is a function of x_0 and Δx, the derivative is a function of x_0 only; and (2) since the derivative is merely a limit of the difference quotient, it must also be of necessity a measure of some rate of change. Since $\Delta x \rightarrow 0$, the rate measured by the derivative is in the nature of an instantaneous rate of change.

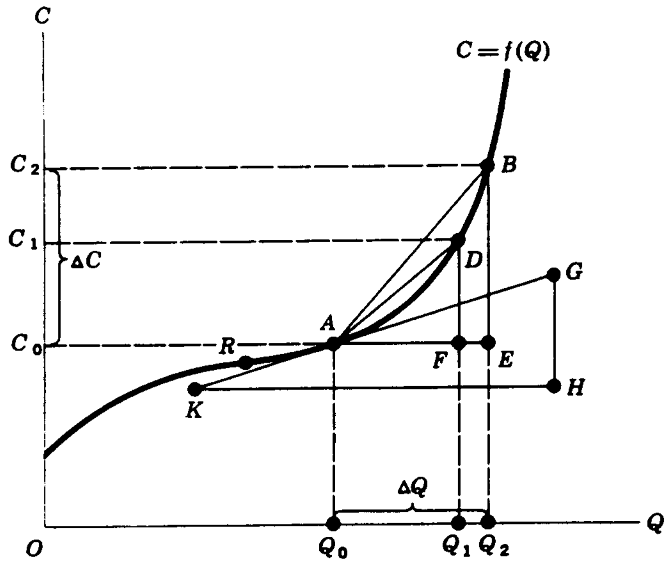
Example: Referring to the function $y=3x^2 - 4$ again. Since

$$\frac{\Delta y}{\Delta x} = 6x + 3\Delta x,$$

we have $\frac{dy}{dx} = 6x$.

6.3 The Derivative and the Slope of a Curve

Elementary economics tells us that, given a total-cost function $C=f(Q)$, where C is the total cost and Q is the output, the marginal cost MC is defined as $MC = \Delta C/\Delta Q$. It is understood that ΔQ is an extremely small change. For the case of a product whose quantity is a continuous variable, ΔQ will refer to an infinitesimal change. It is well known that MC can be measured by the slope of the total cost curve. But the slope of the total-cost curve is nothing but the limit of the ratio $\Delta C/\Delta Q$ as $\Delta Q \to 0$. Thus the concept of the slope of a curve is merely the geometric counterpart of the concept of the derivative.



6.4 The Concept of Limit

In the above, we have defined the derivative of a function $y=f(x)$ as the limit of $\Delta y/\Delta x$ as $\Delta x \to 0$. We now study the concept of limit. For a given function $q=q(v)$, the concept of limit is concerned with the question: What value does q approach as v approaches a specific value? That is, as $v \to N$ (the N can be any number, say $N=0$, $N=+\infty$, $-\infty$), what happens to $\lim_{v \to N} g(v)$.

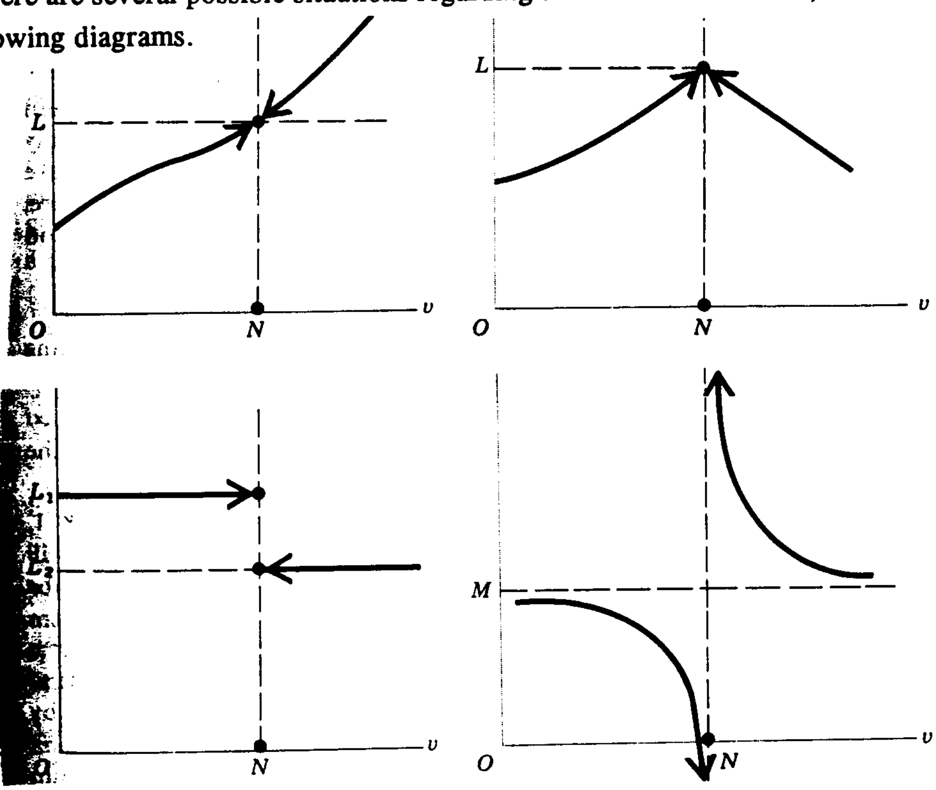
When we say $v \to N$, the variable v can approach the number N either from values greater than N, or from values less than N. If, as $v \to N$ from the left side (from values less than N), q approaches a finite number L, we call L the left-side limit of q. Similarly, we call L the right-side limit of q if $v \to N$ from the right side. The left-side limit and right-side limit of q are denoted by $\lim_{v \to N^-} q$ and $\lim_{v \to N^+} q$, respectively. The limit of q at N is said to exist if

$$\lim_{v \to N^-} q = \lim_{v \to N^+} q$$

and is denoted by $\lim_{v \to N} q$. Note that L must be a finite number. If we have the situation of $\lim_{v \to N} q = \infty$ (or $-\infty$), we shall consider q to possess no limit or an "infinite limit." It is important to realize that the symbol ∞ is not a number, and therefore it cannot be subjected to the usual algebraic operations.

Graphical Illustrations

There are several possible situations regarding the limit of a function, which are shown in the following diagrams.



Evaluation of a Limit

Let us now illustrate the algebraic evaluation of a limit of a given function $q=g(v)$.

Example: Given $q = 2 + v^2$, find $\lim_{v\to 0} q$. It is clear that $\lim_{v\to N^-} q$ and $\lim_{v\to N^+} q$ and $v^2 \to 0$ as $v \to 0$. Thus $\lim_{v\to 0} q = 2$.

Note that, in evaluating $\lim_{v\to N} q$, we only let v tends N but, as a rule, do not let $v=N$. Indeed, sometimes N even is not in the domain of the function $q=g(v)$.

Example: $q = (1-v^2)/(1-v)$. For this function, $N=1$ is not in the domain of the function, and we cannot set $v=1$ since it would involve division by zero. Moreover, even the limit-evaluation procedure of letting v-1 will cuase difficulty since $(1-v) \to 0$ as $v \to 1$.

One way out of this difficulty is to try to transform the given ratio to a form in which v will not appear in the denominator. Since

$$q = \frac{1-v^2}{1-v} = \frac{(1-v)(1+v)}{(1-v)} = 1+v \qquad (v \neq 1).$$

and $v \to 1$ implies $v \neq 1$ and $(1+v) \to 2$ as $v \to 1$, we have $\lim_{v\to 1} q = 2$.

Example: Find $\lim_{V\to\infty} q$ with $q = \frac{2v+5}{v+1}$

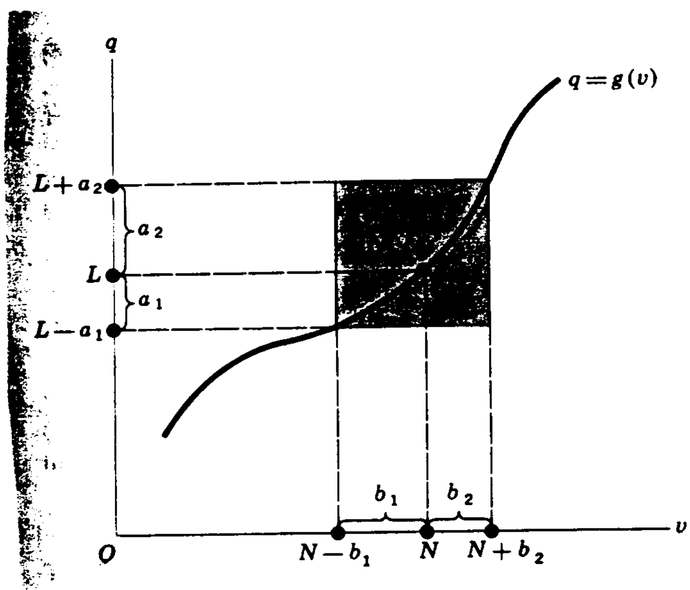
since $\frac{(2v+5)}{v+1} = \frac{2(v+2)+1}{v+1} = 2 + \frac{1}{v+1}$

and $\lim_{v\to\infty} \frac{1}{v+1} = 0$, so $\lim_{v\to\infty} q = 2$.

Formal View of the Limit Concept

Definition of the Limit: The number L is said to be the limit of q=g(v) as v approaches N if, for every neighborhood of L, there can be found a corresponding neighborhood of N (excluding the point v=N) in the domain of the function such that, for every value of v in that N-neighborhood, its image lies in the chosen L-neighborhood. Here a neighborhood of a point L is an open interval defined by

$$(L-a_1, L+a_2) = \{q \mid L-a_1 < q < L+a_2\} \text{ for } a_1 > a_2 > 0$$



6.5 Inequalities and Absolute Values

Rules of Inequalities

Transitivity: a > b and b > c implies a > c
a ≥ b and b ≥ c implies a ≥ c

Addition and Subtraction:

$$a > b \Rightarrow a \pm k > b \pm k$$
$$a \geq b \quad\quad a \pm k \geq b \pm k$$

Multiplication and Division:

$$a > b \Rightarrow \begin{cases} ka > kb & (k > 0) \\ ka < kb & (k < 0) \end{cases}$$

Squaring: $a > b$ with $b \geq 0 \Rightarrow a^2 > b^2$

Absolute Values and Inequalities

For any real number n, the absolute value of n is defined and denoted by

$$|n| \equiv \begin{cases} n & \text{if } n > 0 \\ -n & \text{if } n < 0 \\ 0 & \text{if } n - 0 \end{cases}$$

Thus we can write $|x| < n$ as an equivalent way $-n < x < n$ $(n > 0)$. Also $|x| \leq n$ if and only if $-n \leq x \leq n$ $(n \geq 0)$.

The following properties characterize absolute values:

i) $|m| + |n| \geq |m + n|$

ii) $|m| \cdot |n| = |m \cdot n|$

iii) $\frac{|m|}{|n|} - \left|\frac{m}{n}\right|$

Solution of an Inequality

Example: Find the solution of the inequality $3x - 3 > x + 1$. By adding (3-x) to both sides, we have

$$3x - 3 + 3 - x > x + 1 + 3 - x.$$

Thus, $2x > 4$ so $x > 2$.

Example: Solve the inequality $|1-x| \leq 3$. From $|1-x| \leq 3$, we have $-3 \leq 1-x \leq 3$, or $-4 \leq -x \leq 2$. Thus $4 \geq x \geq -2$, i.e. $-2 \leq x \leq 4$.

6.6 Limit Theorems

Theorems Involving a Single Equation

Theorem I: If $q = av + b$, then $\lim_{v \to N} q = aN + b$

Theorem II: If $q = g(v) = b$, then $\lim_{v \to N} q = b$

Theorem III: $\lim_{v \to N} v^k = N^k$

Example: Given $q = 5v + 7$, then $\lim_{v \to 2} q = 5 \cdot 2 + 7 = 17$

Example: $q = v^3$ Find $\lim_{v \to 2} q$

By Theorem III, we have $\lim_{v \to 2} q = 2^3 = 8$.

Theorems Involving Two Functions

For two functions $q_1 = g(v)$ and $q_2 = h(v)$, if $\lim_{v \to N} q_1 = L_1$ and $\lim_{v \to N} q_2 = L_2$, then we have the following theorems:

Theorem IV: $\lim_{v \to N} (q_1 + q_2) = L_1 + L_2$

Theorem V: $\lim_{v \to N} (q_1 q_2) = L_1 L_2$

Theorem VI: $\lim_{v \to N} \frac{q_1}{q_2} = \frac{L_1}{L_2} \quad (L_2 \neq 0)$

Example: Find $\lim_{v \to 0} \frac{1+v}{2+v}$.

Since $\lim_{v \to 0} (1+v) = 1$ and $\lim_{v \to 0} (2+v) = 2$, so

$$\lim_{v \to 0} \frac{1+v}{2+v} = \frac{1}{2}$$

Remark. Note that L_1 and L_2 represent finite numbers; otherwise theorems do not apply.

Limit of a Polynomial Function

$$\lim_{v \to N} (a_0 + a_1 v + a_2 v^2 + \ldots + a_n v^n)$$
$$= a_0 + a_1 N + a_2 N^2 + \ldots + a_n N^n$$

6.7 Continuity and Differentiability of a Function

Continuity of a Function

A function $q=g(v)$ is said to be continuous at N if $\lim_{v \to N} q$ exists and $\lim_{v \to N} g(v) = g(N)$.

Thus the term continuous involves no less than three requirements: (1) the point N must be in the domain of the function; (2) $\lim_{v \to N} g(v)$ exists; and (3) $\lim_{v \to N} g(v) = g(N)$.

Remark: It is important to note that while--in discussing the limit of a function--the point (N,L) was excluded from consideration, we are no longer excluding it in the present context. Rather, as the third requirement specifically states, the point (N,L) must be on the graph of the function before the function can be considered as continuous at point N.

Polynomial and Rational Functions

From the discussion of the limit of polynomial function, we know that the limit exists and equals the value of the function at N. Since N is a point in the domain of the function, we can conclude that any polynomial function is continuous in its domain. By those theorems involving two functions, we also know any rational function is continuous in its domain.

Example: $q = \frac{4v^2}{v^2+1}$

Then $\lim_{v \to N} \frac{4v^2}{v^2+1} = \frac{\lim_{v \to N} 4v^2}{\lim_{v \to N} (v^2+1)} = \frac{4N^2}{N^2+1}$

Example: The rational function

$$q = \frac{v^3+v^2-4v-4}{v^2-4}$$

is not defined at v=2 and v=-2. Since v=2,-2 are not in the domain, the function is discontinuous at v=2 and v=-2, despite the fact that its limit exists as $v \to 2$ or -2.

Differentiability of a Function

By the definition of the derivative of a function y = f(x), at x_0 we know that $f'(x_0)$ exists if and only if the lim of $\Delta y/\Delta x$ exists at $x=x_0$ as $\Delta x \to 0$, i.e.,

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\equiv \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \quad \text{[differentiability condition]}$$

On the other hand, the function y = f(x) is continuous at x_0 if and only if

$$\lim_{x \to x_0} f(x) = f(x_0) \quad \text{[continuity condition]}.$$

We want to know what is the relationship between the continuity and differentiability of a function. Now we show that the continuity of f is a <u>necessary</u> condition for its differentiability. But this is not sufficient.

Since the notation $x \to x_0$ implies $x \neq x_0$, so $x - x_0$ is a nonzero number, it is permissible to write the following identity:

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0} (x - x_0)$$

Taking the limit of each side of the above equation as $x \to x_0$ yields the following results:

$$\text{Left side} = \lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} f(x) - f(x_0)$$

$$\text{Right side} \quad = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0)$$

$$= f'(x_0)[\lim_{x \to x_0} x - x_0]$$

$$= f'(x_0)(x_0 - x_0) = 0$$

Thus $\lim_{x \to x_0} f(x) - f(x_0) = 0$. So $\lim_{x \to x_0} f(x) = f(x_0)$ which means f(x) is continuous at $x = x_0$.

Although differentiability implies continuity, the converse is not true. That is, continuity is a necessary, but not sufficient, condition for differentiability. The following example shows this fact.

Example: $f(x) = |x|$.

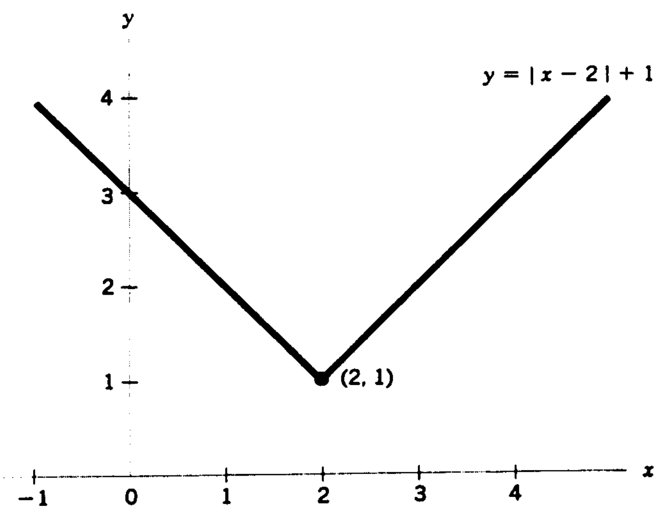
This function is clearly continuous at $x=0$. Now we show that it is not differentiable at $x=0$. This involves the demonstration of a disparity between the left-side and the right-side limits. Since, in considering the right-side limit $X>0$, thus

$$\lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

On the other hand, in considering the left-side limit, $x<0$; thus

$$\lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x}$$
$$= \lim_{x \to 0^-} -1 = -1$$

Thus, $\lim_{x \to 0^-} \frac{\Delta y}{\Delta x}$ does not exist, this means the derivative of $y = |x|$ does not exist at $x=0$.



VII. RULES OF DIFFERENTIATION AND THEIR USE IN COMPARATIVE STATICS

The central problem of comparative-static analysis, that of finding a rate of change, can be identified with the problem of finding the derivative of some function $y=f(x)$, provided only a small change in x is being considered. Before going into comparative-static models, we begin some rules of differentiation.

7.1 Rules of Differentiation for a Function of One Variable

Constant-Function Rule

If $y=f(x)=c$, where c is a constant, then

$$\frac{dy}{dx} \equiv y' \equiv f' = 0.$$

Proof. $\frac{dy}{dx} = \lim_{x' \to x} \frac{f(x') - f(x)}{x' - x} = \lim_{x' \to x} \frac{c - c}{x' - x} = \lim_{x' \to x} = 0.$

We can also write $\frac{dy}{dx} = \frac{df}{dx}$ as

$$\frac{d}{dx}y = \frac{d}{dx}f.$$

So we may consider d/dx as an operator symbol.

Power-Function Rule

If $y=f(x)=x^a$ where a is any real number $-\infty < a < \infty$,

$$\frac{d}{dx}f(x) = ax^{a-1}.$$

Remark. (i) If a = 0, then $\frac{d}{dx}x^0 = \frac{d}{dx}1 = 0$

(ii) If a = 1, then y = x. Thus $\frac{dx}{dx} = 1$.

For simplicity, we prove this rule only for the case where a=n, where n is any positive integer. since

$$x^n - x_0^n = (x-x_0)(x^{n-1} + x_0x^{n-2} + x_0^2x^{n-3} + \ldots + x_0^{n-1}),$$

then

$$\frac{x^n - x_0^n}{x - x_0} = x^{n-1} + x_0x^{n-2} + x_0^2x^{n-3} + \ldots + x_0^{n-1}$$

Thus,

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} \\ &= \lim_{x \to x_0} [x^{n-1} + x_0x^{n-2} + x_0^2x^{n-3} + \ldots + x_0^{n-1}] \\ &= x_0^{n-1} + x_0^{n-1} + x_0^{n-1} + \ldots + x_0^{n-1} \\ &= nx_0^{n-1} \end{aligned}$$

Example: Suppose $y=f(x)=x^{-3}$. Then $y' = -3x^{-4}$.

Example: Suppose $y = f(x) = \sqrt{x}$. Then $y' = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}$. In particular, we can know that

$$f'(2) = \frac{1}{2}\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

Power-Function Rule Generalized

If the function is given by $y = cx^a$, then $\frac{dy}{dx} = \frac{df}{dx} = acx^{a-1}$.

Example: Suppose $y=2x$. Then $\frac{dy}{dx} = 2x^{1-1} = 2x^0 = 2$

Example: Suppose $y=4x^3$. Then $\frac{dy}{dx} = 4 \cdot 3x^{3-1} = 12x^2$

Example: Suppose $y=3x^{-2}$. Then $\frac{dy}{dx} = -6x^{-2-1} = -6x^{-3}$.

7.2 Rules of Differentiation Involving Two or More Functions of the Same Variable

Let f(x) and g(x) be two differentiable functions. We have the following rules:

Sum-Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$$

This rule can easily be extended to more functions

$$\frac{d}{dx}\left[\sum_{i=1}^{n} f_i(x)\right] = \sum_{i=1}^{n} \frac{d}{dx} f_i(x) = \sum_{i=1}^{n} f_i'(x).$$

Example: $\frac{d}{dx}[ax^2 + bx + c] = 2ax + b$

Example: Suppose a short-run total-cost function is given by $c = Q^3 - 4Q^2 + 10Q + 75$. Then the marginal-cost function is the limit of the quotient $\Delta C/\Delta Q$, or the derivative of the C function:

$$\frac{dC}{dQ} = 3Q^2 - 4Q + 10.$$

In general, if a primitive function y=f(x) represents a total function, then the derivative function dy/dx is its marginal function. Since the derivative of a function is the slope of its curve, the marginal function should show the slope of the curve of the total function at each point x. Sometimes, we say a function is smooth if its derivative is continuous.

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$
$$= f(x)g'(x) + g(x)f'(x)$$

Proof.

$$\frac{d}{dx}[f(x_0)g(x_0)] = \lim_{x\to x_0}\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0}$$
$$= \lim_{x\to x_0}\frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0}$$
$$= \lim_{x\to x_0}\frac{f(x)[g(x) - g(x_0)] + g(x_0)[f(x) - f(x_0)]}{x - x_0}$$
$$= \lim_{x\to x_0} f(x)\frac{g(x) - g(x_0)}{x - x_0} + \lim_{x\to x_0} g(x_0)\frac{[f(x) - f(x_0)]}{x - x_0}$$
$$= f(x_0)g'(x_0) + g(x_0)f'(x_0)$$

Since this is true for any $x=x_0$, this proves the rule.

Example: Suppose y = (2x + 3)(3x²). Let f(x) = 2x + 3 and g(x) = 3x². Then f'(x) = 2 g'(x) = 6x. Hence,

$$\frac{d}{dx}[(2x+3)(3x^2)] = (2x+3)6x + 3x^2\cdot 2$$
$$= 12x^2 + 18x + 6x^2$$
$$= 18x^2 + 18x$$

As an extension of the rule to the case of three functions, we have

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

<u>Finding Marginal-Revenue Function from Average-Revenue Function</u>

Suppose the average-revenue (AR) function is specified by

$$AR = 15 - Q.$$

Then the total-revenue (TR) function is

$$TR \equiv AR \cdot Q = 15Q - Q^2.$$

Thus, the marginal-revenue (MR) function is given by

$$MR \equiv \frac{d}{dQ}TR = 15 - 2Q.$$

In general, if $AR = f(Q)$, then

$TR \equiv AR \cdot Q = Qf(Q)$. Thus

$$MR \equiv \frac{d}{dQ}TR = f(Q) + Qf'(Q).$$

From this, we can tell relationship between MR and AR. Since

$$MR - AR = [f(Q) + Qf'(Q)] - f(Q) = Qf'(Q),$$

they will always differ the amount of $Qf'(Q)$. Also, since

$$AR \equiv \frac{TR}{Q} = \frac{PQ}{Q} = P,$$

we can view AR as the inverse demand function for the product of the firm. If the market is perfectly competitive, i.e., the firm takes the price as given, then $P=f(Q)=$constant. Hence $f'(Q) = 0$. Thus MR - AR = 0 or MR = AR. Under imperfect competition, on the other hand, the AR curve is normally downward-sloping, so that $f'(Q) < 0$. Thus MR < AR.

<u>Quotient Rule</u>

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

We will come back to prove this rule after learning the <u>chain rule</u>.

Example:

(i) $$\frac{d}{dx}\left[\frac{2x-3}{x+1}\right] = \frac{2(x+1)-(2x-3)(1)}{(x+1)^2} = \frac{5}{(x+1)^2}$$

(ii) $$\frac{d}{dx}\left[\frac{5x}{x^2+1}\right] = \frac{5(x^2+1)-5x(2x)}{(x^2+1)^2} = \frac{5(1-x^2)}{(x^2+1)^2}$$

(iii) $$\frac{d}{dx}\left[\frac{ax^2+b}{cx}\right] = \frac{2ax(cx)-(ax^2+b)c}{(cx)^2} = \frac{c(ax^2-b)}{(cx)^2} = \frac{ax^2-b}{cx^2}$$

<u>Relationship Between Marginal-Cost and Average-Cost Functions</u>

As an economic application of the quotient rule, let us consider the rate of change of average cost when output varies.

Given a total cost function $C = C(Q)$, the average cost (AC) function and the marginal-cost (MC) function are given by

$$AC \equiv \frac{C(Q)}{Q} \qquad (Q > 0)$$

and

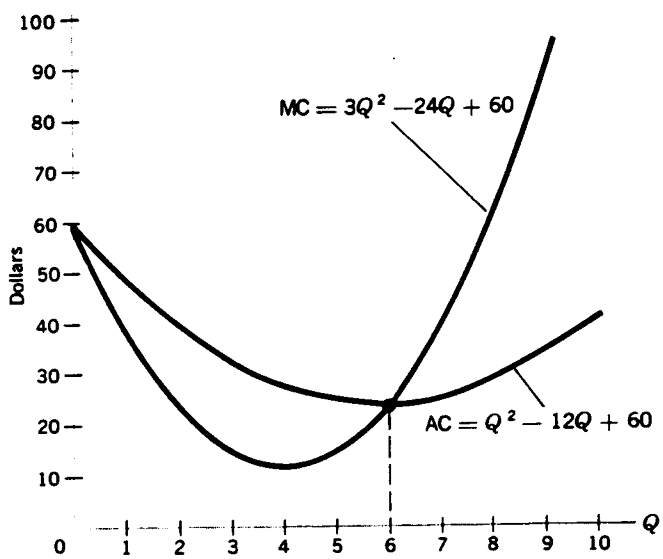
$$MC = C'(Q).$$

The rate of change of AC with respect to Q can be found by differentiating AC:

$$\frac{d}{dQ}\left[\frac{C(Q)}{Q}\right] = \frac{C'(Q)Q - C(Q)}{Q^2} = \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right].$$

From this it follows that, for Q > 0.

$$\frac{d}{dQ} AC \gtreqless 0 \text{ iff } MC(Q) \gtreqless AC(Q).$$



7.3 Rules of Differentiation Involving Functions of Different Variables

Now we consider cases where there are two or more differentiable functions, each of which has a distinct independent variable.

Chain Rule

If we have a function $z=f(y)$, where y is in turn a function of another variable x, say, $y=g(x)$, then the derivative of z with respect to x is given by

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y)g'(x). \qquad \text{[Chain Rule]}$$

The chain rule appeals easily to intuition. Given a Δx, there must result a corresponding Δy via the function $y=g(x)$, but this Δy will in turn being about a Δz via the function $z=f(y)$.

Proof. Note that

$$\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x}.$$

Since $\Delta x \to 0$ implies $\Delta y \to 0$, we have

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y)g'(x). \qquad \text{Q.E.D.}$$

In view of the function $y=g(x)$, we can express the function $z=f(y)$ as $z=f(g(x))$, where the contiguous appearance of the two function symbols f and g indicates that this is a compose function (function of a function). So sometimes, the chain rule is also called the composite-function rule. As an application of this rule, we use it to prove the quotient rule.

Let $z=y^{-1}$ and $y=g(x)$. Then $z = g^{-1}(x) = 1/g(x)$ and

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = -\frac{1}{y^2}g(x) = -\frac{g(x)}{g^2(x)}$$

Thus,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] - \frac{d}{dx}[f(x) \cdot g^{-1}(x)]$$

$$- f'(x)g^{-1}(x) + f(x) \frac{d}{dx} [g^{-1}(x)]$$

$$- f'(x)g^{-1}(x) + f(x) \left[-\frac{g(x)}{g^2(x)} \right]$$

$$- \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \qquad \text{Q.E.D.}$$

Example: If $z = 3y^2$ and $y = 2x + 5$, then

$$\frac{dz}{dx} - \frac{dz}{dy}\frac{dy}{dz} - 6y(2) - 12y - 12(2x + 5)$$

Example: If $z = y - 3$ and $y = y = x^3$, then

$$\frac{dz}{dx} - \frac{dz}{dy}\frac{dy}{dz} - 1 \cdot 3x^2 - 3x^2$$

Example: $z = (x^2 + 3x - 2)^{17}$. Let $z = y^{17}$ and $y = x^2 + 3x - 2$. Then

$$\frac{dz}{dx} - \frac{dz}{dy}\frac{dy}{dx} - 17y^{16} \cdot (2x+3)$$

$$- 17(x^2+3x-2)(2x+3).$$

Example: Suppose $TR = f(Q)$, where output Q is a function of labor input L, or $Q = g(L)$. Then, by the chain rule, the marginal product of labor (MP_L) is

$$MP_L - \frac{dR}{dL} - \frac{dR}{dQ}\frac{dQ}{dL} - f'(Q)g'(L)$$

$$- MR \cdot MP_L$$

Thus the result shown above constitutes the mathematical statement of the well-known result in economics that

$MP_L = MR \cdot MP_L$

Inverse-Function Rule

If the function y=f(x) represents a one-to-one mapping, i.e., if the function is such that a different value of x will always yield a different value of y, then function f will have an inverse function $x = f^{-1}(y)$. Here, the symbol f^{-1} is a function symbol which signifies a function related to the function f; it does not mean the reciprocal of the function f(x). When x and y refer specifically to numbers, the property of one-to-one mapping is seen to be unique to the class of function known as monotonic function. A function f is said to be monotonically increasing (decreasing) if $x_1 > x_2$ implies $f(x_1) > f(x_2)$ $[f(x_1) < f(x_2)]$. In either of these cases, an inverse function f^{-1} exists.

A practical way of ascertaining the monotonicity of a given function y=f(x) is to check whether the f′(x) always adheres to the same algebraic sign for all values. Geometrically, this means that its slope is either always upward or always downward.

Example: Suppose y = 5x + 25. Since y′ = 5 for all x, the function is monotonic and thus the inverse function exists. In fact, it is given by x = 1/5 y - 5.

Generally speaking, if an inverse function exists, the original and the inverse functions must be both monotonic. Moreover, if f^{-1} is the inverse function of f, then f must be the inverse function of f^{-1}. For inverse functions, the rule of differentiation is

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Proof. $$\frac{dx}{dy} = \lim_{y \to y_0} \frac{\Delta x}{\Delta y} = \lim_{y \to y_0} \frac{1}{\frac{\Delta y}{\Delta x}} = \frac{1}{\lim_{x \to x_0} \frac{\Delta y}{\Delta x}} = \frac{1}{y'} \quad \text{Q.E.D.}$$

Example: Suppose $y = x^5 + x$. Since $y' = 5x^4 + 1$, so $\frac{dx}{dy} = \frac{1}{5x^4+1}$

7.4 Partial Differentiation

So far, we have considered only the derivative of functions of a single independent variable. In comparative-static analysis, however, we are likely to encounter the situation in which several parameters appear in a model, so that the equilibrium value of each endogeneous variable may be a function of more than one parameter. Because of this, we now consider the derivative of a function of more than one variable.

Partial Derivatives

Consider a function

$$y = f(x_1, x_2, \ldots, x_n),$$

where the variables $x_i (i=1,2,\ldots,n)$ are all independent of one another, so that each can vary by itself without affecting the others. If the variable x_i changes Δx_i while the other variables remain fixed, there will be a corresponding change in y, namely, Δy. The difference quotient in this case can be expressed as

$$\frac{\Delta y}{\Delta x_i} = \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + \Delta x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{\Delta x_i}.$$

If we take the limit of $\Delta y/\Delta x_i$, that limit will constitute a derivative. We call it the partial derivative of y with respect to x_i. The process of taking partial derivatives is called partial differentiation. Denote the partial derivative of y with respect to x_i by $\frac{\partial y}{\partial x_i}$, i.e.

$$\frac{\partial y}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{\Delta y}{\Delta x_i}.$$

Also we can use f_i to denote $\partial y/\partial x_i$. If the function happens to be written in terms of unsubscripted variables, such as $y=f(u,v,w)$, one also uses, f_u, f_v, f_w to denote the partial derivatives.

Techniques of Partial Differentiation

Partial differentiation differs from the previously discussed differentiation primarily in that we must hold the other independent variables constant while allowing one variable to vary.

Example: Suppose $y = f(x_1,x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$. Find $\partial y/\partial x_1$ and $\partial y/\partial x_2$.

$$\frac{\partial y}{\partial x_1} \equiv \frac{\partial f}{\partial x_1} = 6x_1^2 + x_2$$

$$\frac{\partial y}{\partial x_2} \equiv \frac{\partial f}{\partial x_2} = x_1 + 8x_2^2$$

Example: For $y = f(u,v) = (u+4)(3u+2v)$, we have

$$\frac{\partial y}{\partial u} \equiv f_u = (3u+2v)+(u+4)\cdot 3$$

$$= 6u+2v+12$$

$$= 2(3u+v+6)$$

$$\frac{\partial y}{\partial v} \equiv f_v = 2(u+4)$$

When $u=2$ and $v=1$, then $f_u(2,1) = 2x13 = 26$ and $f_v(2,1) = 2x6 = 12$.

Example: Given $y = (3u - 2v)/(u^2 + 3v)$,

$$\frac{\partial y}{\partial u} = \frac{3(u^2+3v)-(3u-2v)(2u)}{(u^2+3v)^2} = \frac{-3u^2+4uv+9v}{(u^2+3v)^2}$$

$$\frac{\partial y}{\partial v} = \frac{-2(u^2+3v)-(3u-2v)\cdot 3}{(u^2+3v)^2} = \frac{-u(2u+9)}{(u^2+3v)^2}$$

7.5 Applications to Comparative-Static Analysis

Equipped with the knowledge of the various rules of differentiation, we can at last tackle the problem posed in comparative-static analysis: namely, how the equilibrium value of an endogeneous variable will change when there is a change in any of the exogeneous variables or parameters.

<u>Market Model</u>

For the one-commodity market model:

$$Q_d = a - bp \qquad (a,b > 0)$$
$$Q_s = c + dp \qquad (c,d > 0),$$

the equilibrium price and quantity are given by

$$\bar{P} - \frac{a+c}{b+d}$$

$$\bar{Q} - \frac{ad-bc}{b+d}$$

These solutions will be referred to as being in the reduced form: the two endogeneous variables have been reduced to explicit expressions of the four independent parameters, a, b, c, and d.

To find how an infinitesmal change in one of the parameters will affect the value of $\bar{P}$ or $\bar{Q}$, one has only to find out its partial derivatives. If the sign of a partial derivative can be determined, we will know the direction in which $\bar{P}$ will move when a parameter changes; this constitutes a qualitative conclusion. If the magnitude of the partial derivative can be ascertained, it will constitute a quantitative conclusion. To avoid misunderstanding, a clear distinction should be made between the two derivatives, say, $\partial\bar{Q}/\partial a$ and $\partial Q_d/\partial a$. The latter derivative is a concept appropriate to the demand function taken alone, and without regard to the supply function. The derivative $\partial\bar{Q}/\partial a$, on the other hand, to the equilibrium quantity which takes into account interaction of demand and supply together. To emphasize this distinction, we refer to the partial derivatives of $\bar{P}$ and $\bar{Q}$ with respect to the parameters as comparative-static derivatives.

For instance, for $\bar{P}$, we have

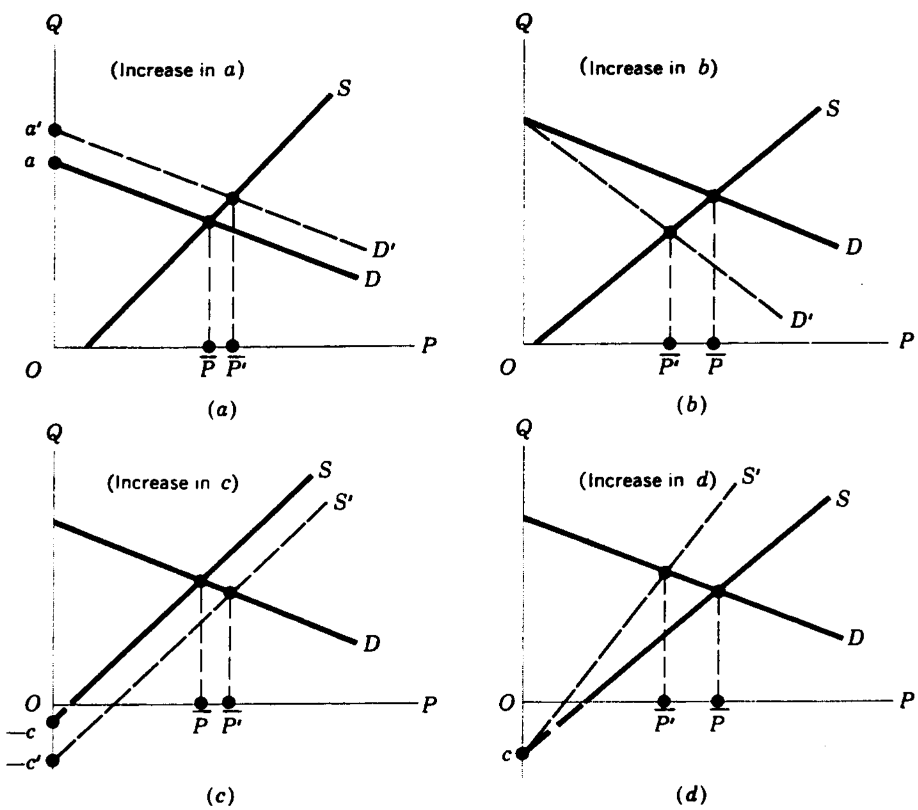
$$\frac{\partial \bar{P}}{\partial a} = \frac{1}{b+d}$$

$$\frac{\partial \bar{P}}{\partial b} = -\frac{(a+c)}{(b+d)^2}$$

$$\frac{\partial \bar{P}}{\partial c} = \frac{1}{b+d}$$

$$\frac{\partial \bar{P}}{\partial d} = \frac{-(a+c)}{(b+d)^2}$$

Thus $\frac{\partial \bar{P}}{\partial a} = \frac{\partial \bar{P}}{\partial c} > 0$ and $\frac{\partial \bar{P}}{\partial b} = \frac{\partial \bar{P}}{\partial d} < 0$.



National-Income Model

$$Y = C + I_o + G_o \qquad \text{(equilibrium condition)}$$
$$C = \alpha + \beta(Y-T) \qquad (\alpha > 0;\ 0 < \beta < 1)$$
$$T = \gamma + \delta Y \qquad (\gamma > 0;\ 0 < \delta < 1)$$

where three endogeneous variables are the national income Y, consumption C, and taxes T. The equilibrium income (in reduced form) is

$$\bar{Y} = \frac{\alpha - \beta\gamma + I_o + G_o}{1 - \beta + \beta\delta}$$

Thus,

$$\frac{\partial \bar{Y}}{\partial G_o} = \frac{1}{1-\beta+\beta\delta} > 0 \qquad \text{[the government-expenditure multiplier]}$$

$$\frac{\partial \bar{Y}}{\partial \gamma} = \frac{-\beta}{1-\beta+\beta\delta} < 0$$

$$\frac{\partial \bar{Y}}{\partial \delta} = \frac{-\beta(\alpha - \beta\gamma + I_o + G_o)}{(1-\beta+\beta\delta)^2} = \frac{-\beta\bar{Y}}{1-\beta+\beta\delta} < 0$$

7.6 Note on Jacobian Determinants

Partial derivatives can also provide a means of testing whether there exists functional (linear or nonlinear) dependence among a set of n functions in n variables. This is related to the notion of Jacobian determinants.

Consider n differentiable functions in n variables not necessary linear

$$y_1 = f^1(x_1, x_2, \ldots, x_n)$$
$$y_2 = f^2(x_1, x_2, \ldots, x_n)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_n = f^2(x_1, x_2, \ldots, x_n)$$

where the symbol f^i denotes the ith function, we can derive a total of n^2 partial derivatives.

$$\frac{\partial y_i}{\partial x_j} \quad (i = 1,2,...,n; \;\; j = 1,2,...,n).$$

We can arrange them into a square matrix, called a Jacobian matrix and denoted by J, and then take its determinant, the result will be what is known as a Jacobian determinant (or a Jacobian, for short), denoted by $|J|$:

$$|J| = \left| \frac{\partial(y_1,y_2,...,y_n)}{\partial(x_1,x_2,...,x_n)} \right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

Example: Consider two functions:

$$y_1 = 2x_1 + 3x_2$$
$$y_2 = 4x_1^2 + 12x_1x_2 + 9x_2^2$$

Then the Jacobian is

$$|J| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ (8x_1 + 12x_2) & (12x_1 + 18x_2) \end{vmatrix}$$

A Jacobian test for the existence of functional dependence among a set of n functions is provided by the following theorem:

Theorem: The Jacobian $|J|$ defined above will be identically zero for all values of $x_1, x_2, \ldots, x_n$ if and only if the n functions $f^1, f^2, \ldots, f^n$ are functionally (linear or nonlinear) dependent.

For the above example, since

$$|J| = \left| \frac{\partial(y_1,y_2)}{\partial(x_1,x_2)} \right| = (24x_1+36x_2) - (24x_1+36x_2) = 0$$

for all x_1 and x_2, y^1 and y^2 are functionally dependent. In fact, y_2 is simply y_1 squared.

Example: Consider the linear-equation system: Ax = d, i.e.,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= d_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= d_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= d_n.
\end{aligned}
$$

We know that the rows of the coefficient matrix A are linearly dependent if and only if $|A| = 0$. This result can now be interpreted as a special application of the Jacobian criterion of functional dependence.

Take the left side of each equation Ax = d as a separate function of the n variables $x_1, x_2, \ldots, x_n$, and denote these functions by $y_1, y_2, \ldots, y_n$. Then we have $\partial y_i/\partial x_j = a_{ij}$. In view of this, the elements of $|J|$ will be precisely the elements of A, i.e., $|J| = |A|$ and thus the Jacobian criterion of functional dependence among $y_1, y_2, \ldots, y_n$ is equivalent to the criterion $|A| = 0$ in the present linear case.

VIII. COMPARATIVE-STATIC ANALYSIS OF GENERAL-FUNCTIONS

The study of partial derivatives has enabled us, in the preceding chapter, to handle the simple type of comparative-static problems, in which the equilibrium solution of the model can be explicitly stated in the reduced form. We note that the definition of the partial derivative requires the absence of any functional relationship among the independent variables. As applied to comparative-static analysis, this means that parameters and/or exogeneous variables which appear in the reduced-form solution must be mutually independent.

However, no such expediency should be expected when, owing to the inclusion of general functions in a model, no explicit reduced-form solution can be obtained. In such a case, we will have to find the comparative-static derivatives directly from the originally given equations in the model. Take, for instance, a simple national-income model with two endogeneous variables Y and C:

$$Y = C + I_o + G_o \qquad \text{[equilibrium condition]}$$
$$C = C(Y,T_o) \qquad [T_o\text{: exogeneous taxes}]$$

which reduces to a single equation

$$Y = C(Y,T_o) + I_o + G_o$$

to be solved for $\bar{Y}$. We must, therefore, find the comparative-static derivatives directly from this equation. How might we approach the problem?

Let us suppose that an equilibrium solution $\bar{Y}$ does exist. We may write the equation

$$\bar{Y} = \bar{Y}(I_o,G_o,T_o)$$

even though we are unable to determine explicitly the form which this function takes. Furthermore, in some neighborhood of $\bar{Y}$, the following identical equality will hold:

$$\bar{Y} \equiv C(\bar{Y},T_o) + I_o + G_o.$$

Since $\bar{Y}$ is a function of T_o, the two arguments of the C function are not independent. T_o can in this case affect C not only directly, but also indirectly via $\bar{Y}$. Consequently, partial differentiation is no longer appropriate for our purposes. In this case, we must resort to <u>total differentiation</u> (as against partial differentiation). The process of total differentiation can lead us to the related concept of total derivative. Once we become familiar with these concepts, we shall be able to deal with functions whose arguments are not all independent so that we can study the comparative-statics of a general-function model.

8.1 Differentials

The symbol dy/dx has been regarded as a single entity. We shall now reinterpret as a ratio of two quantities, dy and dx.

Differentials and Derivatives

Given a function $y=f(x)$, we can use the difference quotient $\Delta y/\Delta x$ to represent the ratio of change of y with respect to x. Since

$$(8.1) \qquad \Delta y \equiv \left[\frac{\Delta y}{\Delta x}\right]\Delta x$$

the magnitude of Δy can be found, once the $\Delta y/\Delta x$ and the variation Δx are known. If we denote the infinitesimal changes in x and y, respectively, by dx and dy, the identity (8.1) becomes

$$(8.2) \qquad dy \equiv \left[\frac{dy}{dx}\right]dx \quad \text{or} \quad dy = f'(x)dx$$

The symbols dy and dx are called the differentials of y and x, respectively.

Dividing the two identities in (8.2) throughout by dx, we have

$$\frac{(dy)}{(dx)} \equiv \left[\frac{dy}{dx}\right] \quad \text{or} \quad \frac{(dy)}{(dx)} = f'(x)$$

This result shows that the derivative $dy/dx \equiv f'(x)$ may be interpreted as the quotient of two separate differentials dy and dx.

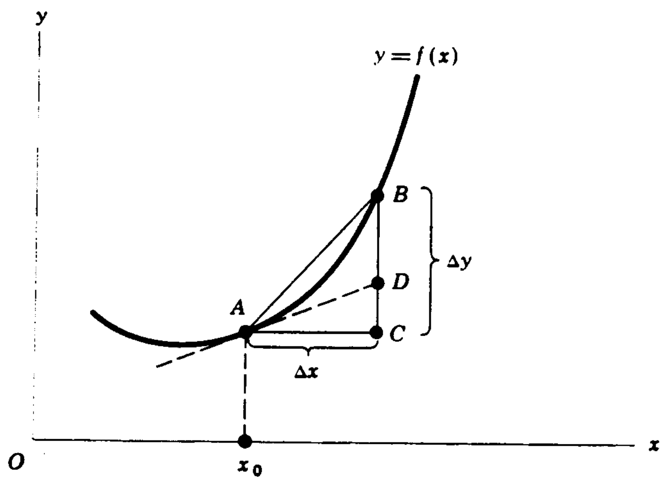
On the basis of (8.2), once we are given $f'(x)$, dy can immediately be written as $f'(x)dx$. The derivative $f'(x)$ may thus be viewed as a "converter" that serves to convert an infinitesimal change dx into a corresponding change dy.

Example: Given $y = 3x^2 + 7x - 5$, find dy. Since $f'(x) = 6x + 7$, the desired differential is

$$dy = (6x + 7)dx$$

<u>Remark</u>. The process of finding the differential dy is called differentiation. Recall that we have used this term as a synonym for derivation. To avoid confusion, the word "differentiation" with the phrase "with respect to x" when we take the derivative dy/dx.

The following diagram shows the relationship between "Δy" and "dy"



$$\Delta y = \left[\frac{\Delta y}{\Delta x}\right]\Delta x = \frac{CB}{AC}AC = CB$$

$$dy = \left[\frac{dy}{dx}\right]\Delta x = \frac{CD}{AC}AC = CD$$

which differs from Δy by an error of DB.

<u>Differentials and Point Elasticity</u>

As an illustration of the application of differentials in economics, let us consider the elasticity of a function. For a demand function Q=f(P), for instance, the elasticity is defined as $(\Delta Q/Q)/(\Delta P/P)$. Now if $\Delta P \rightarrow 0$, the ΔP and ΔQ will reduce to the differential dP and dQ, and the elasticity becomes

$$\epsilon_d \equiv \frac{dQ/Q}{dP/Q} = \frac{dQ/dP}{Q/P} = \frac{\text{marginal demand function}}{\text{average demand function}}$$

In general, for a given function y=f(x), the point elasticity of y with respect to x as

$$\epsilon_{yx} = \frac{dy/dx}{y/x} = \frac{\text{marginal function}}{\text{average function}}$$

Example: Find ϵ_d if the demand function is Q = 100 - 2P. Since dQ/dP = -2 and Q/P = (100-2P)/2, so ϵ_d = (-P)/(50-P). Thus the demand is inelastic ($|\epsilon_d| < 1$) for 0 < P < 25, unit elastic ($|\epsilon_d| = 1$) for P=25, and elastic for 25 < P < 50.

8.2 Total Differentials

The concept of differentials can easily be extended to a function of two or more independent variables. Consider a saving function

$$S = S(Y,i)$$

where S is savings, Y is national income, and i is interest rate. If the function is continuous and possesses continuous partial derivatives, the total differential is defined by

$$dS = \frac{\partial S}{\partial Y} dY + \frac{\partial S}{\partial i} di$$

That is, the infinitesimal change in S is the sum of the infinitesimal change in Y and the infinitesimal change in i.

Remark. If i remains constant, the total differential will reduce to a partial differential:

$$\frac{\partial S}{\partial Y} = \left[\frac{dS}{dY}\right]_{i\ \text{constant}}$$

Furthermore, general case of a function of n independent variables $y=f(x_1,x_2,\ldots,x_n)$, the total differential of this function is given by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n \equiv \sum_{i=1}^{n} f_i dx_i$$

in which each term on the right side indicates the amount of change in y resulting from an infinitesimal change in one of the independent variables.

Similar to the case of one variable, the n partial elasticities can be written as

$$\epsilon_{fx_i} = \frac{\partial f}{\partial x_i}\frac{x_i}{f} \quad (i = 1,2,...,n).$$

8.3 Rules of Differentials

Let c be constant and u and v be two functions of the variables $x_1,x_2,\ldots,x_n$. Then the following rules are valid:

Rule I. $dc = 0$
Rule II. $d(cu^a) = cau^{a-1}du$
Rule III. $d(u \pm v) = du \pm dv$
Rule IV. $d(uv) = vdu + udv$
Rule V. $d(u/v) = 1/v^2\ (vdu - udv)$

Example: Find dy of the function $y = 5x_1^2 + 3x_2$. There are two ways to find dy. One is the straightforward method by finding $\partial f/\partial x_1$ and $\partial f/\partial x_2$: $\partial f/\partial x_1 = 10x_1$ and $\partial f/\partial x_2 = 3$, which will then enable us to write

$$dy = f_1dx + f_2dx = 10x_1dx_1 + 3dx_2.$$

The other way is to use the rules given above by letting $u=5x_1^2$ and $v=3x_2$:

$$\begin{aligned} dy &= d(5x_1^2) + d(3x_2) && \text{[by Rule III]} \\ &= 10x_1dx_1 + 3dx_2 && \text{[by Rule II].} \end{aligned}$$

Example: Find dy of the function $y = 3x_1^2 + x_1x_2^2$. Since $f_1 = 6x_1 + x_2^2$ and $f_2 = 2x_1x_2$, the desired differential is

$$dy = (6x_1 + x_2^2)dx + 2x_1x_2dx_2.$$

By applying the given rules, the same result can be arrived at

$$\begin{aligned} dy &= d(3x_1^2) + d(x_1x_2^2) \\ &= 6x_1dx_1 + x_2^2dx_1 + 2x_1x_2dx_2 \\ &= (6x_1 + x_2^2)dx_1 + 2x_1x_2dx_2 \end{aligned}$$

Example: For the function

$$y = \frac{x_1 + x_2}{2x_1^2},$$

$$f_1 = \frac{-(x_1 + 2x_2)}{2x_1^3} \quad \text{and} \quad f_2 = \frac{1}{2x_1^2},$$

$$\text{then} \quad y = -\frac{(x_1 + 2x_2)}{2x_1^3}\, dx_1 + \frac{1}{2x_1^2}\, dx_2$$

The same result can also be obtained by applying the given rules:

$$\begin{aligned} dy &= \frac{1}{4x_1^4}\,[2x_1^2 d(x_1 + x_2) - (x_1 + x_2)d(2x_1^2)] \quad \text{[by Rule V]} \\ &= \frac{1}{4x_1^4}\,[2x_1^2(dx_1 + dx_2) - (x_1 + x_2)4x_1 dx_1] \\ &= \frac{1}{4x_1^4}\,[-2x_1(x_1 + 2x_2)dx_1 + 2x_1^2 dx_2] \\ &= -\frac{(x_1 + 2x_2)}{2x_1^3}\, dx_1 + \frac{1}{2x_1^2}\, dx_2 \end{aligned}$$

For the case of more than two functions, we have

Rule VI. $d(u \pm v \pm w) = du \pm dv \pm dw$
Rule VII. $d(uvw) = vwdu + uwdv + uvdw$.

8.4 Total Derivatives

Consider any function

$$y = f(x,w) \quad \text{where} \quad x = g(w).$$

Unlike a partial derivative, a total derivative does not require the argument x to main constant as w varies, and can thus allow for the postulated relationship between the two variables. The variable w can affect y through two channels: (1) indirectly, via the function g and then f, and (2) directly, via the function. Whereas the partial derivative f_w is adequate for expressing the direct effect alone, a total derivative is needed to express both effects jointly.

To get the total derivative, we first get the total differential $dy = f_x dx + f_w dw$. Dividing both sides of this equation by dw, we have the total derivative

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w \frac{dw}{dw}$$
$$= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w}$$

Example: Find the dy/dw, given the function

$$y = f(x,w) = 3x - w^2 \text{ where } g(w) = 2w^2 + w + 4.$$

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w = 3(4W+1) - 2w = 10w + 3.$$

As a check, we may substitute the function g into f, to get

$$y = 3(2w^2 + w + 4) - w^2 = 5w^2 + 3w + 12$$

which is now a function of w alone. Then

$$\frac{dy}{dw} = 10w + 3$$

the identical answer.

Example: Find du/dc, given $u = u(c,s)$ with $s=g(c)$. Then $\frac{du}{dc} = \frac{\partial u}{\partial c} + \frac{\partial u}{\partial g(c)} g'(c)$.

A Variation on the Theme

For the function

$$y = f(x_1,x_2,w)$$

with $x_1 = g(w)$ and $x_2 = h(w)$, the total derivative of y is given by

$$\frac{dy}{dw} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dw} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dw} + \frac{\partial f}{\partial w}.$$

Example: Let the production function be

$$Q = Q(K,L,t)$$

where K is the capital input, L is the labor input, and t is the time which indicates that the production function can shift over time in reflection of technological change. Since capital and labor can also change over time, we may write

$$K = K(t) \text{ and } L = L(t).$$

Thus the rate of output with respect to time can be denoted as

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial K}\frac{dK}{dt} + \frac{\partial Q}{\partial L}\frac{dL}{dt} + \frac{\partial Q}{\partial t}$$

<u>Another Variation on the Theme</u>

Now if a function is given,

$$y = f(x_1, x_2, u, v)$$

with $x_1 = g(u,v)$ and $x_2 = h(u,v)$, we can find the total derivative of y with respect to u (while v is held constant). Since

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv.$$

dividing both sides of the above equation by du, we have

$$\frac{dy}{du} = \frac{\partial y}{\partial x_1}\frac{dx_1}{du} + \frac{\partial y}{\partial x_2}\frac{dx_2}{du} + \frac{\partial y}{\partial u}\frac{du}{du} + \frac{\partial y}{\partial v}\frac{dv}{du}$$

$$= \frac{\partial y}{\partial x_1}\frac{dx_1}{du} + \frac{\partial y}{\partial x_2}\frac{dx_2}{du} + \frac{\partial y}{\partial u} \quad \left[\frac{dv}{du} = 0 \text{ since v is constant}\right]$$

Since v is held constant, the above is the partial total derivative, we redenote the above equation by the following notation:

$$\frac{\S y}{\S u} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial u} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial u} + \frac{\partial y}{\partial u}$$

<u>Remark</u>. In the cases we have discussed, the total derivative formulas can be regarded as

expressions of the chain rule, or the composite-function rule. Also the chain of derivatives does not have to be limited to only two "links"; the concept of the total derivative should be extendible to cases where there are three or more links in the composite function.

8.5 Derivatives of Implicit Functions

The concept of total differentials can also enable us to find the derivatives of the so-called "implicit functions."

Implicit Functions

A function given in the form of $y = f(x_1,x_2,\ldots,x_n)$ is called an explicit function, because the variable y is explicitly expressed as a function of x. But in many cases y is not an explicit function of $x_1,x_2,\ldots,x_n$, instead, the relationship between y and $x_1,\ldots,x_n$ is given with the form of

$$F(y,x_1,x_2,\ldots,x_n) = 0.$$

Such an equation may also be defined as implicit function $y = f(x_1,x_2,\ldots,x_n)$. Note that an explicit function $y = f(x_1,\ldots,x_n)$ can always be transformed into an equation $F(y,x_1,\ldots,x_n) \equiv y-f(x_1,x_2,\ldots,x_n) = 0$. The reverse transformation is not always possible.

In view of this uncertainty, we have to impose a certain condition under which we can be sure that a given equation $F(y,x_1,\ldots,x_n) = 0$ does indeed define an implicit function $y=f(x_1,x_2,\ldots,x_n)$. Such a result is given by the so-called "implicit-function theorem."

Implicit-Function Theorem. Given $F(y,x_1,x_2,\ldots,x_n) = 0$, if (a) the function F has continuous partial derivatives $F_y, F_{x_1}, F_{x_2}, \ldots, F_{x_n}$, and if (b) at a point $(y_0,x_{10},\ldots,x_{n0})$ satisfying $F(y_0,x_{10},\ldots,x_{n0}) = 0$, F_y is nonzero, then there exists an n-dimensional neighborhood of $(x_{10},\ldots,x_{n0})$, N, in which y is an implicitly defined function of variables $x_1,\ldots,x_n$, in the form of $y=f(x_1,\ldots,x_n)$, and $F(y,x_1,\ldots,x_n) = 0$ for all points in N. Moreover, the implicit function f is continuous, and has continuous partial derivatives $f_1,\ldots,f_n$.

Derivatives of Implicit Functions

Differentiating F, we have $dF=0$, or

$$F_y dy + F_1 dx_1 + \ldots + F_n dx_n = 0.$$

Suppose that only y and x_i are allowed to vary. Then the above equation reduce to $F_y dy + F_i dy = 0$. Thus

$$\left.\frac{dy}{dx_i}\right|_{\text{other variable constant}} = \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y} \qquad (i = 1,2,\ldots,n)$$

In the simple case where the given equation is $F(y,x) = 0$, the rule gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example: Suppose $y - 3x^4 = 0$. Then $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-12x^3}{1} = 12x^3$

In this particular case, we can easily solve the given equation for y, to get $y=3x^4$ so that $dy/dx = 12x^3$.

Example: $F(x,y) = x^2 + y^2 - 9 = 0$. Thus,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y} = -\frac{x}{y}. \qquad (y \neq 0)$$

Example: For $F(y,x,w) = y^3x^2 + w^3 + yxw - 3 = 0$, we have

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2y^3x + yw}{3y^2x^2 + xw}$$

In particular, at point (1,1,1), $\partial y/\partial x = -3/4$.

Example: Assume that the equation $F(Q,K,L) = 0$ implicitly defines a production function $Q=f(K,L)$. Then we can find MP_K and MP_L as follows:

$$MP_K \equiv \frac{\partial Q}{\partial K} = -\frac{F_K}{F_Q}$$

and

$$MP_L \equiv \frac{\partial Q}{\partial L} = -\frac{F_L}{F_Q}$$

In particular, we can also find the $MRTS_{LK}$ which is given by

$$MRTS_{LK} \equiv \frac{\partial K}{\partial L} = -\frac{F_L}{F_K}$$

Extension to the Simultaneous-Equation Case

Consider a set of simultaneous equations.

$$F^1(y_1,\ldots,y_n;\ x_1,\ldots,x_m) = 0$$
$$F^2(y_1,\ldots,y_n;\ x_1,\ldots,x_m) = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F^n(y_1,\ldots,y_n;\ x_1,\ldots,x_m) = 0$$

Suppose $F^1, F^2, \ldots, F^n$ are differentiable. Then we have

$$\frac{\partial F^1}{\partial y_1}dy_1 + \frac{\partial F^1}{\partial y_2}dy_2 + \ldots + \frac{\partial F^1}{\partial y_n}dy_n = -\left[\frac{\partial F^1}{\partial x_1}dx_1 + \ldots + \frac{\partial F^1}{\partial x_m}dx_m\right]$$
$$\frac{\partial F^2}{\partial y_1}dy_1 + \frac{\partial F^2}{\partial y^2}dy_2 + \ldots + \frac{\partial F^2}{\partial y_n}dy_n = -\left[\frac{\partial F^2}{\partial x_1}dx_1 + \ldots + \frac{\partial F^2}{\partial x_m}dx_m\right]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{\partial F^n}{\partial y_1}dy_1 + \frac{\partial F^n}{\partial y_2}dy_2 + \ldots + \frac{\partial F^n}{\partial y_n}dy_n = -\left[\frac{\partial F^n}{\partial x_1}dx_1 + \ldots + \frac{\partial F^n}{\partial x_m}dx_m\right]$$

Or in matrix form,

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_n \end{bmatrix} = - \begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \frac{\partial F^1}{\partial x_2} & \cdots & \frac{\partial F^1}{\partial x_m} \\ \frac{\partial F^2}{\partial x_1} & \frac{\partial F^2}{\partial x_2} & \cdots & \frac{\partial F^2}{\partial x_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^n}{\partial x_1} & \frac{\partial F^n}{\partial x_2} & \cdots & \frac{\partial F^n}{\partial x_m} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix}$$

If we want to obtain partial derivatives with respect to x_i, we can let $dx_t = 0$ for $t \neq i$. Thus, we have the following equation:

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_i} \\ \frac{\partial y_2}{\partial x_i} \\ \vdots \\ \frac{\partial y_n}{\partial x_i} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F^1}{\partial x_i} \\ \frac{\partial F^2}{\partial x_i} \\ \vdots \\ \frac{\partial F^n}{\partial x_i} \end{bmatrix}$$

Now suppose the following Jacobian determinant is nonzero:

$$|J| = \left| \frac{\partial(F^1,F^2,\ldots,F^n)}{\partial(y_1,y_2,\ldots,y_n)} \right| = \begin{vmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{vmatrix} \neq 0$$

Then, by Cramer's rule, we have

$$\frac{\partial y_j}{\partial x_i} = \frac{|J_j^i|}{|J|} \qquad (j=1,2,\ldots,n,\ i=1,2,\ldots,m).$$

where $|J_j^i|$ is obtained by replacing the jth column of $|J|$ with

$$\left[\frac{\partial F^1}{\partial x_i}, \frac{\partial F^2}{\partial x_i}, \ldots, \frac{\partial F^n}{\partial x_i} \right]'.$$

Of course, we can find these derivatives by inversing the Jacobian matrix J:

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_i} \\ \frac{\partial y_2}{\partial x_i} \\ \vdots \\ \frac{\partial y_n}{\partial x_i} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial F^1}{\partial x_i} \\ \frac{\partial F^2}{\partial x_i} \\ \vdots \\ \frac{\partial F^n}{\partial x_i} \end{bmatrix}$$

In the compact notation,

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}_i} = -J^{-1}F_i$$

Example: Let the national-income model be rewritten in the form:

$$Y - C - I_o - G_o = 0$$
$$C - \alpha - \beta(Y - T) = 0$$
$$T - \gamma - \delta Y = 0$$

Since

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial Y} & \frac{\partial F^1}{\partial C} & \frac{\partial F^1}{\partial T} \\ \frac{\partial F^2}{\partial Y} & \frac{\partial F^2}{\partial C} & \frac{\partial F^2}{\partial T} \\ \frac{\partial F^3}{\partial Y} & \frac{\partial F^3}{\partial C} & \frac{\partial F^3}{\partial T} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{vmatrix} = 1 - \beta + \beta\delta.$$

Suppose all exogeneous variables and parameters are fixed except G_o, then we have

$$\begin{bmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{bmatrix} \begin{bmatrix} \partial \bar{Y}/\partial G_o \\ \partial \bar{C}/\partial G_o \\ \partial \bar{T}/\partial G_o \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so that we can solve the above equation for, say, $\partial \bar{Y}/\partial G_o$ which comes out to be

$$\frac{\partial \bar{Y}}{\partial G_o} = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{vmatrix}}{|J|} = \frac{1}{1-\beta+\beta\delta}$$

8.6 Comparative Statics of General-Function Models

Consider a single-commodity market model:

$$Q_d = Q_s \qquad \text{[equilibrium condition]}$$
$$Q_d = D(P,Y_o) \qquad [\partial D/\partial P < 0;\ \partial D/\partial Y_o > 0]$$
$$Q_s = S(P) \qquad [dS/dP > 0]$$

where Y_o is an exogeneously determined income. From this model, we can obtain a single equation:

$$D(P,Y_o) - S(P) = 0.$$

Even though this equation cannot be solved explicitly for the equilibrium price $\bar{P}$, by the implicit-function theorem, we know that there exists the equilibrium price $\bar{P}$ which is the function of Y_o:

$$\bar{P} = \bar{P}(Y_o)$$

such that

$$D(\bar{P},Y_o) - S(\bar{P}) \equiv 0.$$

It then requires only a straight application of the implicit-function rule to produce the comparative-static derivative, $d\bar{P}/dY_o$:

$$\frac{d\bar{P}}{dY_o} = -\frac{\partial F/\partial Y_o}{\partial F/\partial P} = -\frac{\partial D/\partial Y_o}{\partial D/\partial P - dS/dP} > 0$$

Since $\bar{Q} = S(\bar{P})$, thus we have

$$\frac{d\bar{Q}}{dY_o} = \frac{dS}{dP}\frac{d\bar{P}}{dY_o} > 0$$

IX. OPTIMIZATION: A SPECIAL VARIETY OF EQUILIBRIUM ANALYSIS

From now on, our attention will be turned to the study of goal equilibrium, in which the equilibrium state is defined as the optimal position for a given economic unit and in which the said economic unit will be deliberately striving for attainment of that equilibrium. Our primary focus will be on the classical techniques for locating optimal positions - those using differential calculus.

9.1 Optimal Values and Extreme Values

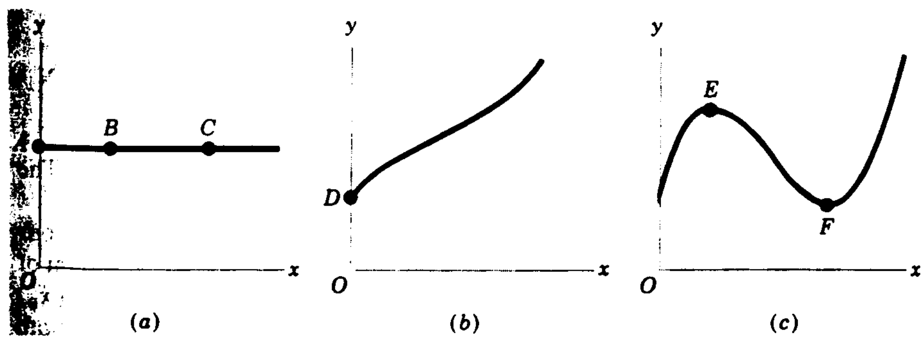
Economics is by large a science of choice. When an economic project is to be carried out, there are normally a number of alternative ways of accomplishing it. One (or more) of these alternatives will, however, be more desirable than others from the stand-point of some criterion, and it is the essence of the optimization problem to choose.

The most common criterion of choice among alternatives in economics is the goal of maximizing something or of minimizing something. Economically, we may categorize such maximization and minimization problems under general heading of optimization. From a purely mathematical point of view, the collective term for maximum and minimum is the more matter-of-fact designation extremum, meaning an extreme value.

In formulating an optimization problem, the first order of business is to delineate an <u>objective function</u> in which the dependent variable represents the object of maximization or minimization and in which the set of independent variables indicates the objects whose magnitudes the economic unit in question can pick and choose. We shall therefore refer to the independent variables as <u>choice variables</u>.

9.2 Relative Maximum and Minimum: First-Derivative Test

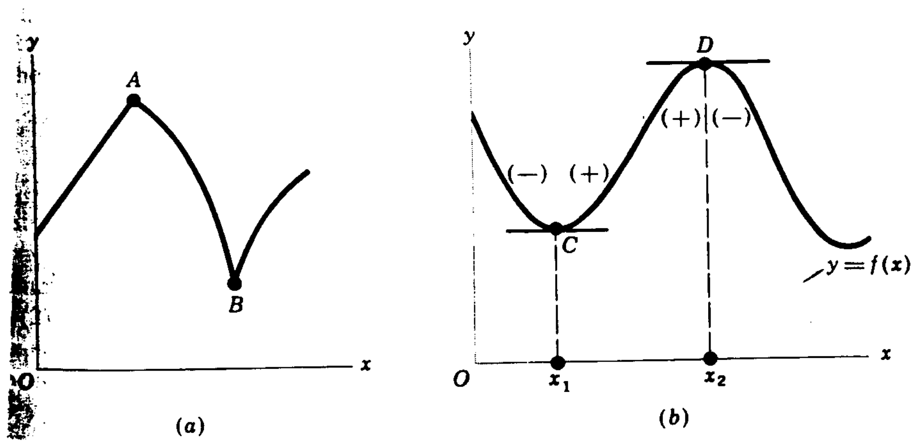
Consider a general-form objective function $y=f(x)$. Three specific cases of functions are depicted in the following Figures:



Remark. The points E and F in (c) are relative (or local) extremum, in the sense that each of these points represents an extremum in some neighborhood of the point only. We shall continue our discussion mainly with reference to the search for relative extrema. Since an absolute (or global) maximum must be either a relative maxima or one of the ends of the function. Thus, if we know all the relative maxima, it is necessary only to select the largest of these and compare it with the end points in order to determine the absolute maximum. Hereafter, the extreme values considered will be relative or local ones, unless indicated otherwise.

First-Derivative Test

Given a function $y=f(x)$, the first derivative $f'(x)$ plays a major role in our search for its extreme values. For smooth functions, relative extreme values can only occur where $f'(x) = 0$, which is a necessary (but not sufficient) condition for a relative extremum (either maximum or minimum).



First-derivative test relative extremum. If $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$, will be

(a) A relative maximum if $f'(x)$ changes its sign from positive to negative from the immediate left of the point x_0 to its immediate right.

(b) A relative minimum if $f'(x)$ changes its sign from negative to positive from the immediate left of x_0 to its immediate right.

(c) No extreme points if $f'(x)$ has the same sign on some neighborhood.

Example: $y = (x-1)^3$. $x=1$ is not an extreme point even $f'(1)=0$.

Example: $y = f(x) = x^3 - 12x^2 + 36x + 8$

Since $f'(x) = 3x^2 - 24 + 36$, to get the critical values, i.e., the values of x satisfying the condition $f'(x) = 0$, we set $f'(x) = 0$, and thus

$$3x^2 - 24x + 36 = 0.$$

Its roots are $\bar{x}_1 = 2$ and $\bar{x}_2 = 6$. It is easy to verify that $f'(x) > 0$ for $x<2$, and $f'(x) < 0$ for $x>2$. Thus $x=2$ is a maximum point and the corresponding maximum value of the function $f(2)=40$. Similarly, we can verify that $x=6$ is a minimum point and $f(6)=8$.

Example: Find the relative extremum of the average-cost function

$$AC = f(Q) = Q^2 - 5Q + 8.$$

Since $f'(2.5)=0$, $f'(Q) < 0$ for $Q < 2.5$, and $f'(Q) > 0$ for $Q > 2.5$, so $\bar{Q} = 2.5$ is an extreme point.

9.3 Second and Higher Derivatives

Since the first derivative $f'(x)$ of a function $y=f(x)$ is also a function of x, we can consider the derivative of $f'(x)$, which is called second derivative. Similarly, we can find derivatives of even higher orders. These will enable us to develop alternative criteria for locating the relative extrema of a function.

The second derivative of the function f is denoted by $f''(x)$ or d^2y/dx^2. If the second derivative $f''(x)$ exists for all x values, $f(x)$ is said to be twice differentiable; if, in addition, $f''(x)$ is continuous, $f(x)$ is said to be twice continuously differentiable.

The higher-order derivatives of $f(x)$ can be similarly obtained and symbolized along the same line as the second derivative:

$$f'''(x),\ f^{(4)}(x),\ \ldots,\ f^{(n)}(x)$$

or

$$\frac{d^3y}{dx^3},\ \frac{d^4y}{dx^4},\ \ldots,\ \frac{d^ny}{dx^n}$$

Remark. d^ny/dx^n can be also written as $(d^n/dx^n)y$, where the d^n/dx^n part serves as an operator symbol instructing us to take the nth derivative with respect to x.

Example: $y = f(x) = 4x^4 - x^3 + 17x^2 + 3x - 1$

Then

$$f'(x) = 16x^3 - 3x^2 + 34x + 3$$
$$f''(x) = 48x^2 - 6x + 34$$
$$f'''(x) = 96x - 6$$
$$f^{(4)}(x) = 96$$
$$f^{(5)}(x) = 0$$

Example. Find the first four derivatives of the function

$$y = g(x) = \frac{x}{1+x} \qquad (x \neq -1)$$

$$g'(x) = (1+x)^{-2}$$
$$g''(x) = -2(1+x)^{-3}$$
$$g'''(x) = 6(1+x)^{-4}$$
$$g^{(4)}(x) = -24(1+x)^{-5}$$

Remark. A negative second derivative is consistently reflected in an inverse U-shaped curve; a positive second derivative is reflected in an U-shaped curve.

9.4 Second-Derivative Test

Second-derivative test for relative extremum. If $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$ will be

(a) A relative maximum if $f''(x_0) < 0$

(b) A relative minimum if $f''(x_0) > 0$.

This test is in general more convenient to use than the first-derivative test, since it does not require us to check the derivative sign to both the left and right of x.

Example: $y = f(x) = 4x^2 - x$.

Since $f'(x) = 8x - 1$ and $f''(x) = 8$, we know f(x) reaches its minimum at $\bar{x} = 1/8$. Indeed, since the function plots as a U-shaped curve, the relative minimum is also the absolute minimum.

Example: $y = g(x) = x^3 - 3x^2 + 2$.

Then $y' = g'(x) = 3x^2 - 6x$ and $y'' = 6x - 6$. Setting $g'(x) = 0$, we obtain the critical values $\bar{x}_1 = 0$ and $\bar{x}_2 = 2$, which in turn yield the two stationary values $g(0)=2$ (a maximum because $g'(0) < 0$) and $g(2)=-2$ (a minimum because $g'(2) > 0$).

Remark. Note that when $f'(x_0)=0$, $f''(x_0) < 0$ ($f''(x_0) > 0$) is a sufficient condition for a relative maximum (minimum) but not a necessary condition. However, the condition $f''(x_0) \leq 0$ ($f''(x_0) \geq 0$) is a necessary condition (even though not sufficient) for a relative maximum (minimum).

Condition for Profit Maximization

Let $R=R(Q)$ be the total-revenue function and let $C=C(Q)$ be the total-cost function, where Q is the level of output. The profit function is then given by

$$\pi = \pi(Q) = R(Q) - C(Q).$$

To find the profit-maximizing output level we need to find $\bar{Q}$ such that

$$\pi'(\bar{Q}) = R'(\bar{Q}) - C'(\bar{Q}) = 0$$

or

$$R'(\bar{Q}) = C'(\bar{Q}), \text{ or } MR(\bar{Q}) = MC(\bar{Q}).$$

To be sure the first-order condition leads to a maximum, we require

$$\frac{d^2\pi}{dQ} \equiv \pi''(\bar{Q}) - R''(\bar{Q}) - C''(\bar{Q}) < 0.$$

Economically, this would mean that, if the rate of change of MR is less than the rate of change of MC at $\bar{Q}$, then that output Q will maximize profit.

Example: Let $R(Q) = 1200Q - 2Q^2$ and $C(Q) = Q^3 - 61.25Q^2 + 1528.5Q + 2000$. Then the profit function is

$$\pi'(Q) = -Q^3 + 59.25Q^2 - 328.5Q - 2000.$$

Setting $\pi'(Q) = -3Q^2 + 118.5Q - 328.5 = 0$, we have $\bar{Q}_1 = 3$ and $\bar{Q}_2 = 36.5$. Since $\pi''(3) = -18 + 118.5 = 100.5 > 0$ and $\pi''(36.5) = -219 + 118.5 < 0$, so the profit-maximizing output is $\bar{Q} = 36.5$.

9.5 Taylor Series and the Mean-Value Theorem

This section considers the so-called "expansion" of a function $y=f(x)$ into what is known as Taylor series (expansion around any point $x=x_0$). To expand a function $y=f(x)$ around a point x_0 means to transform that function into a polynomial form, in which the coefficients of the various terms are expressed in terms of the derivative values $f'(x_0)$, $f''(x_0)$, etc. - all evaluated at the point of expansion x_0.

Taylor's Theorem. Given an arbitrary function $\phi(x)$, if we know the values of $f(x_0)$, $f'(x_0)$, $f''(x_0)$, etc., then this function can be expanded around the point x_0 as follows:

$$\phi(x) = \phi(x_0) + \phi'(x_0)(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{\phi^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$+ R_n \equiv P_n + R_n$$

where P_n represents the nth-degree polynomial and R_n denotes a remainder which can be denoted by the so-called Lagrange form of the remainder:

$$R_n = \frac{\phi^{(n+1)}(P)}{(n+1)!}(x - x_0)^{n+1}$$

with P being some number between x and x_0. Here n! is the "n factorial", defined as

$$n! \equiv n(n-1)(n-2)\ldots(3)(2)(1).$$

Remark. When $n=0$, the Taylor series reduce to the so-called mean-value theorem:

$$\phi(x) = P_0 + R_0 = \phi(x_0) + \phi'(P)(x - x_0)$$

or

$$\phi(x) - \phi(x_0) = \phi'(P)(x - x_0).$$

This mean-value theorem states that the difference between the value of the function ϕ at x_0 and at any other x value can be expressed as the product of the difference $(x - x_0)$ and $\phi'(P)$ with P being some point between x and x_0.

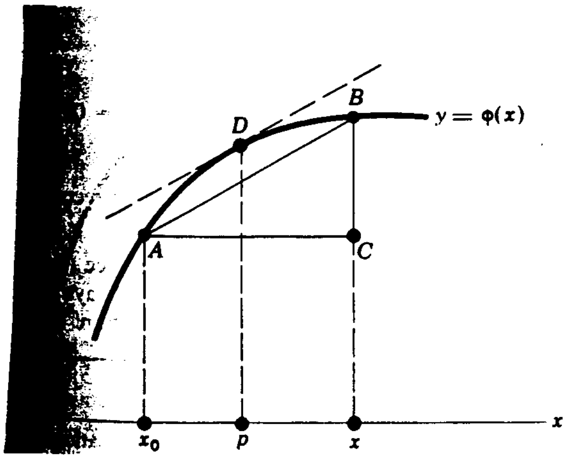


Figure 9.8

Remark. If $x_0 = 0$, then Taylor series reduce to the so-called Maclaurin series:

$$\phi(x) = \phi(0) + \frac{\phi'(0)}{1!}x + \frac{\phi''(0)}{2!}x^2 + \ldots + \frac{\phi^{(n)}(0)}{n!}x^n + \frac{\phi^{(n+1)}(P)}{(n+1)!}x^{n+1}$$

where P is a point between 0 and x.

Example: Expand the function

$\phi(x) = 1/1+x$

around the point $x_0=1$, with $n=4$. Since $\phi(1) = 1/2$ and

$\phi'(x) = -(1+x)^{-2}$, $\phi'(1) = -1/4$
$\phi''(x) = 2(1+x)^{-3}$, $\phi''(1) = 1/4$
$\phi^{(3)}(x) = -6(1+x)^{-4}$, $\phi^{(3)}(1) = -3/8$
$\phi^{(4)}(x) = 24(1+x)^{-5}$, $\phi^{(4)}(1) = 3/4$

we obtain the following Taylor series:

$$\phi(x) = 1/2 - 1/4\ (x-1) + 1/8\ (x-1)^2 - 1/16\ (x-1)^3 + 1/32\ (x-4)^4 + R_n.$$

9.6 Nth-Derivative Test

A relative extremum of the function f can be equivalently defined as follows:

A function f(x) attains a relative maximum (minimum) value at x_0 if $f(x) - f(x_0)$ is nonpositive (nonnegative) for values of x in some neighborhood of x_0.

Assume f(x) has finite, continuous derivatives up to the desired order at $x=x_0$, then the function can be expanded around $x=x_0$ as a Taylor series:

$$f(x) - f(x_0) = f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$$
$$+ \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(P)}{(n+1)!}(x-x_0)^{n+1}$$

From the above expansion, we have the following test:

<u>Nth-Derivative Test</u>. If $f'(x_0) = 0$, and if the first nonzero derivative value at x_0 encountered in successive derivation is that of the Nth derivative, $f^{(N)}(x_0) \neq 0$, then the stationary value $f(x_0)$ will be

(a) A relative maximum if N is an even number and $f^{(N)}(x_0) < 0$.
(b) A relative minimum if N is an even number and $f^{(N)}(x_0) > 0$.
(c) An inflection point if N is odd.

Example: $y = (7-x)^4$.
Since $f'(7) = 4(7-7)^3 = 0$, $f''(7) = 12(7-7)^2 = 0$, $f'''(7) = 24(7-7) = 0$, and $f^{(4)}(7) = 24 > 0$, so $x=7$ is a minimum point such that $f(7) = 0$.

X. DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Exponential functions, as well as the closely related logarithmic functions, have important applications in economics, especially in connection with growth problems, and in economic dynamics in general. This chapter gives some basic properties and derivatives of exponential and logarithmic functions.

10.1 The Nature of Exponential Functions

In its simple version, the exponential function may be represented in the form:

$$y = f(t) = b^t \qquad (b > 1)$$

where b denotes a fixed base of the exponent. Its generalized version has the form:

$$y = ab^{ct}$$

Remark. $y = ab^{ct} = a(b^c)^t$. Thus we can consider b^c as a base of the exponent. It changes exponent from ct to t and changes base b to b^c.

If the base is the irrational number denoted by the symbol e=2.71828..., the function:

$$y = ae^{rt}$$

is referred to the natural exponential function, which can alternatively denote

$$y = a\exp(rt).$$

Remark. It can be proved that e may be defined as the limit:

$$e \equiv \lim_{n\to\infty} f(n) = \lim_{n\to\infty} (1+\frac{1}{n})^n.$$

10.2 Logarithmic Functions

For the exponential function $y=b^t$ and the natural exponential function $y=e^t$, taking the log of y to the base b (denoted by $\log_b y$) and the base e (denoted by $\log_e y$) respectively, we have

$$t = \log_b y$$

and

$$t = \log_e y$$
$$\equiv \ln y.$$

The following rules are familiar to us:

Rules:

(a)	$\ln(uv) = \ln u + \ln v$	[log of product]
(b)	$\ln(u/v) = \ln u - \ln v$	[log of quotient]
(c)	$\ln u^a = a \ln u$	[log of power]
(d)	$\log_b u = (\log_b e)(\log_e u)$	[conversion of log base]
(e)	$\log_b e = 1/(\log_e b)$	[inversion of log base]

Properties of Log.

(a)	$\ln y_1 = \ln y_2$	iff $y_1 = y_2$
(b)	$\ln y_1 > \ln y_2$	iff $y_1 > y_2$
(c)	$0 < y < 1$	iff $\log y < 0$
(d)	$y = 1$	iff $\log y = 0$
(e)	$\log y \to \infty$	as $y \to \infty$
(f)	$\log y \to -\infty$	as $y \to 0$

Remark. $t = \log_b y$ and $t = \ln y$ are the respective inverse functions of the exponential functions $y = b^t$ and $y = e^t$.

10.3 Derivatives of Exponential and Logarithmic Functions

(a) $$\frac{d \ln t}{dt} = \frac{1}{t}$$

(b) $$\frac{de^t}{dt} = e^t$$

(c) $$\frac{de^{f(t)}}{dt} = f'(t)\, e^{f(t)}$$

(d) $\frac{d}{dt} \ln f(t) = \frac{f'(t)}{f(t)}$

Examples:

(a) Let $y = e^{rt}$. Then $dy/dt = re^{rt}$

(b) Let $y = e^{-t}$. Then $dy/dt = -e^{-t}$

(c) Let $y = \ln at$. Then $dy/dt = a/at = 1/t$

(d) $y = \ln t^c$. Since $y = \ln t^c = c \ln t$, so $dy/dt = c(1/t)$

(e) Let $y = t^3 \ln t^2$. Then $dy/dt = 3t^2 \ln t^2 + 2t^3/t = 2t^2(1+3 \ln t)$

The Case of Base b

(a) $\frac{db^t}{dt} = b^t \ln b$

(b) $\frac{d}{dt}\log_b t = \frac{1}{t \ln b}$

(c) $\frac{d}{dt} b^{f(t)} = f'(t) b^t \ln b$

(d) $\frac{d}{dt}\log_b f(t) = \frac{f'(t)}{f(t)} \frac{1}{\ln b}$

Proof of (a). Since $b^t = e^{\ln b^t} = e^{t \ln b}$, then $(d/dt)b^t = (d/dt)e^{t \ln b} = (\ln b)(e^{t \ln b}) = b^t \ln b$.

Proof of (b). Since

$$\log_b t = (\log_b e)(\log_e t) = (1/\ln b)\ln t,$$
$$(d/dt)(\log_b t) = (d/dt)[(1/\ln b)\ln t] = (1/\ln b)(1/t)$$

Examples:

(a) Let $y = 12^{1-t}$. Then $\frac{dy}{dt} = \frac{d(1-t)}{dt} 12^{1-t} \ln 12 = -12^{1-t} \ln 12$

An Application.

Example: Find dy/dx from $y = x^a e^{kx-c}$. Taking the natural log of both sides, we have

$$\ln y = a \ln x + kx - c.$$

Differentiating both with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{a}{x} + k$$

Thus $\frac{dy}{dx} = (a/x + k)y = (a/x + k)x^a e^{kx-c}$

XI. THE CASE OF MORE THAN ONE CHOICE VARIABLE

This chapter develops a way of finding the extreme values of an objective function that involves two or more choice variables. As before, our attention will be focused heavily on relative extrema, and for this reason we should often drop the adjective "relative," with the understanding that, unless otherwise specified, the extrema referred to are relative.

11.1 The Differential Version of Optimation Condition

This section shows the possibility of equivalently expressing the derivative version of first and second conditions in terms of differentials.

Consider the function $z=f(x)$. Recall that the differential of $z=f(x)$ is

$$dz = f'(x)dx.$$

Since $f'(x)=0$, which implies $dz=0$, is the necessary condition for extreme values, so $dz=0$ is also the necessary condition for extreme values. This first-order condition requires that $dz=0$ as x is varied. In such a context, with $dx \neq 0$, $dz=0$ if and only if $f'(x)=0$.

What about the sufficient conditions in terms of second-order differentials?

Differentiating $dz=f'(x)dx$, we have

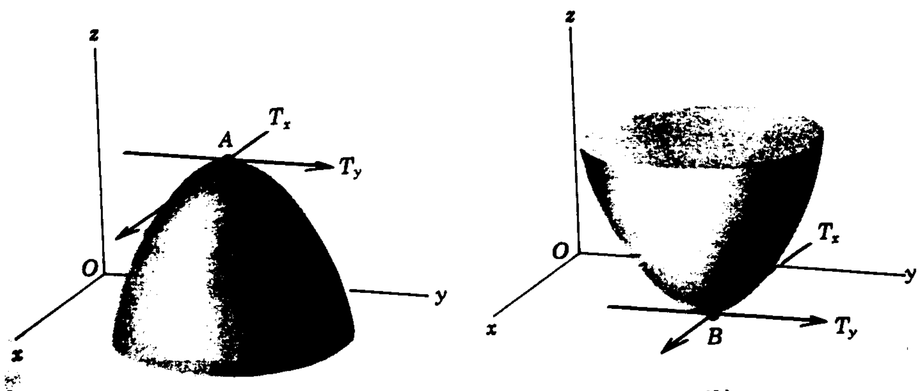
$$\begin{aligned} d^2z &\equiv d(dz) = d[f'(x)dx] \\ &= d[f'(x)]dx \\ &= f''(x)dx^2 \end{aligned}$$

Note that the symbols d^2z and dx^2 are fundamentally different. d^2z means the second-order differential of z; but dx^2 means the squaring of the first-order differential dx.

Thus, from the above equation, we have $d^2z < 0$ ($d^2z > 0$) if and only if $f''(x) < 0$ ($f''(x) > 0$). Therefore, the second-order sufficient condition for maximum (minimum) of $z=f(x)$ is $d^2z < 0$ ($d^2z > 0$).

11.2 Extreme Values of a Function of Two Variables

For a function of one variable, an extreme value is represented graphically by the peak of a hill or the bottom of a valley in a two-dimensional graph. With two choice variables, the graph of the function z=f(x,y) becomes a surface in a 3-space, and while the extreme values are still to be associated with peaks and bottoms.



(a) (b)

First-Order Condition

For the function z=f(x,y), the first-order necessary condition for an extremum again involves dz=0 for arbitrary values of dx and dy: an extremum point must be a stationary point, at which z must be constant for arbitrary infinitesimal changes of two variables x and y.

In the present two-variable case, the total differential is

$$dz = f_x dx + f_y dy$$

Thus the equivalent derivative version of the first-order condition dz=0 is

$$f_x = f_y = 0 \quad \text{or} \quad \partial f/\partial x = \partial f/\partial y = 0$$

As in the earlier discussion, the first-order condition is necessary, but not sufficient. To develop a sufficient condition, we must look to the second-order total, which is related to second-order partial derivatives.

Second-Order Partial Derivatives

From the function $z=f(x,y)$, we can have two first-order partial derivatives, f_x and f_y. Since f_x and f_y are themselves functions of x, we can find second-order partial derivatives:

$$f_{xx} \equiv \frac{\partial}{\partial x} fx \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]$$

$$f_{yy} \equiv \frac{\partial}{\partial y} fy \quad \text{or} \quad \frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]$$

$$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]$$

$$f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]$$

The last two are called cross (or mixed) partial derivatives.

Remark. Even though f_{xy} and f_{yx} have been separately defined, they will--according to Young's theorem, be identical with each other, as long as the two cross partial derivatives are both continuous. In fact, this theorem applies also to functions of three or more variables. Given $z=g(u,v,w)$, for instance, the mixed partial derivatives will be characterized by guv $= g_{vu}$, $g_{uw} = g_{wu}$, etc. provided these partial derivatives are continuous.

Example: Find all second-order partial derivatives of $z = x^3 + 5xy - y^2$. The first partial derivatives of this function are

$$f_x = 3x^2 + 5y \quad \text{and} \quad f_y = 5x - 2y$$

Thus, $f_{xx} = 6x$, $f_{yx} = 5$, $f_{xy} = 5$, and $f_{yy} = 2$. As expected, $f_{yx} = f_{xy}$.

Example: For $z = x^2e^{-y}$, its first partial derivatives are

$$f_x = 2xe^{-y} \quad \text{and} \quad f_y = -x^2e^{-y}$$

Thus, the second partial derivatives are

$$f_{xx} = 2e^{-y},\ f_{yx} = -2xe^{-y},\ f_{xy} = -2xe^{-y},\ f_{yy} = x^2e^{-y}$$

Again $f_{yx} = f_{xy}$.

Second-Order Total Differentials

From the first total differential

$$dz = f_x dx + f_y dx$$

we can obtain the second-order total differential d^2z:

$$\begin{aligned} d^2z \equiv d(dz) &= \frac{\partial(dz)}{\partial x}dx + \frac{\partial(dz)}{\partial y}dy \\ &= \frac{\partial}{\partial x}(f_x dx + f_y dy)dx + \frac{\partial}{\partial y}(f_x dx + f_y dy)dy \\ &= [f_{xx}dx + f_{xy}dy]dx + [f_{yx}dx + f_{yy}dy]dy \\ &= f_{xx}dx^2 + f_{xy}dydx + f_{yx}dxdy + f_{yy}dy^2 \\ &= f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 \quad [\text{if } f_{xy} = f_{yx}] \end{aligned}$$

Example: Given $z = x^3 + 5xy - y^2$, find dz and dz^2

$$\begin{aligned} dz &= f_x dx + f_y dy \\ &= (3x^2 + 5y)dx + (5x - 2y)dy \end{aligned}$$

$$\begin{aligned} d^2z &= f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 \\ &= 6xdx^2 + 10dxdy - 2dy^2 \end{aligned}$$

Second-Order Condition

Using the concept of d^2z, we can state the second-order sufficient condition for:

(a) A maximum of $z = f(x,y)$ if $d^2z < 0$ for any values of dx and dy, not both zero;

(b) A minimum of $z = f(x,y)$ if $d^2z > 0$ for any values of dx and dy, not both zero.

For operational convenience, second-order differential conditions can be translated into equivalent conditions on second-order derivatives. The actual translation would require a knowledge of quadratic forms, which will be discussed in the next section. But we may first introduce the main result here.

Main Result: For any values of dx and dy, not both zero,

$d^2z < 0$ iff $f_{xx} < 0$; $f_{yy} < 0$; and $f_{xx}f_{yy} > (f_{xy})^2$
$d^2z > 0$ iff $f_{xx} > 0$; $f_{yy} > 0$; and $f_{xx}f_{yy} > (f_{xy})^2$

From the first- and second-order conditions, we obtain conditions for relative extremum:

Conditions for Maximum: $f_x = f_y = 0$ (necessary condition) and $f_{xx} < 0$; $f_{yy} < 0$; and $f_{xx}f_{yy} > (f_{xy})^2$

Conditions for Minimum: $f_x = f_y = 0$ (necessary condition) and $f_{xx} > 0$; $f_{yy} > 0$; and $f_{xx}f_{yy} > (f_{xy})^2$

Example: Find the extreme values of $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$.

$f_x = 24x^2 + 2y - 6x$, $\quad f_y = 2x + 2y$
$f_{xx} = 48x - 6$, $\quad f_{yy} = 2 \quad f_{xy} = 2$

Setting $f_x = 0$ and $f_y = 0$, we have

$$24x^2 + 2y - 6x = 0$$
$$2y + 2x = 0$$

Then $y = -x$ and thus from $24x^2 + 2y - 6x$, we have $24x^2 - 8x = 0$ which yields two solutions for x: $\bar{x}_1 = 0$ and $\bar{x}_1 = 1/3$.

Since $f_{xx}(0,0) = -6$ and $f_{yy}(0,0) = 2$, it is impossible $f_{xx}f_{yy} \geq (f_{xy})^2 = 4$, so the point $(\bar{x}_1,\bar{y}_1) = (0,0)$ is not extreme point. For the solution $(\bar{x}_2,\bar{y}_1) = (1/3, -1/3)$, we find that $f_{xx} = 10 > 0$, $f_{yy} = f_{xy} = 2 > 0$, and $f_{xx}f_{yy} - (f_{xy})^2 = 20 - 4 > 0$, so $(\bar{x},\bar{y},\bar{z}) = (1/3, -1/3, 23/27)$ is a relative minimum point.

Example: $z = x + 2ey - e^x - e^{2y}$. Letting $f_x = 1 - e^x = 0$ and $f_y = 2e - 2e^{2y} - 0$, we have $\bar{x} = 0$ and $\bar{y} = 1/2$. Since $f_{xx} = -e^{-x}$, $f_{yy} = -4e^{2y}$, and $f_{xy} = 0$, we have $f_{xx}(0, 1/2) = -1 < 0$,

$f_{yy}(0, 1/2) = -e^{-1} < 0$, and $f_{xx}f_{yy} - (f_{xy})^2 = e^{-1} > u$, so $(\bar{x},\bar{y},\bar{z}) = (0, 1/2, -1)$ is the maximum point.

11.3 Quadratic Forms

Quadratic Forms

A function q with n-variables is said to have the quadratic form if it can be written as

$$
\begin{aligned}
q(u_1,u_2,\ldots,u_n) = {} & d_{11}u_1^2 + 2d_{12}u_1u_2 + \ldots + 2d_{1n}u_1u_n \\
& + d_{22}u_2^2 + 2d_{23}u_2u_3 + \ldots + 2d_{2n}u_2u_n \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + d_{nn}u_n^2
\end{aligned}
$$

If we let $d_{ji} = d_{ij}$, $i < j$, then $q(u_1,u_2,\ldots,u_n)$ can be written as

$$
\begin{aligned}
q(u_1,u_2,\ldots,u_n) = {} & d_{11}u_1^2 + d_{12}u_1u_2 + \ldots + d_{1n}u_1u_n \\
& d_{21}u_2u_1 + d_{22}u_2^2 + \ldots + d_{2n}u_2u_n \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& d_{n1}u_nu_1 + d_{n2}u_nu_2 + \ldots + d_{nn}u_n^2 \\
& = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}u_iu_j \\
& = u'Du
\end{aligned}
$$

where

$$
D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}
$$

which is called quadratic-form matrix.

Since $d_{ij} = d_{ji}$, D is a symmetric matrix.

Positive and Negative Definiteness

A quadratic form $q(u_1,\ldots,u_n) = u'Du$ is said to be

(a) positive definite if $q(u) > 0$ for all $u \neq 0$;
(b) positive semidefinite if $q(u) \geq 0$ for all $u \neq 0$;
(c) negative definite if $q(u) < 0$ for all $u \neq 0$;
(d) negative definite if $q(u) \leq 0$ for all $u \neq 0$.

Sometimes, we say that a matrix D is, for instance, positive definite if the corresponding quadratic form $q(u) = u'Du$ is positive definite.

Determinantal Test for Sign Definiteness

We state without proof that for the quadratic form $q(u) = u'Du$, the necessary and sufficient condition for positive definiteness is the principal minors of $|D|$, namely,

$$|D| = d_{11} > 0 \; ; \qquad |D| = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} > 0, \cdots \; ;$$

$$|D_n| = \begin{vmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{vmatrix} > 0.$$

The corresponding necessary and sufficient condition for negative definiteness is that the principal minors alternate in sign as follows:

$$|D_1| < 0 \, , \; |D_2| > 0 \, , \; |D_3| < 0 \, , \text{ etc.}$$

Two-Variable Quadratic Form

Example: Consider the second-order total differential

$$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$$

$$= [dx,dy]\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix}$$

Thus, for the function $z = f(x,y)$,

(a) d^2z is positive definite iff $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$

(b) d^2z is negative definite iff $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$

From the inequality $f_{xx}f_{yy} - (f_{xy})^2 > 0$, it implies that f_{xx} and f_{yy} are required to take the same sign, we see that this is precisely the second-order sufficient condition presented in the last section.

Remark. The determinant

$$|H| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

is the Hessian determinant (or simply a Hessian).

Example: Is $q = 5u^2 + 3uv + 2v^2$ either positive or negative? The symmetric matrix is

$$D = \begin{bmatrix} 5 & 1.5 \\ 1.5 & 2 \end{bmatrix}.$$

Since the principal minors of $|D|$ is $|D_1| = 5$ and $|D_2| = \begin{vmatrix} 5 & 1.5 \\ 1.5 & 2 \end{vmatrix} = 10 - 3 = 7 > 0$,

so q is positive definite.

Example: Given $f_{xx} = -2$, $f_{xy} = 1$, and $f_{yy} = -1$ at a certain point on a function $z = f(x,y)$, does d^2z have a definite sign at that point regardless of the values of dx and dy? The Hessian determinant is in this case $\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}$, with principal minors $|H_1| = -2 < 0$ and

$$|H_2| = \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 2 - 1 = 1 > 0 .$$

Thus d^2z is negative definite.

Three-Variable Quadratic Form

Example: Determine whether $q = u_1^2 + 6u_2^2 + 3u_3^2 - 2u_1u_2 - 4u_2u_3$ is either positive or negative definite. The matrix D corresponding this quadratic form is

$$D = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{bmatrix}.$$

and the principal minors of $|D|$ are $|D_1| = 1 > 0$, $|D_2| = \begin{vmatrix} 1 & -1 \\ -1 & 6 \end{vmatrix} = 6 - 1 = 5 > 0$, and

$$|D_3| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{vmatrix} = 11 > 0 .$$

Thus, the quadratic form is positive definite.

11.4 Objective Functions with More than Two Variables

When there are n choice variables, the objective function may be expressed as

$$z = f(x_1,x_2,\ldots,x_n)$$

The total differential will then be

$$dz = f_1dx_1 + f_2dx_2 + \ldots + f_ndx_n.$$

so that the necessary condition for extremum is $dz = 0$ for arbitratry dx_i, which in turn means that all the n first-order partial derivatives are required to be zero:

$$f_1 = f_2 = \ldots = f_n = 0.$$

It can be verified that the second-order differential d^2z can be written as

$$d^2z = [dx_1, dx_2, \cdots, dx_n] \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}$$

$$\equiv (dx)'Hdx.$$

Thus the Hessian determinant is

$$|H| = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

and the second-order sufficient condition for extremum is, as before, that all the n principal minors be positive for a minimum in z and that they duly alternate in sign for a maximum in z, the first one being negative.

In summary,

Conditions for Maximum:

(1) $f_1 = f_2 = \dots = f_n = 0$ (necessary condition)
(2) $|H_1| < 0$, $|H_2| > 0$, $|H_3| < 0$, ... $(-1)^n\ |H_n| > 0$. [d^2z negative definite].

Conditions for Minimum:

(1) $f_1 = f_2 = \dots = f_n = 0$ (necessary condition)
(2) $|H_1| > 0$, $|H_2| > 0$, ..., $|H_n| > 0$. [d^2z positive definite].

Example: Find the extreme values of

$$z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2.$$

From the first-order conditions:

$$\begin{aligned} f_1 = 0: &\quad 4x_1 + x_2 + x_3 = 0 \\ f_2 = 0: &\quad x_1 + 8x_2 + 0 = 0 \\ f_3 = 0: &\quad x_1 + 0 + 2x_3 = 0, \end{aligned}$$

we can find a unique solution $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$. This means that there is only one stationary value, $\bar{z} = 2$. The Hessian determinant of this function is

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

the principal minors of which are all positive: $|H_1| = 4$, $|H_2| = 31$, and $|H_3| = 54$. Thus we can conclude that $\bar{z} = 2$ is a minimum.

11.5 Second-Order Conditions in Relation to Concavity and Convexity

Second-order conditions which are always concerned with whether a stationary point is the peak of a hill or the bottom of a valley are closely related to the so-called (strictly) concave or convex functions.

A function that gives rise to a hill (valley) over the entire domain is said to be a concave (convex) function. If the hill (valley) pertains only to a subset S of the domain, the function is said to be concave (convex) on S.

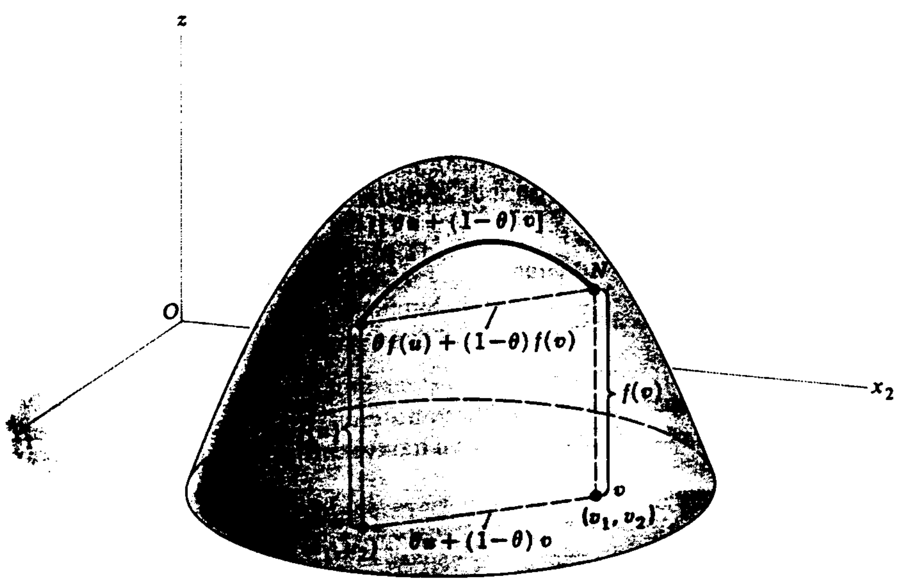
Mathematically, we have

A function f is concave (convex) if and only if, for any pair of distinct points u and v in the domain of f, and for any $0 < \theta < 1$,

$$\theta f(u) + (1-\theta)f(v) \leq f(\theta u + (1-\theta)v)$$
$$\left(\theta f(u) + (1-\theta)f(v) \geq f(\theta u + (1-\theta)v)\right).$$

Further, if the weak inequality "$\leq$" ("$\geq$") is replaced by the strictly inequality "$<$" ("$>$"), the function is said to be strictly concave (strictly convex).

Remark. θu + (1-θ)v consists of line segments between points u and v when θ takes values of $0 \leq \theta \leq 1$. Thus, in the sense of geometry, the function f is concave (convex) iff, the line segment of any two points u and v lies either on or below (above) the surface. The function is strictly concave (strictly convex) iff the line segment lies entirely below (above) the surface, except at M and N.



From the definitions of concavity and convexity, we have the following three theorems:

Theorem I (linear functions). If f(x) is a linear function, then it is a concave function as well as a convex function, but not strictly so.

Theorem II (negative of a function). If f(x) is a (strictly) concave function, then -f(x) is a (strictly) convex function, and vice versa.

Theorem III (sum of functions). If f(x) and g(x) are both concave (convex) functions, then f(x) + g(x) is also a concave (convex) functions. Further, in addition, either one or both of them are strictly concave (strictly convex), then f(x) + g(x) is strictly concave (strictly convex).

In view of the association of concavity (convexity) with a global hill (valley) configuration, an extremum of a concave (convex) function must be a peak -a maximum (a bottom - a minimum). Moreover, the maximum (minimum) must be an absolute maximum (minimum). Further, the maximum (minimum) is unique if the function is strictly concave (strictly convex).

In the preceding paragraph, the properties of concavity and convexity are taken to be global in scope. If they are valid only for a portion of surface (only in a subset S of domain), the associated maximum and minimum are relative to that subset of the domain.

We know that when $z = f(x_1,\ldots,x_n)$ is twice continuously differentiable, $z = f(x_1,\ldots x_n)$ reaches its maximum (minimum) if d^2z is negative (positive) definite.

The following proposition shows the relationship between concavity (convexity) and negative definiteness.

Proposition. A twice continuously differentiable function $z = f(x_1,x_2,\ldots x_n)$ is concave (convex) if and only if d^2z is everywhere negative (positive) semidefinite. The said function is strictly concave (convex) if (but not only if) d^2z is everywhere negative (positive) definite.

This proposition is useful. It can easily verify whether a function is strictly concave (strictly convex) by checking whether its Hessian matrix is negative (positive) definite.

Example: Check $z = -x^4$ for concavity or convexity by the derivative condition.

Since $d^2z = -12x^2 \leq 0$ for all x, it is concave. This function, in fact, is strictly concave.

Example: Check $z = x_1^2 + x_2^2$ for concavity or convex.

Since

$$H = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}.$$

$|H_1| = 2 > 0, \quad |H_2| = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0.$ Thus, by the proposition, the function is strictly convex.

11.6 Economic Applications

Problem of a Multiproduct Firm

Example: Suppose a competitive firm produces two products. Let Q_i represent the output level of the i-th product and let the prices of the products are denoted by P_1 and P_2. Since the firm is a competitive firm, it takes the prices as given. Accordingly, the firm's revenue function will be

$$TR = P_1Q_1 + P_2Q_2$$

The firm's cost function is assumed to be

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2.$$

Thus, the profit function of this hypothetical firm is given by

$$\pi = TR - C = P_1Q_1 + P_2Q_2 - 2Q_1^2 - Q_1Q_2 - 2Q_2^2.$$

The firm wants to maximize the profit by choosing the levels of Q_1 and Q_2. For this purpose, setting

$$\frac{\partial \pi}{\partial Q_1} = 0: \qquad 4Q_1 + Q_2 = P_1$$

$$\frac{\partial \pi}{\partial Q_2} = 0: \qquad Q_1 + 4Q_2 = P_2 \, ,$$

We have

$$\bar{Q}_1 = \frac{4P_1 - P_2}{15} \quad \text{and} \quad \bar{Q}_2 = \frac{4P_2 - P_1}{15}.$$

Also the Hessian matrix is

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix}.$$

Since $|H_1| = -4 < 0$ and $|H_4| = \begin{vmatrix} -4 & -1 \\ -1 & -4 \end{vmatrix} = 16 - 1 = 15 > 0$, the Hessian matrix (or d^2z) is negative definite, and the solution does maximize. If fact, since H is everywhere negative definite, the maximum profit found above is actually a unique absolute maximum.

Example: Let us now transplant the problem in the above example into the setting of a monopolistic market.

Suppose that the demands facing the monopolist firm are as follows:

$Q_1 = 40 - 2P_1 + P_2$
$Q_2 = 15 + P_1 - P_2$

Again, the cost function is given by

$C = Q_1^2 + Q_1Q_2 + Q_2^2.$

From the monopolistic's demand function, we can express prices P_1 and P_2 as functions of Q_1 and Q_2. That is, solving

$$-2P_1 + P_2 = Q_1 - 40$$
$$P_1 - P_2 = Q_2 - 15,$$

we have

$$P_1 = 55 - Q_1 - Q_2$$
$$P_2 = 70 - Q_1 - 2Q_2.$$

Consequently, the firm's total revenue function TR can be written as

$$\begin{aligned} TR &= P_1Q_1 + P_2Q_2 \\ &= (55 - Q_1 - Q_2)Q_1 + (70 - Q_1 - 2Q_2)Q_2 \\ &= 55Q_1 + 70Q_2 - 2Q_1Q_2 - Q_1^2 - 2Q_2^2. \end{aligned}$$

Thus the profit function is

$$\begin{aligned} \pi &= TR - C \\ &= 55Q_1 + 70Q_2 - 3Q_1Q_2 - 2Q_1^2 - 3Q_2^2 \end{aligned}$$

which is an objective function with two choice variables. Setting

$$\frac{\partial \pi}{\partial Q_1} = 0 : \qquad 4Q_1 + 3Q_2 = 55$$

$$\frac{\partial \pi}{\partial Q_2} = 0 : \qquad 3Q_1 + 6Q_2 = 70.$$

we can find the solution output level are

$$(\bar{Q}_1, \bar{Q}_2) = (8,\ 7\ 2/3).$$

The prices and profit are

$$\bar{P}_1 = 39\ 1/3, \quad \bar{P}_2 = 46\ 2/3, \quad \text{and } \bar{\pi} = 488\ 1/3.$$

Inasmuch as the Hessian determinant is

$$\begin{vmatrix} -4 & -3 \\ -3 & -6 \end{vmatrix},$$

we have $|H_1| = -4 < 0$ and $|H_2| = 15 > 0$ so that the value of $\bar{\pi}$ does represent the maximum. Also, since Hessian matrix is everywhere negative definite, it is a unique absolute maximum.

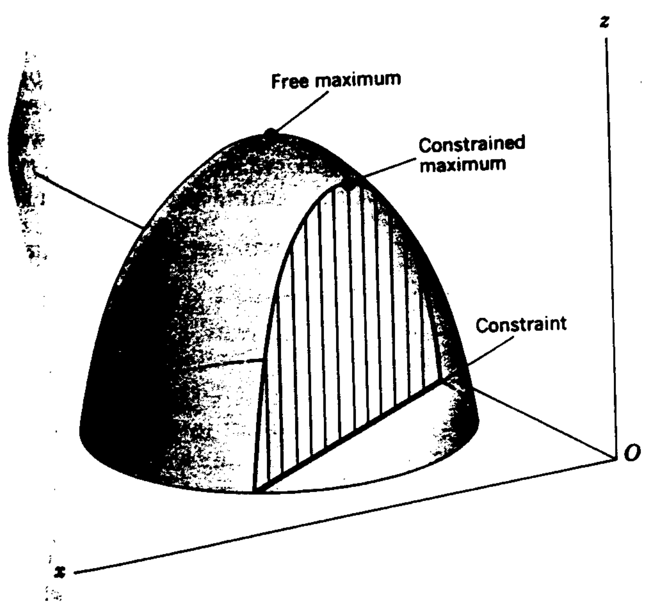
XII. OPTIMIZATION WITH EQUALITY CONSTRAINTS

The last chapter presented a general method for finding the relative extrema of an objective function of two or more choice variables. One important feature of that discussion is that all the choice variables are independent of one another, in the sense that the decision made regarding one variable does not impinge upon the choices of the remaining variables. However, in many cases, optimization problems are the constrained optimization problem. Say, every consumer maximizes his utility subject to his budget constraint. A firm minimizes the cost of production with the constraint of production technology.

In the present chapter, we shall consider the problem of optimization with equality constraints. Our primary concern will be with relative constrained extrema.

12.1 Effects of a Constraint

In general, for a function, say $z = f(x,y)$, the difference between a constrained extremum and a free extremum may be illustrated in the following graph:



The free extremum in this particular graph is the peak point of entire dome, but the constrained extremum is at the peak of the inverse U-shaped curve situated on top of the constraint line. In general, a constraint maximum can be expected to have a lower value than the free maximum, although by coincidence, the two maxima may happen to have the same value. But the constrained maximum can never exceed the free maximum. In general, the number of constraints should be less than the number of choice variables.

12.2 Finding the Stationary Values

For illustration, let us consider a consumer choice problem: maximizes his utility:

$$U(x_1,x_2) = x_1x_2 + 2x_1$$

subject to the budget constraint.

$$4x_1 + 2x_1 = 60$$

Even without any new technique of solution, the constrained maximum in this problem can easily be found. Since the budget line implies

$$x_2 = \frac{60 - 4x_1}{2} = 30 - 2x_1$$

we can combine the constraint with the objective function by substitution. The result is an objective function in one variable only:

$$u = x_1(30 - 2x_1) + 2x_1 = 32x_1 - 2x_1^2$$

which can be handled with the method already learned. By setting $\frac{\partial u}{\partial x_1} = 32 - 4x_1 = 0$, we get the solution $\bar{x}_1 = 8$ and thus, by the budget constraint,
$\bar{x}_2 = 30 - 2\bar{x}_1 = 30 - 16 = 14$ since $d^2u/dx^2 = -4 < 0$, that stationary value constitutes a (constrained) maximum.

However, when the constraint is itself a complicated function, or when the constraint cannot be solved to express one variable as an explicit function of the other variables, the technique of substitution and elimination of variables could become a burdensome task or would in fact be of no avail. In such cases, we may resort to a method known as the method of Lagrange multiplier.

Lagrange-Multiplier Method

The essence of the Lagrange-multiplier method is to convert a constrained-extremum problem into a form such that the first-order condition of the free-extremum problem can still be applied.

In general, given an objective function

$$z = f(x,y)$$

subject to the constraint

$$g(x,y) = c$$

where c is a constant, we can write the Lagrange function as

$$Z = f(x,y) + \lambda[c - g(x,y)].$$

The symbol λ, representing some as yet undermined number, is called a Lagrange multiplier. If we can somehow be assured that $g(x,y) = c$, so that the constraint will be satisfied, then the last term of Z will vanish regardless of the value of λ. In that event, Z will be identical with u. Moreover, with the constraint out of the way, we only have to seek the free maximum. The question is: How can we make the parenthetical expression in Z vanish?

The tactic that will accomplish this is simply to treat λ as an additional variable, i.e., to consider $Z = Z(\lambda,x,y)$. For stationary values of Z, then the first-order condition for free extremum is

$$Z_\lambda \equiv \frac{\partial Z}{\partial \lambda} = c - g(x,y) = 0$$

$$Z_x \equiv \frac{\partial Z}{\partial x} = f_x - \lambda g_x = 0$$

$$Z_y \equiv \frac{\partial Z}{\partial y} = f_y - \lambda g_y = 0$$

and the first equation will automatically guarantee the satisfaction of the constraint. And since the expression $\lambda[c - g(x,y)] = 0$, the stationary values of Z must be identical with those of $Z = f(x,y)$, subject to $g(x,y) = C$.

Example: Let us again consider the consumer's choice problem above. The Lagrange function is

$$Z = x_1x_2 + 2x_1 + \lambda[60 - 4x_1 - 2x_2]$$

for which the necessary condition for a stationary value is

$$Z_\lambda = 60 - 4x_1 - 2x_2 = 0$$

$$Z_1 = x_2 + 2 - 4\lambda = 0$$

$$Z_2 = x_1 - 2\lambda = 0$$

Solving the critical values of the variables, we find that $\bar{x}_1 = 8$, $\bar{x}_2 = 14$, and $\lambda = 4$. As expected, $\bar{x}_1 = 8$ and $\bar{x}_2 = 14$ are the same obtained by the substitution method.

Example: Find the extremum of $z = xy$ subject to $x + y = 6$. The Lagrange function is

$$Z = xy + \lambda(6 - x - y)$$

The first-order condition is

$$Z_\lambda = 6 - x - y = 0$$

$$Z_x = y - \lambda = 0$$

$$Z_y = x - \lambda = 0$$

Thus, we find $\bar{\lambda} = 3$, $\bar{x} = 3$, $\bar{y} = 3$.

Example: Find the extremum of $z = x_1^2 + x_2^2$ subject to $x_1 + 4x_2 = 2$.

The Lagrange function is

$$Z = x_1^2 + x_2^2 + \lambda(2 - x_1 - 4x_2).$$

The first-order condition (FOC) is

$$Z_\lambda = 2 - x_1 - 4x_2 = 0$$

$$Z_1 = 2x_1 - \lambda = 0$$

$$Z_2 = 2x_2 - 4\lambda = 0$$

The stationary value of Z, defined by the solution

$$\bar{\lambda} = \frac{4}{17},\ \bar{x}_1 = \frac{2}{17},\ \bar{x}_2 = \frac{8}{17},\ \bar{z} = \bar{z} = \frac{4}{17}\ .$$

To tell whether $\bar{z}$ is a maximum or minimum, we need to consider the second-order condition.

An Interpretation of the Lagrange Multiplier

The Lagrange multiplier $\bar{\lambda}$ measures the sensitivity of Z to change in the constraint. If we can express the solutions $\bar{\lambda}$, $\bar{x}$, and $\bar{y}$ all as implicit functions of the parameter C:

$$\bar{\lambda} = \bar{\lambda}(c),\ \bar{x} = \bar{x}(c),\ \text{and}\ \bar{y} = \bar{y}(c)$$

all of which will have continuous derivatives. Also we have the identities

$$c - g(\bar{x}, \bar{y}) \equiv 0$$

$$f_x(\bar{x}, \bar{y}) - \bar{\lambda} g_x(\bar{x}, \bar{y}) \equiv 0$$

$$f_y(\bar{x}, \bar{y}) - \bar{\lambda} g_y(\bar{x}, \bar{y}) \equiv 0$$

Thus, we can consider Z as a function c:

$$Z = f(\bar{x}, \bar{y}) + \bar{\lambda}[c - g(\bar{x}, \bar{y})].$$

Then

$$
\begin{aligned}
\frac{d\bar{Z}}{\partial c} &= f_x \frac{d\bar{x}}{dc} + f_y \frac{d\bar{y}}{dc} + [c - g(\bar{x}, \bar{y})] \frac{d\bar{\lambda}}{dc} \\
&\quad + \lambda \left[1 - g_x \frac{d\bar{x}}{dc} - g_y \frac{d\bar{y}}{dc} \right] \\
&= (f_x - \lambda g_x) \frac{d\bar{x}}{dc} + (f_y - \lambda g_y) \frac{d\bar{y}}{dc} + [c - g(\bar{x}, \bar{y})] \frac{d\bar{x}}{dc} + \lambda \\
&= \lambda
\end{aligned}
$$

n-Variable and Multiconstraint Cases

The generalization of the Lagrange-multiplier method to n variable can be easily carried. The objective function is

$$z = f(x_1, x_2, \ldots, x_n)$$

subject to

$$g(x_1, \ldots, x_n) = c.$$

It follows that the Lagrange function will be

$$Z = f(x_1, x_2, \ldots, x_n) - \lambda g(x_1, x_2, \ldots, x_n)$$

for which the first-order condition will be given by

$$Z_\lambda = c - g(x_1, x_2, \ldots, x_n)$$

$$Z_i = f_i(x_1, x_2, \ldots, x_n) - \lambda g_i(x_1, \ldots, x_n) \qquad [i = 1, 2, \ldots, n].$$

If the objective function has more than one constraint, say, two constraints

$$g(x_1, \ldots, x_n) = c \quad \text{and} \quad h(x_1, \ldots, x_n) = d$$

The Lagrange function is

$$Z = f(x_1,\ldots,x_n) + \lambda[c - g(x_1,\ldots,x_n)] + \mu[d - h(x_1,\ldots,x_n)],$$

for which the first-order condition consists of (n + 2) equations:

$$Z_\lambda = c - g(x_1,x_2,\ldots,x_n) = 0$$

$$Z_\mu = d - h(x_1,x_2,\ldots,x_n) = 0$$

$$Z_i = f_i(x_1,\ldots,x_n) - \lambda g_i(x_1,\ldots,x_n) - \mu h_i(x_1,\ldots,x_n) = 0$$

12.3 Second-Order Condition

From the last section, we know that finding the constrained extremum is equivalent to find the free extremum of the Lagrange function Z and gave the first-order condition. This section gives the second-order condition for the constrained extremum of f.

We only consider the case where the objective functions take form

$$z = f(x_1,x_2,\ldots,x_n)$$

subject to

$$g(x_1,x_2,\ldots,x_n).$$

The Lagrange function is then

$$Z = f(x_1,\ldots,x_n) + \lambda[c - g(x_1,\ldots,x_n)].$$

Define the bordered Hessian determinant $|\bar{H}|$ by

$$|H| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & Z_{11} & Z_{12} & \cdots & Z_{1n} \\ g_2 & Z_{21} & Z_{22} & \cdots & Z_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n & Z_{n1} & Z_{n2} & \cdots & Z_{nn} \end{vmatrix}$$

where $Z_{ij} = f_{ij} - \lambda g_{ij}$. Note that by the first-order condition, $\lambda = \frac{f_1}{g_1} = \frac{f_2}{g_2} = \cdots = \frac{f_n}{g_n}$.

The bordered principal minors can be defined as

$$|\bar{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & Z_{11} & Z_{12} \\ g_2 & Z_{21} & Z_{22} \end{vmatrix} \qquad |\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & Z_{11} & Z_{12} & Z_{13} \\ g_2 & Z_{21} & Z_{22} & Z_{23} \\ g_3 & Z_{31} & Z_{32} & Z_{33} \end{vmatrix} \qquad \text{(etc.)}.$$

The Conditions for Maximum:

(1) $Z_\lambda = Z_1 = Z_2 = \ldots = Z_n = 0$ [necessary condition]

(2) $|\bar{H}_2| > 0 \,;\, |\bar{H}_3| < 0 \,;\, |\bar{H}_4| > 0 \,;\, \ldots\, (-1)^n|\bar{H}_n| > 0.$

The Conditions for Minimum:

(1) $Z_\lambda = Z_1 = Z_2 = \ldots = Z_n = 0$ [necessary condition]

(2) $|\bar{H}_2| < 0 \,;\, |\bar{H}_2| < 0 \,;\, \ldots\, |\bar{H}_n| < 0.$

Example: For the objective function $z = xy$ subject to $x + y = b$, we have shown that $(\bar{x},\bar{y},\bar{z}) = (3,3,9)$ is an extremum point. Since $Z_x = y - \lambda$ and $Z_y = x - \lambda$, then $Z_{xx} = 0$, $Z_{xy} = 1$, and $Z_{yy} = 0$, $g_x = g_y = 1$. Thus, we find that

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2 > 0.$$

which establishes the value of $\bar{z} = 9$ as an maximum.

Example: For the objective function of $Z = x_1^2 + x_2^2$ subject to $x_1 + 4x_2 = 2$, we have shown that $(\bar{x}, \bar{y}, \bar{z}) = (\frac{2}{17}, \frac{8}{17}, \frac{4}{7})$ is the extremum point. To tell whether it is maximum or minimum, we check the second-order sufficient condition. Since $Z_1 = 2x_1 - \lambda$ and $Z_2 = 2x_2 - \lambda$ as well as $g_1 = 1$ and $g_2 = 4$, we have $Z_{11} = 2$, $Z_{22} = 2$, and $Z_{12} = Z_{21} = 0$. It thus follows that the bordered Hessian is

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & 4 \\ 1 & 2 & 0 \\ 4 & 0 & 2 \end{vmatrix} = -34 < 0$$

and the value $\bar{Z} = \frac{4}{17}$ is a minimum.

12.4 Quasiconcavity and Quasiconvexity

For a problem of free extremum, we know that the concavity (convexity) of the objective function guarantees the existence of absolute maximum (absolute minimum). For a problem of constrained optimization, we will demonstrate the quasiconcavity (quasiconvexity) of the objective function guarantees the existence of absolute maximum (absolute minimum).

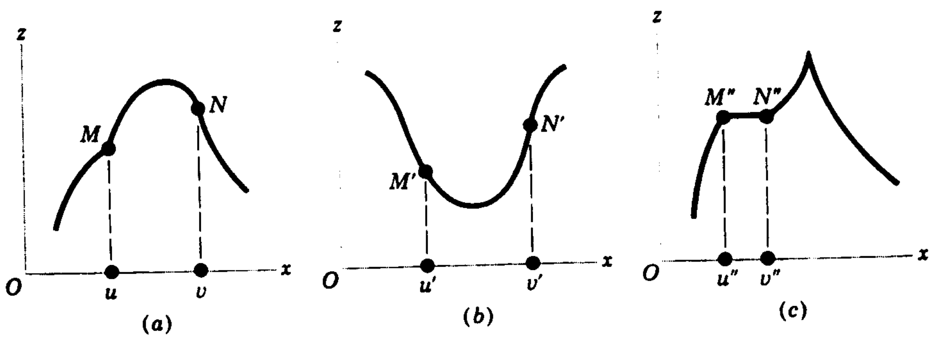
Algebraic Characterization

Quasiconcavity and quasiconvexity, like concavity and convexity, can be either strict or nonstrict:

Definition: A function is quasiconcave (quasiconvex) iff, for any pair of distinct points u and v in the convex domain of f, and for $0 < \theta < 1$, $f(v) \geq f(u)$ implies

$$f(\theta u + (1 - \theta)v) \geq f(u)$$
$$[f(\theta u + (1 - \theta)v) \leq f(v)].$$

Further, if the weak inequality " ≥ " (" ≤ ") is replaced by the strict inequality " > " (" < "), f is said to be strictly quasiconcave (strictly quasiconvex).



<u>Remark:</u> From the definition of quasiconcavity (quasiconvexity), we can know that quasiconcavity (quasiconvexity) is a weaker condition than concavity (convexity).

<u>Theorem I (negative of a function)</u>. If f(x) is quasiconcave (strictly quasiconcave), then f(x) is quasiconvex (strictly concave).

<u>Theorem II (concavity versus quasiconcavity)</u>. Any (strictly) concave (convex) function is (strictly) quasiconcave (quasiconvex), but the converse is not true.

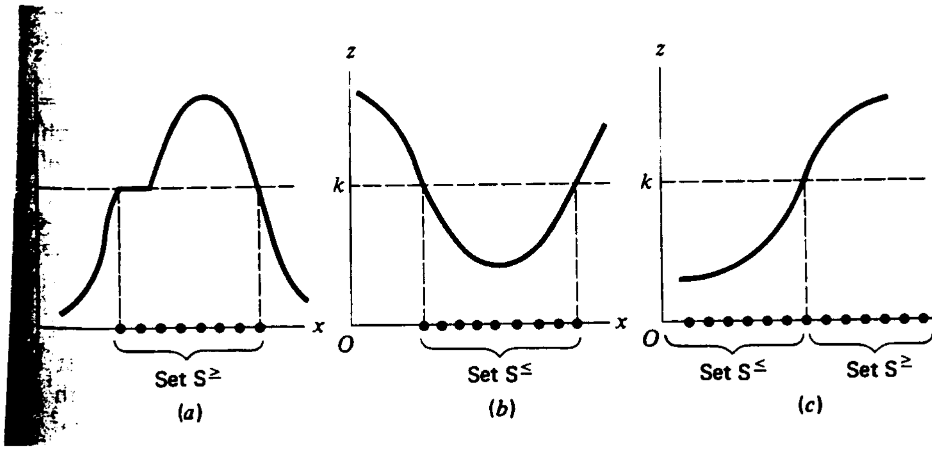
<u>Theorem III (linear function)</u>. If f(x) is linear, then it is quasiconcave as well as quasiconvex.

<u>Theorem IV (monotone function with one variable)</u>. If f is a function of <u>one variable</u>, then it is quasiconcave as well as quasiconvex.

<u>Remark:</u> Note that, unlike concave (convex) functions, a sum of two quasiconcave (quasiconvex) functions is not necessarily quasiconcave (quasiconvex).

Sometimes it may prove easier to check quasiconcavity and quasiconvexity by the following alternative definitions.

A function f(x), where x is a vector of variables, is quasiconcave (quasiconvex) iff, for any constant k, the set $S^{\geq} \equiv \{x \mid f(x) \geq k\}$ ($S^{\leq} \equiv \{x \mid f(x) \leq k\}$) is convex.



Examples:

(1) $Z = x^2$ is quasiconvex since $S^{\leq}$ is convex.

(2) $Z = f(x,y) = xy$ is quasiconcave since $S^{\geq}$ is convex.

(3) $Z = f(x,y) = (x - a)^2 + (y - b)^2$ is quasiconcave since $S^{\leq}$ is convex.

The above facts can be seen by looking at graphs of these functions.

Differentiable Functions

If a function $z = f(x_1,\ldots,x_n)$ is twice continuously differentiable, quasiconcavity and quasiconvexity can be checked by means of the first and second partial derivatives of the function.

Define a bordered determinant as follows:

$$|B| = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

Remark: The determinant $|B|$ is different from the bordered Hessian $|H|$. Unlike $|H|$, the border in $|B|$ is composed of the first derivatives of the function f rather than an extraneous constraint function g.

We can define successive principal minors of B as follows:

$$|B_1| = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix} \qquad |B_2| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} \qquad \dots \qquad |B_n| = |B|.$$

A necessary condition for a function $z = f(x_1,\dots,x_n)$ defined the nonnegative orthant to be quasiconcave is that

$$|B_1| \le 0,\ |B_2| \ge 0,\ |B_3| \le 0,\ \dots,\ (-1)^n|B_n| \ge 0.$$

For quasiconvexity, it is necessary that

$$|B_1| \le 0,\ |B_2| \le 0,\ \dots,\ |B_n| \le 0.$$

A sufficient condition for f to be quasiconcave on the nonnegative orthant is that

$$|B_1| < 0,\ |B_2| > 0,\ |B_3| < 0,\ \dots,\ (-1)^n|B_n| > 0.$$

For quasiconvexity, the corresponding sufficient condition is that

$$|B_1| < 0,\ |B_2| < 0,\ \dots,\ |B_n| < 0.$$

Example: $z = f(x_1,x_2) = x_1x_2$. Since $f_1 = x_2$, $f_2 = x_1$, $f_{11} = f_{22} = 0$, and $f_{12} = f_{21} = 1$, the relevant principal minors turn out to be

$$|B_1| = \begin{vmatrix} 0 & x_2 \\ x_2 & 0 \end{vmatrix} = -x_2^2 \le 0 \qquad |B_2| = \begin{vmatrix} 0 & x_2 & x_1 \\ x_2 & 0 & 1 \\ x_1 & 1 & 0 \end{vmatrix} = 2x_1x_2 \ge 0.$$

Thus $z = x_1x_2$ is quasiconcave on the positive orthant.

Example: Show that $z = f(x,y) = x^a y^b$ $(x, y > 0, 0 < a, b < 1)$ is quasiconcave.

Since

$$f_x = ax^{a-1}y^b \qquad f_y = bx^a y^{b-1}$$

$$f_{xx} = a(a-1)x^{a-2}y^b \qquad f_{xy} = f_{yx} = abx^{a-1}y^{b-1} \qquad f_{yy} = b(b-1)x^a y^{b-2}.$$

thus

$$|B_1| = \begin{vmatrix} 0 & f_x \\ f_x & f_{xx} \end{vmatrix} = -(ax^{a-1}y^b)^2 < 0$$

$$|B_2| = \begin{vmatrix} 0 & f_x & f_y \\ f_x & f_{xx} & f_{xy} \\ f_y & f_{yx} & f_{yy} \end{vmatrix} = [2a^2b^2 - a(a-1)b^2 - a^2b(b-1)]x^{3a-2}y^{3b-2} > 0.$$

So it is quasiconcave.

Remark: When the constraint g is linear: $g(x) = a_1x_1 + \ldots + a_nx_n = c$, the bordered determinant $|B|$ and the bordered Hessian have the following relationship:

$$|B| = \lambda^2 |\bar{H}|$$

Consequently, in the linear-constraint case, the two bordered determinants always have the same sign at the stationary of z. The same is true for principal minors.

Absolute versus Relative Extrema

If a function is quasiconcave (quasiconvex), by the similar reasons for concave (convex) functions, its relative maximum (relative minimum) is an absolute maximum (absolute minimum).

12.5 Utility Maximization and Consumer Demand

Let us now re-examine the consumer choice problem--utility maximization problem. For simplicity, only consider the two-commodity case. The consumer wants to maximize his utility

$$u = u(x,y) \qquad (u_x > 0,\ u_y > 0)$$

subject to his budget constraint

$$P_x x + P_y y = B$$

by taking prices P_x and P_y as well as his income as given.

<u>First-Order Condition</u>

The Lagrange function is

$$Z = u(x,y) + \lambda(B - P_x x - P_y y).$$

At the first-order condition, we have the following equations:

$$Z_\lambda = B - P_x x - P_y y = 0$$

$$Z_x = u_x - \lambda P_x = 0$$

$$Z_y = u_y - \lambda P_y = 0$$

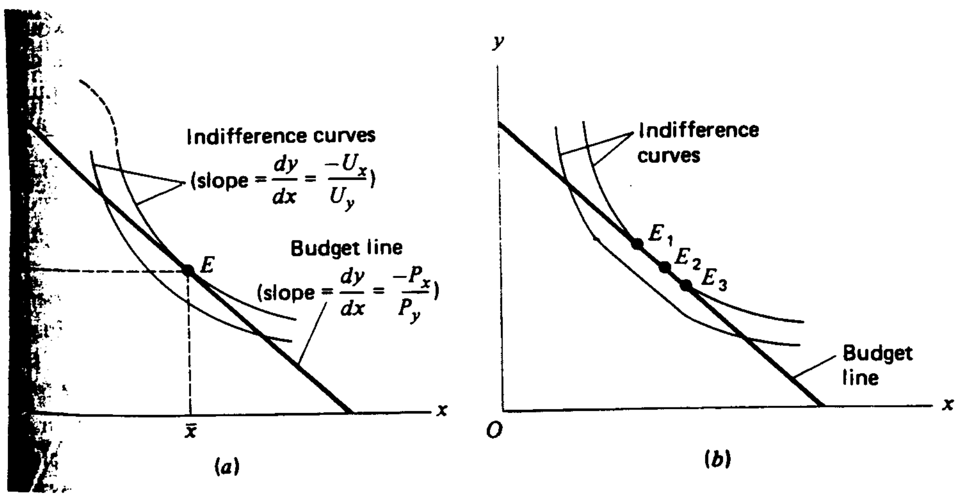
From the last two equations, we have

$$\frac{u_x}{P_x} = \frac{u_y}{P_y} = \lambda$$

Or

$$\frac{u_x}{u_y} = \frac{P_x}{P_y}.$$

The term $\frac{u_x}{u_y} = MRS_{xy}$ is the so-called marginal rate of substitution of x for y. Thus, we obtain the well-known equality: $MRS_{xy} = \frac{P_x}{P_y}$ which is the necessary condition for the interior solution.



Second-Order Condition

If the bordered Hessian in the present problem is positive, i.e., if

$$|\bar{H}| = \begin{vmatrix} O & P_x & P_y \\ P_x & U_{xx} & U_{xy} \\ P_y & U_{yx} & U_{yy} \end{vmatrix} = 2P_xP_yu_{xy} - P_y^2U_{xx} - P_x^2U_{yy} > 0,$$

(with all the derivatives evaluated at the critical value of $\bar{x}$ and $\bar{y}$), then the stationary value of U will assuredly be maximum.

Since the budget constraint is linear, from the result in the last section, we have

$$|B| = \lambda^2 |\bar{H}|$$

Thus, as long as $|B| > 0$, we know the second-order condition holds.