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ON THE CONSTRAINED WALRASIAN AND LINDAHL CORRESPONDENCES

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This paper establishes conditions on preferences under the constrained Walrasian correspondence W_c, the constrained Lindahl correspondence L_c, and the weak Lindahl correspondence L_w that is introduced in this paper are subcorrespondences of the (weak) Pareto-efficient correspondence P (P_w).

1. Introduction

The purpose of this paper is to clear up some confusion which has apparently arisen in the literature on implementation of the constrained Walrasian correspondence W_c and the constrained Lindahl correspondence L_c. For the private good economies, W_c is an extension of the Walrasian correspondence W introduced in the implementation literature to deal with possible boundary violations by W of Maskin's (1977) monotonicity condition. It is generally believed that W_c, like W, is a subcorrespondence of the Pareto-efficient correspondence [1] P under the same conditions where the W is Pareto efficient. An example given by Thomson (1985) shows that this is not true. Even so, we will show that the W_c is a subcorrespondence of the weak Pareto-efficient correspondence P_w, and of the Pareto-efficient correspondence if preferences satisfy strict monotonicity. For the public goods economies, the constrained Lindahl correspondence L_c is an extension of the Lindahl correspondence L [cf. Hurwicz (1986, pp. 1470–1471) or Hurwicz, Maskin, and Postlewaite (1984)]. It is also believed that L_c is monotonic and Pareto efficient [see Hurwicz (1986, pp. 1470–1471)]. We will show that this is not true. For this reason, we introduce the weak Lindahl correspondence L_w, so as to be distinguished from L_c, which will be shown to be monotone and weakly Pareto optimal. For the sake of convenience, we first give some notation and definitions. The main results are given in section 3.

2. Economic environments and allocations

A public goods economy has L private goods and K public goods (if $K = 0$, the economy becomes a private goods economy), x being private and y public. There are n agents. Denote by

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[1] Pareto efficiency means that there is no way of making everyone at least as well off and one person better off while weak Pareto efficiency means that there is no way of making everyone better off.

$N = \{1, 2, \ldots, n\}$ the set of agents. Each agent's initial endowment is $w_i \geq 0$.[2] Each consumer has a preference ordering R_i defined on R_+^{L+K} (P_i denotes the strict preference ordering). We assume that there is no initial endowment of public goods, but that the public goods can be produced by the private goods. Let Y be the production set. Its element is (r, y), where r is the vector of private goods inputs which is non-positive and y is the vector of public goods outputs. It is assumed that the production possibility set Y is a closed, convex cone; $0 \in Y$; and for any $y \in R_+^K$, there is an $r \in -R_+^L$ such that $(r, y) \in Y$. As above, we call $e_i = (w_i, R_i)$ the characteristic of consumer i and we call the full vector $e = (e_1, \ldots, e_n, Y)$ an economy and the set of all such economies is denoted by E.

An allocation $z = (x, y)$ is *feasible* if $(x, y) \in R_+^{nL+K}$ and $(r, y) \in \gamma$, where $\sum_{i=1}^n x_i - r \leqq \sum_{i=1}^n w_i$ and $x = (x_1, \ldots, x_n)$. Denote by $A(e)$ all such allocations. An allocation (x^*, y^*) is a *Lindahl allocation* for an economy e, if it is feasible and there is a price vector $p^* \in R_+^L$ and price vectors $q_i^* \in R_+^K$, one for each i, such that (i) $p^* \cdot x_i^* + q_i^* \cdot y^* \leq p^* \cdot w_i$ for all $i \in N$, (ii) $(x_i, y)\ P_i\ (x_i^*, y^*)$ implies $p^* \cdot x_i + q_i^* \cdot y > p^* \cdot w_i$ for all $i \in N$, (iii) $q^* \cdot y^* + p^* \cdot (\sum_{i=1}^n x_i^* - \sum_{i=1}^n w_i) \geq p^* \cdot r + q^* \cdot y$ for all $(r, y) \in Y$, where $q^* = \sum_{i=1}^n q_i^*$. Denote by $L(e)$ the set of all such allocations. A *constrained Lindahl allocation* (x^*, y^*) for an economy e differs from an ordinary Lindahl allocation in that condition (ii) is replaced by the following (ii'): $(x_i, y)P_i(x_i^*, y^*)$ and $x_i \leqq \sum_{j=1}^n w_j$ imply $p^* \cdot x_i + q_i^* \cdot y > p^* \cdot w_i$ for all $i \in N$. Denote by $L_c(e)$ all such allocations. A *weak Lindahl allocation* (x^*, y^*) for an economy e differs from an ordinary Lindahl allocation in that condition (ii) is replaced by the following (ii''): $(x_i, y)P_i(x_i^*, y^*)$ and $x_i - r \leqq \sum_{j=1}^n w_j$ imply $p^* \cdot x_i + q_i^* \cdot y > p^* \cdot w_i$ for all $i \in N$. Denote by $L_w(e)$ all such allocations. Analogously, for the private goods economies, an allocation $x^* \in R_+^{nL}$ *is a Walrasian allocation* for an economy e, if there is a price vector $p^* \in R_+^L$ such that (i) $p^* \cdot x_i^* \leq p^* \cdot w_i$ for all $i \in N$, (ii) $x_i P_i x_i^*$ implies $p^* \cdot x_i > p^* \cdot w_i$ for all $i \in N$, and *(iii)* $\sum_{j=1}^n x_j^* \leqq \sum_{j=1}^n w_j$. A *constrained Walrasian allocation* x^* for an economy e differs from an ordinary Walrasian allocation in that condition (ii) is replaced by the following (ii'): $x_i P_i x_i^*$ and $x_i \leqq \sum_{j=1}^n w_j$ imply $p^* \cdot x_i > p^* \cdot w_i$ for all $i \in N$.

3. Main results

It is known that the Pareto-efficient correspondence and the weak Pareto-efficient correspondence coincide for the private goods economies under the conditions of strict monotonicity and continuity. It seems (to my knowledge) nobody points out that the weak Pareto-efficient correspondence and the Pareto-efficient correspondence do not coincide for the public goods economies even if preferences satisfy strict monotonicity and continuity.

Theorem 1. *For the public goods economies, $P \neq P_w$ even if preferences satisfy strict monotonicity and continuity.*

Proof. The proof is by way of an example. Consider an economy with $(n, L, K) = (3, 1, 1)$, constant returns in producing y from x (the input–output coefficient normalized to one), and the following endowments and utility functions: $w_1 = w_2 = w_3 = 1$, $u_1(x_1, y) = x_1 + y$, and $u_i(x_i, y) = x_i + 2y$ for $i = 2, 3$. Then $z = (x, y)$ with $x = (0.5, 0, 0)$ and $y = 2.5$ is weakly Pareto-efficient but not Pareto-efficient because $z' = (x', y') = (0, 0, 0, 3)$ Pareto-dominates z by agents 2 and 3. Q.E.D.

[2] As usual, vector inequalities are defined as follows: Let $a, b \in R^m$. Then $a \geqq b$ means $a_s \geq b_s$ for all $s = 1, \ldots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \ldots, m$.

The main results on W_c, L_c, and L_w are summarized in the following two theorems. Notice that though some results on W_c are not new, we still state them in *(i), (iii)*, and *(v)* in Theorem 2 in order to compare W_c with L_c and L_w.

Theorem 2. For the private goods economies, the following statements hold:
(i) W_c *is monotonic.* [3]
(ii) $W \subseteq W_c \subseteq P_w$.
(iii) $W_c \not\subseteq P$*, even if preferences satisfy local non-satiation.*
(iv) $W_c \subseteq P$ *if preferences satisfy strict monotonicity.*
(v) $\mathrm{int} W_c = \mathrm{int} W$ *if preferences satisfy convexity.* [4]

Theorem 3. For the public goods economies, the following statements hold:
(i′) *(a)* L_c *is not monotonic;*
(b) L_w *is monotonic.*
(ii′ $L \subseteq L_c \subseteq L_w \subseteq P_w$.
(iii′) $L_c \not\subseteq P$ *even if preferences satisfy local non-satiation.*
(iv′) *(a)* $L_c \subseteq P$ *if preferences satisfy strict monotonicity;*
(b) $L_w \not\subseteq P$ *even if preferences satisfy strict monotonicity.*
(v′) $L_w = L_c = L$ *if preferences satisfy convexity and* $L_w \subseteq R^L_{++} \times R^K_{+}$.

The proofs of (*i*), (*iii*), and (*v*) of Theorem 2 can be seen in Thomson (1985). In any case, the proof of Theorem 2 is similar to that of Theorem 3. We will only give proofs of the statements in Theorem 3.

Proof of (i′a). The proof is by way of an example. Consider an economy e with $(n, L, K) = (2, 1, 1)$, constant returns in producing y from x (the input–output coefficient normalized to one), and $w_1 = w_2 = 1$, illustrated in fig. 1. $z^* = (x^*, y^*)$ with $x^* = (0, 0.5)$ and $y^* = 1.5$ is a constrained Lindahl allocation with associated $p^* = 1$, $q^* = (2/3, 1/3)$. So z^* is on the boundary of $A(e)$. Now we choose R_2' so that $C(z^*, e_2) \subseteq C(z^*, e_2')$ (see fig. 1). We find that (x_2^*, y^*) is not an R_2'-maximizing consumption satisfying his/her budget constraint $x_2 + q_2^* y \leqq 1$ and the aggregate constraint: $x_2 \leqq w_1 + w_2$ because $(x_2, y) P_2' (x_2^*, y^*)$, and q^* is the only candidate personalized price vector, then z^* is not in $L_c(e_1, e_2')$. Hence L_c is not monotone. Similar examples could be easily found for any other $(n, L,)$. Q.E.D.

Remark 1. For the public goods economies with one private good and one public good, we can easily see $L_c = L$. Thus L is not monotonic, either. Notice that the (constrained) Lindahl correspondence violates monotonicity only on the boundary of A. This observation motivates the weak Lindahl correspondence.

Proof of (i′b). This follows directly from the definition of L_w. Q.E.D.

The proof of (*ii′*) is straightforward.

Proof of (iii′). The proof is by way of an example. Consider an economy e with $(n, L, K) = (2, 3, 1)$, the production relation: $y = x_3$, and the following endowments and utility functions: $w_l = w_2 =$

[3] A correspondence F: $E \to\to A$ is monotone on E if and only if for any $e, e' \in E$ and $a \in F(e)$, $C(a, e_i) \subseteq C(a, e_i')$ for all $i \in N$ implies $a \in F(e')$, where $C(a, e_i) = \{b \in A(e)\colon aR_i b\}$.

[4] preference R_i is convex if, for a, b, c in R^m_+ and $0 < t \leq 1$ and $c = ta + (1-t)b$, the relation $aP_i b$ implies $cP_i b$.

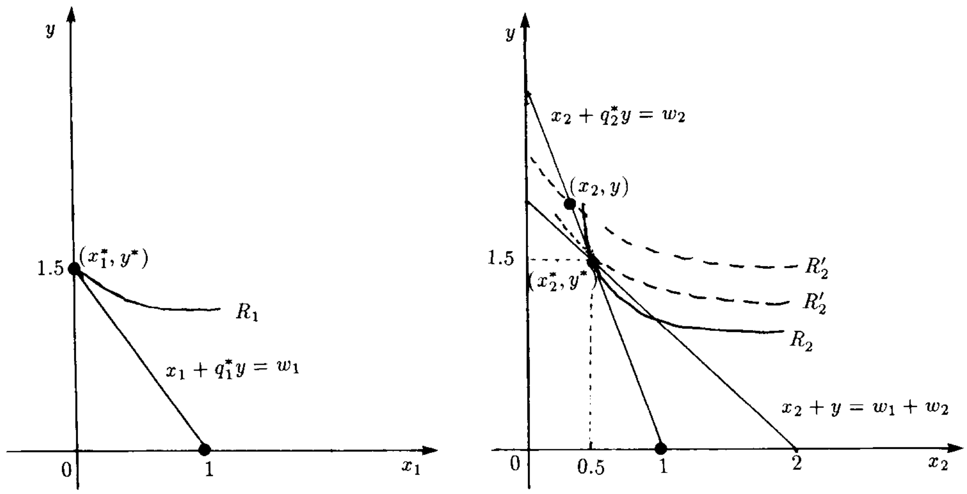

Fig. 1.

(1, 1, 1), $u_1(x_1, y) = x_{11} + x_{31}$, $u_2(x_2, y) = x_{22} + y$. Then $z = (x_1, x_2, y)$ with $x_1 = (1, 0, 1)$, $x_2 = (1, 2, 0)$ and $y = 1$ is a constrained Lindahl allocation with associated $p^* = (1, 0, 1)$ and $q^* = (0, 1)$. But z is not Pareto-efficient because it is Pareto-dominated by $z' = (x_1', x_2', y')$ with $x_1' = (2, 0, 1)$, $x_2' = (0, 2, 0)$ and $y' = 1$. Q.E.D.

Proof of (iv'a). We argue by contradiction. Suppose that there is an allocation (x^*, y^*) which is in $L_c(e)$ but not in $P(e)$ for some $e \in E$. Then there is a feasible allocation (x, y), such that $(x_i, y) R_i ici(x_i^*, y^*)$ for all $i \in N$, $(x_j, y) P_j(x_j^*, y^*)$ for at least one j. We first show $p^* \cdot x_i + q_i^* \cdot y \geq p^* \cdot w_i$ for all $i \in N$. Suppose, by way of contradiction, that $(x_i, y) R_i(x_i^*, y^*)$ and $p^* \cdot x_i + q_i^* \cdot y < p^* \cdot w_i$ for some $i \in N$. Then $x_{il} < \sum_{j=1}^n w_{jl}$ for some good l. Therefore, there is x_i' such that $x_i' \geq x_i$, $p^* \cdot x_i' + q_i^* \cdot y \leq p^* \cdot w_i$ and $x_i' \leq \sum_{j=1}^n w_j$. Thus, we have $(x_i', y) P_i(x_i, y) R_i(x_i^*, y^*)$ by strict monotonicity of preferences. This contradicts $(x^*, y^*) \in (L_c(e)$. So we must have $p^* \cdot x_i = q_i^* \cdot y \geq p^* \cdot w_i$ for all $i \in N$. Also, by the definition of $L_c(e)$, we have $p^* \cdot x_j + q_j^* \cdot y > p^* \cdot w_j$ for at least one j and thus $\sum_{k=1}^n p^* \cdot x_k + q^* \cdot y > \sum_{k=1}^n p^* \cdot w_k$ or $q^* \cdot y + q^* \cdot r > 0$. But the profit-maximizing condition for the weak Lindahl equilibrium asserts that $q^* \cdot y + p^* \cdot r \leq q^* \cdot y^* + p^* \cdot (\sum_{k=1}^n x_k^* - \sum_{k=1}^n w_k) \leq 0$ for all $(r, y) \in Y$, a contradiction. Q.E.D.

Proof of (iv'b). The proof is by way of an example. Consider an economy with $(n, L, K) = (3, 1, 1)$, constant returns in producing y from x (the input–output coefficient normalized to one), and the following endowments and utility functions: $w_1 = w_2 = w_3 = 1$, $u_1(x_1, y) = x_1 + y$, and $u_i(x_i, y) = x_i + 2y/3$ for $i = 2, 3$. Then $z = (x, y)$ with $x = (1/3, 0, 0)$ and $y = 8/3$ is a weak Lindahl allocation, with associated $q^* = (1/4, 3/8, 3/8)$, but not a Pareto-efficient allocation because $z' = (x, y) = (0, 0, 0, 3)$ Pareto-dominates z by agents 2 and 3. Q.E.D.

Proof of (v'). By (ii'), we only need to show that L_w is a subcorrespondence of L. Suppose, by way of contradiction, that there is an allocation (x^*, y^*) which is in $L_w(e)$ with associated price vectors $p^* \in R_+^L$ and $q_i^* \in R_+^K$ but not in $L(e)$ for some $e \in E$. Then there is $(x_i, y) \in R_+^{L+K}$ such that $(x_i, y) P_i(x_i^*, y^*)$ and $p^* \cdot x_i + q_i^* \cdot y \leq p^* \cdot w_i$ for some $i \in N$. By convexity of preferences, for all $(x_{it}, y_t) = (tx_i + (1-t)x_i^*, ty + (1-t)y^*)$ where $0 < t \leq 1$, we have $(x_{it}, y_t) P_i(x_i^*, y^*)$ and $p^* \cdot x_{it}$

$+ q_i^* \cdot y_t \leq p^* \leq w_i$. Since $x^* > 0$ and thus $x_i^* - r^* < \sum_{j=1}^n w_j$, (x_{it}, y_t) can be chosen to satisfy $x_{it} - r_t < \sum_{j=1}^n w_j$, but this contradicts $z^* \in L_w(e)$. Therefore $L_w \subseteq L$. Q.E.D.

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