Economics Letters 33 (1990) 203–206
North-Holland

EQUILIBRIUM IN ABSTRACT ECONOMIES WITH A NON-COMPACT INFINITE DIMENSIONAL STRATEGY SPACE, AN INFINITE NUMBER OF AGENTS AND WITHOUT ORDERED PREFERENCES

Guoqiang TIAN
Texas A&M University, College Station, TX 77843-4228, USA

Received 17 October 1989
Accepted 18 December 1989

The purpose of this note is to prove the existence of an equilibrium for abstract economies with a non-compact infinite-dimensional strategy space, a countably infinite number of agents and without ordered preferences. We do so by generalizing the result of Yannelis and Prabhakar to a non-compact strategy space.

1. Introduction

The abstract economy defined by Debreu (1952) generalizes the Nash non-cooperative game in that a player's strategy set depends on the strategy choices of all the other players. Debreu (1952) proved the existence of equilibrium in abstract economies with finitely many agents, finite dimensional strategy space, and quasi-concave utility functions. Since then, the Debreu's result has been extended in several directions. For the finite number of agents and the finite dimensional strategy space case, Shafer and Sonnenschein (1975) and Borglin and Keiding (1976) extended the Debreu's result to abstract economies without ordered preferences. For the infinite dimensional space and finite or infinitely many agents case, the existence results were given by Yannelis and Prabhakar (1983), Khan and Vohra (1984), Toussaint (1984), Kim, Prikry, and Yannelis (1985), Khan (1986), and Yannelis (1987), among others.

However, all existence theorems mentioned above are proved upon the compactness of choice sets. In the finite-dimensional setting, the usual closed boundedness assumptions imply compactness of the feasible sets. However, in the infinite-dimensional setting, the usual closed boundedness assumptions do not imply compactness of the feasible sets and thus the feasible sets will not generally be (weakly) compact, a typical situation in infinite dimensional linear space. [1] To avoid this difficulty in the literature, people explicitly assume that the feasible set is (weakly) compact in some topology and ignore the case of non-compact feasible sets.

Recently, Tian (1988) considered the existence of equilibrium for abstract economies with a non-compact infinite dimensional choice space and a countably infinite number of agents by the quasi-variational inequality approach. However, the quasi-variational inequality approach requires that preferences be representable by a concave utility function.

[1] For some infinite-dimensional spaces such as L_1, l_1, L_2, l_2, L_∞, and l_∞ spaces, the feasible sets with finite number of agents, however, may be (weakly) compact in some weak topologies if some additional conditions are imposed [cf. Zame (1987, pp. 1084–1085)].

0165-1765/90/$3.50

This note gives another existence theorem on an equilibrium for abstract economies with a non-compact infinite-dimensional strategy space, a countably infinite number of agents and without ordered preferences. Even though this theorem only considers generalization of the existence theorem of Yannelis and Prabhakar (1983) by relaxing the compactness of strategy space, we think the techniques developed in the note can be used to generalize the other existence theorems on abstract economies and/or competitive equilibrium in the existing literature by relaxing the compactness of choice sets.

This paper is organized as follows. Notation and definitions are given in section 2. Our main existence theorem and its proof are given in section 3. Some remarks are presented in section 4.

2. Notation and definitions

Let X and Y be two topological spaces, and let 2^Y be the set of all subsets of Y. A correspondence $F\colon X \to 2^Y$ is said to be *upper semi-continuous* (henceforth u.s.c.) if the set $\{x \in X\colon F(x) \subset V\}$ is open in X for every open subset V of Y. A correspondence $F\colon X \to 2^Y$ is said to be *lower semi-continuous* (henceforth l.s.c.) if the set $\{x \in X\colon F(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y. A correspondence $F\colon X \to 2^Y$ is said to be *continuous* if it is both u.s.c. and l.s.c. A correspondence $F\colon X \to 2^Y$ is said to have open lower sections if the set $F^{-1}(y) = \{x \in X\colon y \in F(x)\}$ is open in X for every $y \in Y$.

Let the set of agents be any countable set denoted by I. Each agent has a choice set X_i, a constraint correspondence $A_i\colon X \to 2^{X_i}$, and a preference correspondence $P_i\colon X \to 2^{X_i}$, where $X = \prod_{i \in I} X_i$. Denote by X_{-i} and A the product $\prod_{j \in I\setminus\{i\}} X_j$ and the product $\prod_{j \in I} A_j$. Denote by con B and $\bar{B}$ the convex hull and the closure of the set B.

An *abstract economy* $\Gamma = (X_i, A_i, P_i)_{i \in I}$ is defined as a family of ordered triples (X_i, A_i, P_i). An *equilibrium* for Γ is an $x^* \in X$ such that $x^* \in \bar{A}(x^*)$ and $P_i(x^*) \cap A_i(x^*) = \emptyset$ for each $i \in I$.

3. Existence of equilibrium

In this section we prove the existence of equilibrium for abstract economies with a non-compact infinite-dimensional strategy space, with a countable number of agents, and with nontotal-nontransitive preferences. We begin by stating Theorem 6.1 of Yannelis and Prabhakar (1983).

Theorem 1. *Let $\Gamma = (Z_i, A_i, P_i)_{i \in I}$ be an abstract economy satisfying for each $i \in I$:*

(*i*) *Z_i is a non-empty, compact, convex, metrizable subset in a locally convex Hausdorff topological vector space,*
(*ii*) *$A_i(x)$ is non-empty and convex for all $x \in Z$,*
(*iii*) *the correspondence $\bar{A}_i\colon Z \to 2^{Z_i}$ is an upper semi-continuous,*
(*iv*) *A_i has open lower sections,*
(*v*) *P_i has open lower sections,*
(*vi*) *$x_i \notin$ con $P_i(x)$ for all $x \in Z$.*

Then Γ has an equilibrium.

The following result is a generalization of Theorem 1 by relaxing the compactness condition.

Theorem 2. *Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy satisfying for each $i \in I$:*

(*i*) *X_i is a non-empty, convex, metrizable subset in a locally convex Hausdorff topological vector space,*
(*ii*) *$A_i(x)$ is non-empty and convex for all $x \in X$,*
(*iii*) *the correspondence $\bar{A}_i$: $X \to 2^{X_i}$ is upper semi-continuous and $\bar{A}_i(x)$ is compact for all $x \in X$,*
(*iv*) *A_i has open lower sections,*
(*v*) *P_i has open lower sections,*
(*vi*) *$x_i \notin \text{con}\, P_i(x)$ for all $x \in X$,*
(*vii*) *there exists a non-empty compact convex set $C_i \subset X_i$ such that:*
  (*vii.1*) *$A_i(C)$ is contained in a compact convex set $D_i \subset X_i$, where $C = \prod_{i \in I} C_i$;*
  (*vii.2*) *$A_i(x) \cap Z_i \neq \emptyset$ for all $x \in X_{-i} \times Z_i$, where $Z_i = \text{con}\{D_i \cup C_i\}$;*
  (*vii.3*) *for each $x_i \in Z_i \setminus C_i$ and $x_{-i} \in X_{-i}$ there exists $y_i \in A_i(x) \cap Z_i$ such that $y_i \in P_i(x)$.*

Then Γ has an equilibrium.

Proof. Since C_i and D_i are compact and convex subsets of X_i, then Z_i is compact convex. Let $Z = \prod_{i \in I} Z_i$. For each $i \in I$, define a correspondence $K_i : Z \to 2^{Z_i}$ by, for each $x \in Z$,

$$K_i(x) = A_i(x) \cap Z_i. \tag{1}$$

Then, by condition (ii) and properties of Z_i, $K_i(x)$ is non-empty and convex for all $x \in Z$. Since Z is compact and $\bar{A}_i$ is closed (i.e., its graph is closed) by Proposition 3.7 in Aubin and Ekeland (1984, p. 111), $\bar{K}_i$ is closed and therefore is upper semi-continuous on Z by Theorem 3.8 in Ekeland (1984, p. 111). Also, note that

$$K_i(x) = \begin{cases} A_i(x) & \text{if} \quad x_i \in C_i \\ A_i(x) \cap Z_i & \text{otherwise} \end{cases}. \tag{2}$$

Therefore, the abstract economy $\Gamma = (Z_i, K_i, P_i \mid Z)_{i \in I}$ satisfies all assumptions of Theorem 1 and thus there exists $x^* \in Z$ such that $\bar{K}(x^*) = \prod_{i \in I} \bar{K}_i(x^*)$ and $P_i(x^*) \cap K_i(x^*) = \emptyset$. Now $x^* \in C$, for otherwise (vii.3) of Theorem 2 would be violated, and hence $K(x^*) = A(x^*)$. Then, we have $x^* \in \bar{A}(x^*)$ and $P_i(x^*) \cap A_i(x^*) = \emptyset$ for all $i \in I$. So $x^* \in X$ is an equilibrium of the abstract economy Γ. □

4. Remarks

Remark 1. Observe that in case of a compact convex X_i, assumption (vii) in Theorem 2 is satisfied by $C_i = X_i$. Thus Theorem 2 is indeed a generalization of Theorem 1 by relaxing the compactness condition. Also, when X_i is a subset in a finite dimension topological vector space, we can only assume C_i is non-empty compact and let $Z_i = \text{con}\{A_i(C) \cup C_i\}$ so that assumption (vii.1) is unnecessary.

Remark 2. Assumption (vii.2) is needed to guarantee the existence of a fixed point of $\bar{A}(x)$. [2] Assumption (vii.3) guarantees the maximal element x^* exists and is in the compact subset C of X.

[2] Tian (1990) proved that a necessary and sufficient condition for the existence of fixed points of an upper semi-continuous correspondence with non-empty closed values F defined on any subset (which may be non-compact and non-convex) of a locally convex Hausdorff topological vector space is that there exists a compact convex subset $B \subset X$ such that $F(x) \cap B \neq \emptyset$ for all $x \in B$.

This assumption is similar to an assumption which was imposed by Border (1985, pp. 34–35) and is used to prove the existence of maximal elements of preferences on a non-compact choice set.

References

Aubin, J.P. and I. Ekeland, 1984, Applied nonlinear analysis (Wiley, New York).

Border, K.C., 1985, Fixed point theorems with application to economics and game theory (Cambridge University Press, Cambridge).

Borglin, A. and H. Keiding, 1976, Existence of equilibrium actions and of equilibrium: A note on the 'new' existence theorem, Journal of Mathematical Economics 3, 313–316.

Debreu, G., 1952, A social equilibrium existence theorem, Proceedings of the National Academy of Sciences of the U.S.A. 38, 386–393.

Khan, M. Ali, 1986, Equilibrium points of nonatomic games over a Banach space, Transactions American Mathematical Society 29, 737–749.

Khan, M. Ali and R. Vohra, 1984, Equilibrium in abstract economies without ordered preferences and with a measure of agents, Journal of Mathematical Economics 13, 133–142.

Kim, T., K. Prikry and N.C. Yannelis, 1985, Equilibrium in abstract economies with a measure space and with an infinite dimensional strategy space (University of Minnesota, Minneapolis, MN).

Nash, J., 1950, Equilibrium points in N-person games, Proceedings of the National Academy of Sciences 36, 48–49.

Shafer, W. and H. Sonnenschein, 1975, Equilibrium in abstract economies without ordered preferences, Journal of Mathematical Economics 2, 345–348.

Tian, G., 1988, An equilibrium existence theorem on abstract economies, Working paper no. 30 (Texas A&M University, College Station, TX).

Tian, G., 1990, Fixed points theorems for mappings with non-compact and non-convex domains, Journal of Mathematical Analysis and Applications, forthcoming.

Toussaint, S., 1984, On the existence of equilibria in economics with infinitely many commodities, Journal of Economic Theory 33, 98–115.

Yannelis, N.C., 1987, Equilibria in noncooperative models of competition, Journal of Economic Theory 41, 96–111.

Yannelis, N.C. and N.D. Prabhakar, 1983, Existence of maximal elements and equilibria in linear topological spaces, Journal of Mathematical Economics 12, 233–245.

Zame, W.R., 1987, Competitive equilibria in production with an infinite-dimensional commodity space, Econometrica 55, 1075–1108.