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:TITLE: Economic theory.
:IMPRINT: Berlin : Springer-Verlag, c1991-
:ARTICLE: : A General-Equilibrium Intertemporal Model of an Open Economy
:VOL: 2 :NO: :DATE: 1992 :PAGES: 215-246
:VERIFIED: <TN:233348>OCLC ISSN: 0938-2259 [Format: Serial]
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Econ. Theory 2, 215–246 (1992)

Economic Theory

A general-equilibrium intertemporal model of an open economy*

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Received: August 15, 1990

Summary. This paper develops a very general (general-equilibrium) intertemporal model of a country engaged in international trade which can be used to address a wide variety of issues of interest – in particular, econometric application – under the assumption that prices of tradable commodities (consumer goods and capital goods) and the interest rate are exogenous to the country. It allows for an arbitrarily large number of commodities which are distinguished into seven categories and for finite or infinite periods of time. This model can be used to draw various policy conclusions. We investigate how current net imports, the balance of payments on current account, current consumption expenditure, next-period bondholdings, current wealth, and current internal prices will react to exogenous changes in current external prices, the current interest rate, current taxes, current factor endowments, and current-period bondholdings. This paper also considers the integrability of net-import demand functions.

1 Introduction

This paper develops a multi-commodity, multi-period general-equilibrium model of an open economy under the assumption that current and future prices of tradable commodities (consumer goods and capital goods) are exogenous to the country. It has been developed as an intertemporal extension to the multi-commodity static model of Chipman [6–9], which has been applied to West German and Swedish trade and price data for the periods 1958–84 and 1971–84 respectively, and is designed to study the responses of imports and exports to the kinds of exogenous shocks that industrial countries have experienced in recent decades. Static models have of course proved deficient for such purposes because they must necessarily treat capital movements as exogenous. While this may be a reasonable assumption

* Work supported by National Science Foundation grant SES-8607652. We wish to thank the editor and an anonymous referee for useful comments and suggestions.

in the case of capital movements induced by fiscal policies (e.g., the U.S. budget deficit),[1] clearly it is inappropriate in the case of an unexpected price increase that may cause consumers to dissave, leading to a movement into deficit in the balance of payments on current account.

Intertemporal models in international trade are of course not new, and in fact a large literature has developed of which mention may be made of Obstfeld [28, 29] Lucas [26], Sachs [30], Dornbusch [13], Svensson and Razin [36], Kehoe [20], Blanchard [3], Svensson [35], Frenkel and Razin [14] among many others. Some of these assume fixed production, and some are limited to a single consumer good; most have been developed as special-purpose models with enough structure to permit the drawing of qualitative policy conclusions without requiring knowledge of numerical values of parameters. These are of course very valuable in providing insight into the relevant relationships. But in many cases, special-purpose models make very restrictive assumptions to get wanted conclusions. These models are very likely unrealistic and/or conclusions can be totally different when the structures of the models are set forth differently. For instance, the so-called Harberger–Laursen–Metzler effect has aroused considerable theoretical controversy. Some economists, including Harberger [16], Laursen and Metzler [24], Svensson and Razin [36], and many others have believed that a terms-of-trade deterioration entails a deterioration of the trade balance. Other economists, including White [38], Day [12], Obstfeld [28, 29], and many others have believed that a terms-of-trade deterioration will lead to an improvement, rather than a deterioration, in the trade balance.

Our purpose in the present paper is different. A model designed to describe the real world well or to be applied to data must necessarily be a general-purpose model which should be very general in structure and take account of the many kinds of shocks that may be expected to occur over the period of time considered. In empirical applications one cannot assume that certain things are held constant which are not in fact constant in the real world. For these reasons a more general formulation is required. Moreover, while we do not wish to underestimate the role of theory in merely providing insight, it is our belief that the proper role of theory is to generalize, so that knowledge gained in one application can be put to advantage in another. The disadvantage (as some conceive it) of this approach is that it is not possible to arrive at many neat qualitative conclusions in the absence of further specifications on utility and production functions – or numerical estimates of the relevant parameters – and that confrontation of the model with empirical data is therefore required. However, although much remains to be done to transform the present model into a tractable econometric one, we believe that it presents a sounder approach to estimation of "elasticities of demand for imports" than that provided by the usual partial-equilibrium formulations.

[1] Even in this case it may be more appropriate to take account of the fact that the way in which fiscal policies induce capital movements is typically by affecting real interest rates, and to treat the latter as the appropriate exogenous variable. On the other hand it should be noted if, e.g., the U.S. budget deficit increases while those of other countries simultaneously decrease, a capital movement will be induced which need not involve any changes in real interest rates.

The main features of the model may be briefly described. Seven categories of commodities are distinguished: (1) tradable consumer goods produced at home; (2) tradable consumer goods not produced at home; (3) nontradable consumer goods produced at home; (4) tradable capital goods produced at home; (5) tradable capital goods not produced at home; (6) nontradable capital goods produced at home; and (7) nontradable and nonproduced primary factors of production; and it allows for the existence of intermediate inputs. A limitation of the model that must be mentioned is that it assumes that over the period of time considered, exogenous shocks are not sufficiently great to cause commodities to switch categories. At a very detailed level of disaggregation, this would of course be unrealistic: the U.S. may stop producing Volkswagens and start producing Hondas. And of course, our model does not allow for introduction of new products. However, at the coarser level of disaggregation that is usually dictated by availability of data, the assumption does not appear too limiting; it is unlikely that exogenous shocks will be sufficiently great to cause the U.S. to cease production of automobiles altogether, or to start producing bananas and mining tin. It should also be kept in mind that our model is not intended to represent the "stages of growth" of a developing country over centuries of time, but is intended to depict its structural adjustment to disturbances over at most a 20- or 30-year period.

On the assumption that consumer preferences are aggregable, i.e., that aggregate consumer demand can be generated by maximization of an aggregate utility function, we show that the net-import demand function (or "trade-demand function") can be generated by maximization of a trade-utility function with import and export quantities as arguments, subject to a balance-of-payments constraint. In this model bondholdings (except for initial bondholdings) are treated as an endogenous variable, and the balance-of-payments constraint is a difference equation in terms of bondholdings. Thus, the trade-demand function, the expenditure on consumer goods, and next-period bondholdings are the solutions of a dynamic-programming problem with current external prices, current bondholdings, current taxes, the current interest factor, and current factor endowments as arguments. Under suitable assumptions on utility and production functions to be specified below,[2] we investigate how current internal prices, the balance of payments on current account, current wealth, current consumption expenditure, next-period bondholdings, and the current net-demand for tradable goods will react when the economy is faced with exogenous changes in current external prices (terms of trade), current factor endowments, the current interest factor, current taxes, and current bondholdings. For instance, we conclude that an increase in current taxes (assumed to be lump-sum) will cause current wealth, consumption expenditure, prices of nontradable consumer goods, net-import demand, the next period bondholdings, and the balance-of-payments deficit to decrease; an increase in current bondholdings will cause all of the above variables (current wealth, consumption

[2] That these assumptions are nonvacuous is shown in Sect. 3 below by the fact that they are satisfied by Cobb–Douglas utility and production functions (the former with the transformation $F(u) = 1 - e^{-u}$) when there is only one nontradable consumer good. See Footnote 19 below.

expenditure, prices of nontradable consumer goods, net-import demand, next-period bondholdings, and the balance-of-payments deficit) to increase; an increase in the current interest factor will cause all of the above variables to increase if the country is lending or to decrease if it is borrowing. It seems to us that casual empirical observation provides good support to these conclusions. From our model, one can also determine under what conditions on utility and production functions the Harberger–Laursen–Metzler effect holds or does not hold. Since our model is closer to the real world, these qualitative policy conclusions drawing from such a general model are more reliable than those derived from special-purpose models.

The model developed in this paper is very general, and includes many models in the literature as special cases; it can thus be used to address a wide variety of issues of interest. The main advantage of the intertemporal model is that it no longer treats international capital movements as exclusively exogenous, but allows for capital inflows or outflows in response to exogenous changes in current and future prices of tradable goods. (It also allows for capital inflows or outflows in response to changes in initial bondholdings.) Another virtue is that the equilibrium solution of the model is a stationary stochastic process, and therefore it is possible to estimate and test the same model at the empirical stage that is developed at the theoretical stage.

The plan of this paper is as follows. Section 2 sets forth a general form of the intertemporal model of an open economy. Section 3 provides some theoretical results on policy functions. Section 4 considers the integrability of the net-import demand function. Proofs are given in the Appendix.

2 The model

We assume that our country uses seven categories of commodities (consumer goods and capital goods):[3]

1. n_1 tradable consumer goods produced at home;
2. n_2 tradable consumer goods not produced at home (hence imported);
3. n_3 nontradable consumer goods produced at home;
4. n_4 tradable capital goods produced at home;
5. n_5 tradable capital goods not produced at home (hence imported);
6. n_6 nontradable capital goods produced at home;
7. n_7 nontradable and non-produced primary factors of production (such as labor, natural resources, land, etc.).

We classify capital goods used as inputs as produced primary factors of production, in contrast to raw materials and goods-in-process which are classified as produced intermediate factors of production.

Let $N_k = \{1, 2, \ldots, n_k\}$, $k = 1, 2, \ldots, 7$ and let t denote the time period, $t = 0, 1, \ldots, T$ (T can be set to $T = \infty$). The basic variables of the model are indicated

[3] In this paper, we rule out switching between categories. See the remarks in the introduction.

by the following notation:

$x_j^k(t)$	= consumption or capital accumulation of the j-th good in category $k, x_j^k(t) \geqq 0$ $(j \in N_k, k=1,2,\ldots,6)$;
$q_j^k(t)$	= gross output of the j-th commodity in category k, $q_j^k(t) \geqq 0$ $(j \in N_k, k=1,2,\ldots,6)$;
$y_j^k(t)$	= net output of the j-th commodity in category k $(j \in N_k, k=1,2,\ldots,6)$;
$z_j^k(t) = x_j^k(t) - y_j^k(t)$	= import (if positive) or export (if negative) of the j-th commodity in category $k (j \in N_k, k=1,2,\ldots,6)$;
$u_{ij}^{rk}(t)$	= input of the i-th commodity in category r into the output of the j-th commodity in category k, $u_{ij}^{rk}(t) \geqq 0$ $(i \in N_r, j \in N_k, r=1,2,\ldots,6$, and $k=1,3,4,6)$;
$v_{ij}^{rk}(t)$	= input of the i-th primary factor in category r into the output of the j-th commodity in category $k, v_{ij}^{rk}(t) \geqq 0$ $(i \in N_r, j \in N_k, r=4,5,6,7$, and $k=1,3,4,6)$;
$l_i^k(t)$	= endowment of the i-th primary factor in category $k (i \in N_k, k=4,5,6,7)$;
$p_j^k(t)$	= price of the j-th consumer good (or rental of the j-th factor of production) in category $k (j \in N_k, k=1,2,\ldots,7)$;
$\pi_j^k(t)$	= the "Lerner price" of the j-th consumer good (or rental of the j-th factor of production) in category $k (j \in N_k, k=1,2,\ldots,7)$;
$f_j^k(\cdot)$	= production function for the j-th commodity in category k;
$g_j^k(\cdot)$	= minimum-unit-cost function for the j-th commodity in category k;
$e_j^k(t)$	= random error in the production function $f_j^k(\cdot)$;
$e^k(t)$	= random-error vector with $e_j^k(t)$ as component;
$e(t)$	= random-error vector with $e^k(t)$ as component;
$h_j^k(\cdot)$	= aggregate demand for the j-th consumer good in category k $(j \in N_k, k=1,2,3)$;
$C(t)$	= current consumption expenditure;
$\mathscr{Y}(l(t), e(t), t)$	= the production-possibility set;
$Y(t)$	= gross national product (GNP);
$\Pi(\cdot)$	= the gross-national-product (GNP) function;
$I(t)$	= gross national capital formation;
$\Gamma(t)$	= the current national non-portfolio income;
Ω	= the intertemporal national-wealth (in the absence of borrowing or lending);
$\Upsilon(\cdot)$	= the intertemporal national-wealth function;
	= the interest rate on a one-period bond;
$R(t)$	= the interest factor on a one-period bond, i.e., $R(t) = 1 + r(t)$;
$S(t)$	= the current rate of saving;
$W(t)$	= the current wealth position;
$b_P(t)$	= end-of-period private bondholdings;
$b_G(t)$	= end-of-period government bondholdings;

$b(t)$	= end-of-period total bondholdings; thus $b(t) = b_P(t) + b_G(t)$;
$G(t)$	= government expenditure;
$\Theta(t)$	= lump-sum taxes;
$Z(t)$	= the deficit in the balance of payments on current account.

In order to avoid the assumption of perfect foresight, throughout the paper, we assume that the external price vector $(p^1(t), p^2(t), p^4(t), p^5(t))$, the interest factor $R(t)$ and the taxes $\Theta(t)$ are stochastic, and that they follow, together with the stochastic term $e(t)$ in the production functions (2), a stochastic difference equation

$$u(t+1) = \Psi(u(t), \varepsilon(t+1)), \tag{1}$$

where $u(t) = (p^1(t), p^2(t), p^4(t), p^5(t), R(t), \Theta(t), e(t)) \in R^m$ with $m = 2 + 2n_1 + n_2 + n_3 + 2n_4 + n_5 + n_6$, and where $\Psi\colon R^m \times R^m \to R^m$ is continuous and $\{\varepsilon(t)\}$ is a sequence of independent random variables with identical distribution functions F which are assumed to be known (hence agents have rational expectations).

Production of the j-th commodity in category $k (k = 1, 2, \ldots, 6)$ at time t is governed by a stochastic industry (aggregate) production function

$$\begin{aligned} q_j^k(t) &= f_j^k(u_j^k(t), v_j^k(t), e_j^k(t), t) \\ &= \lambda_j^k(t, e_j^k(t)) f_j^k(u_j^k(t), v_j^k(t), e_j^k(0), 0) \quad (j \in N_k), \end{aligned} \tag{2}$$

which is concave, homogeneous of degree one,[4] and increasing in the input vectors $u_j^k(t)$ and $v_j^k(t)$, where

$$u_j^k(t) = (u_j^{1k}(t), u_j^{2k}(t), \ldots, u_j^{6k}(t))$$

with

$$u_j^{rk}(t) = (u_{1j}^{rk}(t), u_{2j}^{rk}(t), \ldots, u_{n_r j}^{rk}(t)) \quad (r = 1, \ldots, 6),$$

and

$$v_j^k(t) = (v_j^{4k}(t), \ldots, v_j^{7k}(t))$$

with

$$v_j^{rk}(t) = (v_{1j}^{rk}(t), \ldots, v_{n_r j}^{rk}(t)) \quad (r = 4, 5, 6, 7).$$

The second expression in (2) specifies the assumption of Hicks-neutral technical change $\lambda_j^k(t, e_j^k(t))$, which is an unknown function to be estimated and is assumed to satisfy $\lambda_j^k(0, e_j^k(0)) = 1$ and $\lambda_j^k(t, e_j^k(t)) > 0$ for $t > 0$. Here $e_j^k(t)$ is a random error.

We assume that capital goods can be used only to augment the capital stock, and hence

$$u_j^{rk}(t) = 0 \quad (r = 4, 5, 6 \text{ and } k = 1, \ldots, 6), \tag{3}$$

and net output $y_j^k(t)$ equals gross output $q_j^k(t)$ for capital goods $(k = 4, 5, 6)$.

[4] That is, the industry production function is assumed to have the property of constant returns to scale. See Stigler [33, Ch. 8–9] for a classic discussion of conditions under which this may be a reasonable assumption at the industry level (pp. 131–144) or the firm level (pp. 169–70). The methods of this paper could readily be extended to allow for external economies or diseconomies of scale along the lines of Inoue's [19] treatment. For a discussion of this approach see Chipman [10, pp. 933–4].

Dual to (2) are the Samuelson [31]–Shephard [32] minimum-unit-cost functions $g_j^k(p(t), e_j^k(t), t)$ for commodity j in category k which are also homogeneous of degree one and concave in $p(t)$, where $p(t) = (p^1(t), \ldots, p^7(t))$ is a vector of commodity prices and factor rentals with $p^k(t) = (p_1^k(t), \ldots, p_{n_k}^k(t))$.

The profit-maximization condition is then

$$\begin{aligned} p_j^k(t) &= g_j^k(p(t), e_j^k(t), t) \\ &= \lambda_j^k(t, e_j^k(t))^{-1} g_j^k(p(t), e_j^k(0), 0) \quad (j \in N_k; k = 1, 3, 4, 6) \end{aligned} \tag{4}$$

in the producing industries (i.e., the vectors $p^1(t), p^3(t), p^4(t)$, and $p^6(t)$ lie in the range of the corresponding cost functions, owing to the assumption of perfect competition and the definition of producing industries) and

$$\begin{aligned} p_j^k(t) &< g_j^k(p(t), e_j^k(t), t) \\ &= \lambda_j^k(t, e_j^k(t))^{-1} g_j^k(p(t), e_j^k(0), 0) \quad (j \in N_k; k = 2, 5) \end{aligned} \tag{5}$$

in the nonproducing industries, owing to the assumption that these are non-producing.[5]

By Shephard's [32] envelope theorem, the input-output and factor-output coefficients are given by

$$\partial g_j^k(\cdot, t)/\partial p_i^r(t) = \hat{u}_{ij}^{rk}(\cdot, t)/q_j^k(\cdot, t) \equiv a_{ij}^{rk}(\cdot, t) \tag{6}$$

for $r = 1, 2, 3$, and

$$\partial g_j^k(\cdot, t)/\partial p_i^s(t) = \hat{v}_{ij}^{sk}(\cdot, t)/q_j^k(\cdot, t) \equiv b_{ij}^{sk}(\cdot, t) \tag{7}$$

for $i \in N_s$ and $s = 4, 5, 6, 7$, where $\hat{u}_{ij}^{rk}(\cdot, t)$ and $\hat{v}_{ij}^{sk}(\cdot, t)$ are the demand functions for intermediate inputs and primary factors, and $a_{ij}^{rk}(\cdot, t)$ and $b_{ij}^{sk}(\cdot, t)$ are the corresponding input-output and factor-output coefficients satisfying

$$\begin{aligned} a_{ij}^{rk}(\cdot, t) &= a_{ij}^{rk}(p(t), e_j^k(t), t) \\ &= \lambda_j^k(t, e_j^k(t))^{-1} a_{ij}^{rk}(p(t), e_j^k(0), 0) \end{aligned} \tag{8}$$

and

$$\begin{aligned} b_{ij}^{sk}(\cdot, t) &= b_{ij}^{sk}(p(t), e_j^k(t), t) \\ &= \lambda_j^k(t, e_j^k(t))^{-1} b_{ij}^{sk}(p(t), e_j^k(0), 0), \end{aligned} \tag{9}$$

which are homogeneous of degree zero in $p(t)$.

Net outputs of the i-th commodity in category r are defined by[6]

$$\begin{aligned} y_i^r(t) = q_i^r(t) &- \sum_{j=1}^{n_1} a_{ij}^{r1}(\cdot, t) q_j^1(t) - \sum_{j=1}^{n_3} a_{ij}^{r3}(\cdot, t) q_j^3(t) \\ &- \sum_{j=1}^{n_4} a_{ij}^{r4}(\cdot, t) q_j^4(t) - \sum_{j=1}^{n_6} a_{ij}^{r6}(\cdot, t) q_j^6(t) \end{aligned} \tag{10}$$

---

[5] This assumption is satisfied if the country does not start off too close to the switching points and if the random errors are sufficiently small.

[6] Note that $y_i^2(t) \leqq 0$ for $i \in N_2$ since $q_i^2(t) = 0$ by definition.

for $r = 1, 2, 3$, and

$$y_j^r(t) = q_j^r(t) \tag{11}$$

for $r = 4, 5, 6$. Thus $y_j^5(t) = 0$ since $q_j^5(t) = 0$ by definition.

The assumption of full employment entails

$$\sum_{j=1}^{n_1} b_{ij}^{r1}(\cdot, t) q_j^1(t) + \sum_{j=1}^{n_3} b_{ij}^{r3}(\cdot, t) q_j^3(t) + \sum_{j=1}^{n_4} b_{ij}^{r4}(\cdot, t) q_j^4(t) + \sum_{j=1}^{n_6} b_{ij}^{r6}(\cdot, t) q_j^6(t) = l_i^r(t), \tag{12}$$

which is the factor-market equilibrium condition. Here $l_i^r(t)$ is the country's endowment of factor i in category $r (r = 4, 5, 6, 7)$ at time t.

For each period t and each capital good $(k = 4, 5, 6)$, it is assumed that gross capital accumulation $x_j^k(t)$ is nonnegative (i.e., capital cannot be dismantled once installed) and $x_j^k(T) = 0$ ($\lim_{T\to\infty} x_j^k(T) = 0$ when $T = \infty$). Let the i-th capital good in category r, when installed as a factor, depreciate at the rate of $\delta_i^r(t)$ $(0 < \delta_i^r(t) \leqq 1)$ per period; then the amounts of the produced factors satisfy the difference equation

$$l_i^r(t+1) = [1 - \delta_i^r(t)] l_i^r(t) + x_i^r(t) \tag{13}$$

for $r = 4, 5, 6$ with $l_i^r(0)$ given. The unproduced factors $l_i^7(t)$ will be considered to be exogenous functions of time.

Let $z_j^k(t) = x_j^k(t) - y_j^k(t)$ denote the trade (import if positive, export if negative) in the j-th good in category k; then we have

$$x_j^1(t) = y_j^1(t) + z_j^1(t); \quad x_j^2(t) = y_j^2(t) + z_j^2(t); \quad x_j^3(t) = y_j^3(t);$$

$$x_j^4(t) = y_j^4(t) + z_j^4(t); \quad x_j^5(t) = y_j^5(t) + z_j^5(t); \quad x_j^6(t) = y_j^6(t).$$

Define the input-output and factor-output matrices $A^{rk}(\cdot, t) = [a_{ij}^{rk}(\cdot, t)]$ and $B^{sk}(\cdot, t) = [b_{ij}^{sk}(\cdot, t)]$, that is,

$$A^{rk}(\cdot, t) = A^{rk}(p(t), e^k(t), t);$$

$$B^{sk}(\cdot, t) = B^{sk}(p(t), e^k(t), t),$$

where $e^k(t) = (e_1^k(t), \ldots, e_{n_k}^k(t))$.

Since $g_j^k(t)$ is homogeneous of degree one, we have by Euler's theorem

$$Hp(t) = (A(\cdot, t)', B(\cdot, t)')p(t), \tag{14}$$

where the superscript "′" denotes the transpose of a matrix;

$$H = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix};$$

$$A(\cdot, t) = \begin{bmatrix} A^{11}(\cdot, t) & A^{13}(\cdot, t) & A^{14}(\cdot, t) & A^{16}(\cdot, t) \\ A^{21}(\cdot, t) & A^{23}(\cdot, t) & A^{24}(\cdot, t) & A^{26}(\cdot, t) \\ A^{31}(\cdot, t) & A^{33}(\cdot, t) & A^{34}(\cdot, t) & A^{36}(\cdot, t) \end{bmatrix};$$

$$B(\cdot,t)=\begin{bmatrix} B^{41}(\cdot,t) & B^{43}(\cdot,t) & B^{44}(\cdot,t) & B^{46}(\cdot,t) \\ B^{51}(\cdot,t) & B^{53}(\cdot,t) & B^{54}(\cdot,t) & B^{56}(\cdot,t) \\ B^{61}(\cdot,t) & B^{63}(\cdot,t) & B^{64}(\cdot,t) & B^{66}(\cdot,t) \\ B^{71}(\cdot,t) & B^{73}(\cdot,t) & B^{74}(\cdot,t) & B^{76}(\cdot,t) \end{bmatrix}.$$

Equation (14) can also be written as

$$(p^1(t)', p^2(t)', p^3(t)', p^4(t)', p^6(t)')M(\cdot,t) = (p^4(t)', p^5(t)', p^6(t)', p^7(t)')B(\cdot,t), \quad (15)$$

where

$$M(\cdot,t)=\begin{bmatrix} I-A^{11}(\cdot,t) & -A^{13}(\cdot,t) & -A^{14}(\cdot,t) & -A^{16}(\cdot,t) \\ -A^{21}(\cdot,t) & -A^{23}(\cdot,t) & -A^{24}(\cdot,t) & -A^{26}(\cdot,t) \\ -A^{31}(\cdot,t) & I-A^{33}(\cdot,t) & -A^{34}(\cdot,t) & -A^{36}(\cdot,t) \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

We may write (10)–(12) in matrix form

$$\tilde{y}(t) = M(\cdot,t)q(t); \quad (16)$$

$$B(\cdot,t)q(t) = l(t), \quad (17)$$

where

$$\tilde{y}(t) = (y^1(t)', y^2(t)', y^3(t)', y^4(t)', y^6(t)')';$$
$$q(t) = (q^1(t)', q^3(t)', q^4(t)', q^6(t)')';$$
$$l(t) = (l^4(t)', l^5(t)', l^6(t)', l^7(t)').$$

Post-multiplying (15) by $q(t)$ and applying (16) and (17), we obtain immediately the identity of gross national product and gross national income at time t:

$$Y(t) \equiv \sum_{\substack{k=1 \\ k\neq 5}}^{6} p^k(t)\cdot y^k(t) = \sum_{k=4}^{7} p^k(t)\cdot l^k(t). \quad (18)$$

Then we may call $\Sigma_{k=4}^{7} p^k(t)\cdot l^k(t)$ the current national-cost function and $l^k(t) = [I-\delta^k(t-1)]l^k(t-1) + x^k(t-1)$ the national factor-demand function ($k=4,\ldots,7$). Define the national production-possibility set as

$$\mathscr{Y}(l(t)) = \{(y^1(t), y^2(t), y^3(t), y^4(t), y^6(t))\}$$

satisfying the constraints (2), (10–12).

The GNP function is defined as

$$\Pi(p(t), l(t), e(t), t) = \max\{Y(t): (y^1(t), y^2(t), y^3(t), y^4(t), y^6(t)) \in \mathscr{Y}(l(t))\}, \quad (19)$$

where $e(t) = (e^1(t), e^3(t), e^4(t), e^6(t))$. This is the problem that a central planner would attempt to solve given the price vector $p(t)$ and the endowment vector $l(t)$. It can be shown (cf., e.g., Chipman [10]; Woodland [39, Ch. 5]) that the solution of this central planner's optimization problem for $y(t)$ is precisely the equilibrium output vector obtained by the profit-maximization condition and the factor-market equilibrium condition (a result that goes back to Pareto and Barone and of course forms part of the so-called "fundamental theorem of welfare economics").

Remark 1. If the number of primary factors of production exceeds or equals the number of commodities (i.e., $n_4+n_5+n_6+n_7 \geqq n_1+n_3+n_4+n_6$, or $n_5+n_7 \geqq n_1+n_3$), the country's production-possibility frontier will in all but exceptional cases be strictly concave to the origin (cf. Chipman [4, pp. 218–219], Khang [22], Kemp, Khang and Uekawa [21]) and we can define the single-valued Rybczynski function $\hat{y}^k(p(t), l(t), e(t), t)$ for $k = 1, 2, 3, 4, 6$. Note that the long vector $e(t)$ (not merely $e^k(t)$) is in $\hat{y}^k(\cdot)$ for $k = 1, 2, 3$ by (10). For convenience we can write $\hat{y}^k(\cdot)$ ($k = 4, 5, 6$) as a function of $e(t)$ without loss of generality.

The gross-national-product (GNP) function can be shown to have the properties specified in the following lemma (cf., e.g., Chipman [10, pp. 928–931], Woodland [39, Ch. 5]):

Lemma 1. (*a*) *$\Pi(p(t), l(t), e(t), t)$ is defined and is non-negative for all $p(t) > 0$ and $l(t) \geqq 0$ as well as positive if $l(t) > 0$;*
(*b*) *$\Pi(p(t), l(t), e(t), t)$ is a continuous, linearly homogeneous, convex function in $p(t)$ for all $l(t)$;*
(*c*) *$\Pi(p(t), l(t), e(t), t)$ is a continuous, linearly homogeneous, non-decreasing concave function in $l(t)$ for all $p(t)$;*
(*d*) *If $\Pi(p(t), l(t), e(t), t)$ is differentiable at $l(t)$, then*

$$\frac{\partial \Pi(p(t), l(t), e(t), t)}{\partial l^k(t)} = p^k(t) \quad (k = 4, 5, 6, 7); \tag{20}$$

(*e*) *If $\Pi(p(t), l(t), e(t), t)$ is differentiable at $p(t)$ (in this case the number of primary factors of production is required to exceed or equal the number of commodities), then*

$$\frac{\partial \Pi(p(t), l(t), e(t), t)}{\partial p^k(t)} = \hat{y}^k(p(t), l(t), e(t), t) \quad (k = 1, 2, 3, 4, 6). \tag{21}$$

Since $x^r(t)$ is endogenous and thus so is $l^r(t)$, $l^r(t)$ has to be determined by the model. However, once $x^r(t)$ is known for all $t \geqq 0$, the law of motion (13) together with the initial endowment $l(0)$ determines the sequence $\{l(t)\}_{t=1}^T$. Of course, $x^r(t)$ has to be determined by economic behavior. To do so, define gross national capital formation (investment) $I(t)$ by

$$I(t) = \sum_{k=4}^{6} p^k(t) \cdot x^k(t), \tag{22}$$

and the intertemporal national-wealth (in the absence of borrowing or lending) by

$$\Omega = \sum_{t=0}^{T} \Phi(t)[\Pi(p(t), l(t), e(t), t) - I(t)], \tag{23}$$

where $\Phi(t) = \Pi_{\tau=0}^{t}(1 + r(\tau))^{-1}$.

The intertemporal national-wealth function $\Upsilon(p, l(0), e)$ is defined as

$$\Upsilon(p, l(0), e) = \max\{\Omega\colon l_i^r(t+1) = (1 - \delta_i^r(t))l_i^r(t) + x_i^r(t)\} \tag{24}$$

by appropriate choice of sequences $\{x^r(t)\}_{t=1}^T$ and $\{l^r(t)\}_{t=1}^T$ $(r=4,5,6)$.[7] Here $p=\{(p^1(t),\ldots,p^7(t))\}_{t=1}^T$ and $e=\{e(t)\}_{t=1}^T$.[8] When $T=\infty$, in order to guarantee that $\Upsilon(\pi,l(0),e)$ is finite, we may assume that $r(t)\geqq\varepsilon$ for some $\varepsilon>0$.

Similarly, we can prove the following lemma:

Lemma 2. *If the intertemporal national-wealth function $\Upsilon(p,l(0),e)$ is differentiable at $p>0$, then the Rybczynski functions $\hat{y}^k(p(t),\tilde{l}(\cdot,t),e^k(t),t)$ $(k=1,\ldots,4)$ and the national-investment functions $\hat{x}^k(p,l(0),e)$ $(k=4,5)$ satisfy*

$$\frac{\partial\Upsilon(p,l(0),e)}{\partial p_j^k(t)}=\Phi(t)\hat{y}_j^k(p(t),\tilde{l}(\cdot,t),e(t),t)\quad(k=1,2,3);\tag{25}$$

$$\frac{\partial\Upsilon(p,l(0),e)}{\partial p_j^4(t)}=\Phi(t)[\hat{y}_j^4(p(t),\tilde{l}(\cdot,t),e(t),t)-\hat{x}_j^4(p,l(0),e,t)];\tag{26}$$

$$\frac{\partial\Upsilon(p,l(0),e)}{\partial p_j^5(t)}=-\Phi(t)\hat{x}_j^5(p,l(0),e,t),\tag{27}$$

where $\tilde{l}(\cdot,t)=\tilde{l}(p,e,t)$ which is the solution of (24) *for $l(t)$.*

Define $A^{rk}(\cdot,0)=[a_{ij}^{rk}(p(t),e_j^k(0),0)]$ and $B^{sk}(\cdot,0)=[b_{ij}^{sk}(p(t),e_j^k(0),0)]$. Deleting the rows corresponding to $y^2(t)$ of (16), we can obtain

$$\tilde{M}(\cdot,0)\Lambda(t)^{-1}q(t)=\check{y}(t),\tag{28}$$

where

$$\tilde{M}(\cdot,0)=\begin{bmatrix}\Lambda^1(t)-A^{11}(\cdot,0) & -A^{13}(\cdot,0) & -A^{14}(\cdot,0) & -A^{16}(\cdot,0)\\ -A^{31}(\cdot,0) & \Lambda^3(t)-A^{33}(\cdot,0) & -A^{34}(\cdot,0) & -A^{36}(\cdot,0)\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix},$$

$$\Lambda(t)=\begin{bmatrix}\Lambda^1(t,e^1(t)) & & & \\ & \Lambda^3(t,e^3(t)) & & \\ & & \Lambda^4(t,e^4(t)) & \\ & & & \Lambda^6(t,e^6(t))\end{bmatrix},$$

(where $\Lambda^k(t,e^k(t))=\operatorname{diag}\{\lambda_j^k(t,e^k(t))\},j\in N_k)$, and

$$\check{y}(t)=(y^1(t)',y^3(t)',y^4(t)',y^6(t)')'.$$

---

[7] By an argument similar to that of Remark 2 below, we can show that the wealth-maximization problem (24) is equivalent to the maximization problem

$$\max\sum_{t=0}^{N}\Phi(t)\vartheta(t)\text{ subject to }\vartheta(t)=\Gamma(t)+R(t)b(t)-b(t+1)\text{ and (44)},$$

where $b(t)$ is total one-period bondholdings (which will be defined below) and $R(t)=1+r(t)$ is the interest factor on $b(t)$. This approach is more common in the macroeconomic literature.

[8] Even though $\Upsilon(\cdot)$ is a function of $(p^1,\ldots,p^6)$, for convenience we can write it as a function of p without loss of generality.

Assume that the Leontief matrix in (28) satisfies the usual Hawkins–Simon [17] condition of having positive principal minors (which is so if the matrix

$$\begin{pmatrix} A^1(t) - A^{11}(\cdot,0) & -A^{13}(\cdot,0) \\ -A^{31}(\cdot,0) & A^3(t) - A^{33}(\cdot,0) \end{pmatrix}$$

has positive principal minors). By the results of Gale and Nikaido [15] there is a unique solution for the gross outputs. This solution can be written in the form

$$q(t) = \Lambda(t)\tilde{M}(\cdot,0)^{-1}\check{y}(t). \tag{29}$$

Substituting (29) into (17), one obtains

$$B(\cdot,0)\tilde{M}(\cdot,0)^{-1}\check{y}(t) = l(t). \tag{30}$$

Also from (16) and (29) we have

$$y^2(t) = A^2(\cdot,0)\tilde{M}(\cdot,0)^{-1}\check{y}(t), \tag{31}$$

where

$$A^2(\cdot,0) = -(A^{21}(\cdot,0), A^{23}(\cdot,0), A^{24}(\cdot,0), A^{26}(\cdot,0)).$$

Thus we have the following basic set of equations from the production side:

$$\begin{aligned} g^1(p(t), e^1(t), t) &= p^1(t) \quad (t = 0, 1, \ldots, T) \\ g^3(p(t), e^3(t), t) &= p^3(t) \quad (t = 0, 1, \ldots, T) \\ g^4(p(t), e^4(t), t) &= p^4(t) \quad (t = 0, 1, \ldots, T) \\ g^6(p(t), e^6(t), t) &= p^6(t) \quad (t = 0, 1, \ldots, T-1) \\ B(\cdot,0)\tilde{M}(\cdot,0)^{-1}\check{y}(t) &= l(t) \quad (t = 0, 1, \ldots, T) \\ y^2(t) &= A^2(\cdot,0)\tilde{M}(\cdot,0)^{-1}\check{y}(t) \quad (t = 0, 1, \ldots, T) \\ x^6(t) &= y^6(t) \quad (t = 0, 1, \ldots, T-1). \end{aligned} \tag{32}$$

To close the model, we need to consider the consumption side. We first introduce a government that spends on goods that do not affect the utility of consumers and finances spending either by lump-sum taxes or by debt (government bonds). Its dynamic budget constraint is[9]

$$R(t)b_G(t) + G(t) = b_G(t+1) + \Theta(t), \tag{33}$$

where $b_G(t)$ is government bondholdings (with $b_G(0)$ given), $G(t)$ is government expenditure, and $\Theta(t)$ denotes lump-sum taxes. The government is required to satisfy the transversality condition

$$\Phi(T)b_G(T) = 0.$$

We also assume that consumers' preferences are aggregable. The aggregate consumption sequences $(x^1, x^2, x^3) = (\{x^1(t), x^2(t), x^3(t)\}_{t=0}^T)$ are obtained by solving

[9] Extension to the case where the interest factors for government bonds are different from those for private bondholdings is straightforward.

an intertemporal maximization problem which consists in choosing a path of consumption (x^1, x^2, x^3) and bondholdings $\{b(t+1)\}_{t=0}^{T}$ that maximizes the discounted lifetime expected utility function[10]

$$E \sum_{t=0}^{T} \beta^t U(x^1(t), x^2(t), x^3(t)) \quad (0 < \beta < 1) \tag{34}$$

subject to the dynamic budget constraints[11,12]

$$\sum_{k=1}^{5} p^k(t) \cdot x^k(t) + b(t+1) = R(t)b(t) + \sum_{k=1}^{4} p^k(t) \cdot y^k(t) - \Theta(t) \quad (t = 0, \ldots, T). \tag{35}$$

Here $b(t)$ is the sum of private bondholdings $b_P(t)$ (with $b_P(0)$ given) and government bondholdings $b_G(t)$, i.e., $b(t) = b_G(t) + b_P(t)$. The country is a net lender or borrower according as $b(t) > 0$ or < 0; in the former case the bondholders may be considered as all being domestic residents, and in the latter case foreign residents. The instantaneous utility function $U(\cdot)$ is assumed to be bounded, twice continuously differentiable, increasing in each argument, and strictly concave.

For convenience of exposition, denote the national non-portfolio (net) income available for consumption and saving by[13]

$$\Gamma(t) = Y(t) - I(t); \tag{37}$$

the current wealth position by

$$W(t) = \Gamma(t) + R(t)b(t) - \Theta(t); \tag{38}$$

the current consumer expenditure by

$$\begin{aligned} C(t) &= \sum_{k=1}^{3} p^k(t) \cdot x^k(t) \\ &= \Gamma(t) + R(t)b(t) - b(t+1) - \Theta(t) \end{aligned} \tag{39}$$

(in which we get from the first to the second equation by using the dynamic budget

[10] Extension to the case where consumers' discount rates and preferences vary over time is possible by using the approach given in Tian and Chipman [37]. Also, the model of this paper can be directly extended to the more general version of the overlapping-generations model of Blanchard [3], Frenkel and Razin [14], and Chipman and Tian [11] in which the finiteness of the horizon is introduced through the assumption that at each point in time individuals face a given probability of death.

[11] Note that $y^5(t) = q^5(t) = 0$ and $x^6(t) = y^6(t)$ (see footnote 6 and eq. (11)).

[12] From the government budget constraint (33) and the consumers budget constraint (35), we can obtain the national-income identity or feasibility condition:

$$\sum_{k=1}^{5} p^k(t) \cdot x^k(t) + G(t) + b_P(t+1) = R(t)b_P(t) + \sum_{k=1}^{4} p^k(t) \cdot y^k(t).$$

[13] Denoting the current rate of saving by $S(t) = I(t) + b(t+1) - R(t)b(t)$ we see from the following development that it satifies

$$Y(t) - \Theta(t) = C(t) + S(t). \tag{36}$$

constraint (35) together with (18), (22), and the definition of category 6);[14] and the deficit in the balance of payments on current account by

$$Z(t) = C(t) + I(t) + G(t) - \Theta(t) - Y(t), \tag{41}$$

i.e., $Z(t)$ is equal to private absorption $[C(t) + I(t)]$ plus government absorption $[G(t) - \Theta(t)]$ minus national product $Y(t)$.

Then from the budget constraint we have in terms of the above notation

$$C(t) + b(t+1) = W(t) \tag{42}$$

and, by using (37) and (39),

$$Z(t) = R(t)b(t) - b(t+1) + G(t) - 2\Theta(t). \tag{43}$$

In addition to the above budget constraint (35), the transversality condition

$$\Phi(T)b(T) = 0 \tag{44}$$

is needed to prevent the country from borrowing arbitrarily large amounts.

Define the "Lerner price" $\pi_j^k(t)$ of the j-th commodity in category k, following Lerner [25, Ch. 20], as the price that would subsist in a competitive intertemporal economy in the absence of an interest rate, by

$$\pi_j^k(t) = \Phi(t)p_j^k(t). \tag{45}$$

Remark 2. The maximization problem (34), together with (35) and (44), is equivalent to the maximization problem

$$\max E \sum_{t=0}^{T} \beta^t U(x^1(t), x^2(t), x^3(t)) \tag{46}$$

subject to

$$\sum_{t=0}^{T} \sum_{k=1}^{3} \pi^k(t) \cdot x^k(t) = \Omega - \sum_{t=0}^{T} \Phi(t)\Theta(t), \tag{47}$$

where Ω is given by (23).[15] In fact, solving the difference equation (35) recursively and using the "transversality condition" (44), we can obtain (47). Conversely, let the end-of-period bondholdings in period t be defined as[16]

$$b(0) = 0;$$

$$b(t) = R(t-1)b(t-1) + \Gamma(t-1) - \Theta(t) - \sum_{k=1}^{3} p^k(t-1) \cdot x^k(t-1) \tag{48}$$

for $t = 1, \ldots, T$. Then we obtain the budget constraint (35).

[14] From the government budget constraint (33) and (35), we can write $C(t)$ as:

$$C(t) = \Gamma(t) + R(t)b_P(t) - b_P(t+1) - G(t), \tag{40}$$

which can be used to study effects of government spending and private bondholdings on policy functions.

[15] The budget constraint (47) provides an interpretation as to why the central planner should want to maximize Ω.

[16] The formulation justifies the introduction of one-period bonds in (35) which, after all, are not observed in the real world.

Solving (45) for the $p^k(t)$ and substituting them into (48), and using the definition $z^k(t) = x^k(t) - y^k(t)$, we obtain the difference equation

$$b(t+1) = R(t)b(t) - \Theta(t) - \sum_{s}^{t} R(s) \sum_{k=1}^{5} \pi^k(t)\cdot z^k(t). \tag{49}$$

Solving (49) recursively, we obtain the "transversality condition" (44).

Given $(u(0), b(0))$, the maximization problem (34) is to choose a path of consumption $\{(x^1(t), x^2(t), x^3(t))\}_{t=0}^{T}$ and bondholdings $\{b(t+1)\}_{t=0}^{T}$ so as to maximize the additive objective function (34) subject to (35) and (44).

If T is finite, the problem "maximize (34) subject to (35) and (44)" is an entirely standard concave programming problem. The set of $(x^1(t), x^2(t), x^3(t))$ and $b(t+1)$ satisfying $\Sigma_{k=1}^{3} p^k(t)\cdot x^k(t) + b(t+1) \leqq W(t)$ is compact and convex, and the objective function $E\Sigma_{t=0}^{T} \beta^t U(x^1(t), x^2(t), x^3(t))$ is continuous and strictly concave. Hence there is exactly one optimal choice, which is completely characterized by the Kuhn–Tucker conditions. If T is infinite, we need to use dynamic programming to solve the maximization problem.

To reduce the dimensionality of the dynamic programming problem and focus our interests on the primary variables $W(t)$, $C(t)$, and $Z(t)$, our simplification is to replace the utility function in (34) by the indirect utility function

$$V(p_c(t), C(t)) = \max \{U(x^1(t), x^2(t), x^3(t)) : \sum_{k=1}^{3} p^k(t)\cdot x^k(t) \leqq C(t)\}, \tag{50}$$

where $p_c(t) = (p^1(t), p^2(t), p^3(t))$.

Then the maximization problem becomes

$$\max E \sum_{t=0}^{T} \beta^t V(p_c(t), C(t)) \tag{51}$$

subject to

$$C(t) + b(t+1) \leqq W(t). \tag{52}$$

Since the utility function is strictly concave in $x^k(t)$, the indirect utility function is also strictly concave in $C(t)$. As demonstrated in the Appendix, the above problem is a stationary discounted dynamic programming problem. Thus we replace the original maximization problem by the optimal equation (called Bellman's equation)

$$\mathfrak{R}(b, u, l) = \max_{(C, b') \in \gamma(b, u, l)} \{V(p_c, C) + \beta \int \mathfrak{R}(b', \Psi(u, \varepsilon'), l')dF(\varepsilon')\}, \tag{53}$$

where primes denote the variables b, u, and l in the next period,[17] $p_c = (p^1(t), p^2(t), p^3(t))$, $C = C(t)$, $b = b(t)$, $u = u(t)$, $l = l(t)$ and

$$\gamma(b, u, l) = \{(C, b') : C + b' \leqq W, C \geqq 0\}. \tag{54}$$

Note that $W(t) = \Gamma(t) + R(t)b(t) - \Theta(t)$, which is given in (38).

[17] Note that we have also used primes to denote the transpose of a matrix. But as long as primes are used with the variables b, u, and l, the prime "′" means the next period.

Let L denote the space of continuous, bounded functions $\mathfrak{R}$, normed by

$$\|\mathfrak{R}\| = \sup_{(b,u,l)} \mathfrak{R}(b,u,l).$$

Then L is a Banach space. Define the operator $\mathcal{T}$ as follows:

$$\mathcal{T}\mathfrak{R}(b,u,l) = \max_{(C,b')\in y(b,u,l)} \{V(p_c, C) + \beta E\mathfrak{R}(b',u',l')\}.$$

To solve (53) means to find a fixed point. In the Appendix, we will prove

Lemma 3. *$\mathcal{T}$: $L \to L$, that is, $\mathcal{T}$ maps bounded continuous functions into themselves.*

Lemma 4. *Equation* (53) *has exactly one continuous bounded solution, say, $\tilde{\mathfrak{R}} \in L$, and for all $\mathfrak{R}^0 \in L$ and $n = 1, 2, \ldots,$*

$$\|\mathcal{T}^n \mathfrak{R}^0 - \tilde{\mathfrak{R}}\| \leqq \alpha^n \|\mathfrak{R}^0 - \tilde{\mathfrak{R}}\|$$

for $0 < \alpha < 1$.

Lemma 5. *The unique solution $\tilde{\mathfrak{R}}(b,u,l)$ to* (53) *is strictly increasing and strictly concave in b.*

Lemma 6. *For each fixed (b,u,l), the solution $\tilde{\mathfrak{R}}(b,u,l)$ is attained by a unique C and b′ denoted by*

$$C = \tilde{C}(b,u,l) \tag{55}$$

and

$$b' = \tilde{b}(b,u,l), \tag{56}$$

and the functions $\tilde{C}(\cdot)$ and $\tilde{b}(\cdot)$ are both continuous.

The motion of the state variables (b,u) of the system is then given by the initial state $(b(0), u(0))$, the distribution $F(\cdot)$ of the stochastic variables $\varepsilon(t)$, and the two difference equations which are eq. (1) and

$$b(t+1) = \tilde{b}(b(t), u(t), l(t)). \tag{57}$$

Lemma 7. *For given $(b(0), u(0))$, $\{C(t), b(t+1)\}$ defined by* (55) *and* (56) *solves the maximization problem*

$$\max E \sum_{t=0}^{T} \beta^t V(C(t), p_c(t)) \tag{58}$$

subject to

$$C(t) + b(t+1) = W(t). \tag{59}$$

Substituting $C(t) = \tilde{C}(b(t), u(t), l(t))$ into (50), we obtain the demand functions by the Antonelli–Allen–Roy partial differential equation

$$x^k(t) = h^k(p_c(t), C(t)) = -\frac{\partial V(p_c(t), C(t))/\partial p^k(t)}{\partial V(p_c(t), C(t))/\partial C(t)} \tag{60}$$

for $k = 1, 2, 3$.

From Lemmas 3–7, we know that (55), (56), and (60) solve the maximization problem (34). By definition of category 3, the demand for nontradable consumer goods is equal to the supply, i.e., $x^3(t) = y^3(t)$, so that in equilibrium we have

$$y_j^3(t) = x_j^3(t) = h_j^3(p_c(t), \tilde{C}(\cdot)) \tag{61}$$

for $j \in N_3$.

Solving the system of eqs. (32) and (61) notationally for $\{y^1(t), y^2(t), y^3(t), y^4(t), y^6(t), p^3(t), p^6(t), p^7(t)\}$, we have

$$y^k(t) = \tilde{y}^k(b(t), u(t), l(t), t) \tag{62}$$

for $k = 1, 2, 3, 4, 6$ and

$$p^k(t) = \tilde{p}^k(b(t), u(t), l(t), t) \tag{63}$$

for $k = 3, 6, 7$.

We define the net import of commodity j in category k as

$$z_j^k(t) = x_j^k(t) - y_j^k(t) \tag{64}$$

for $j \in N_k$ and $k = 1, 2, 4, 5$. The net-import demand functions for consumer goods are defined by

$$\begin{aligned}\tilde{z}^k(b(t), u(t), l(t), t) = h^k(&p^1(t), p^2(t), \tilde{p}^3(b(t), u(t), l(t), t),\\ &\tilde{C}(b(t), u(t), l(t))) - \tilde{y}^k(b(t), u(t), l(t), t) \quad (k = 1, 2).\end{aligned} \tag{65}$$

3 Some theoretical results on policy functions

In this section and the following one, we will apply our model to undertake some theoretical analyses of the responses of next-period bondholdings, net imports, the balance of payments on current account, the current wealth position, current expenditure on consumption, and current internal prices to current external prices, the current interest factor, current factor endowments, current taxes, and current bondholdings. We will also investigate the integrability of the net-import demand functions. To do so, we restrict ourselves to the special cases where $u(t)$ follows an independent and identically distributed random process and there are no capital goods (i.e., $n_4 = n_5 = n_6 = 0$)[18] so that $\Gamma(t)$ reduces to

$$\tilde{\Gamma}(p(t), l(t), e(t), t) = \Pi(p(t), l(t), e(t), t). \tag{66}$$

Also we only consider the case $n_7 \geqq n_1 + n_2$ so that we have a single-valued net supply function $\hat{y}^k(p(t), l(t), e(t), t)$, and thus the function $\tilde{y}^k$ of (62) becomes

$$\tilde{y}^k(b(t), u(t), l(t), t) = \hat{y}^k(p^1(t), p^2(t), \tilde{p}^3(b(t), u(t), l(t), t), l(t), e(t), t) \tag{67}$$

for $k = 1, 2, 3$ and the net-import demand function becomes

$$\begin{aligned}\tilde{z}^k(b(t), u(t), l(t), t) = h^k(p^1(t), p^2(t), \tilde{p}^3(b(t), u(t), l(t), t), \tilde{C}(b(t), u(t), l(t)))\\ - \hat{y}^k(p^1(t), p^2(t), \tilde{p}^3(b(t), u(t), l(t), t), l(t), e(t), t)\end{aligned} \tag{68}$$

[18] Extension to include capital goods may be possible but the exposition is simpler if attention is restricted to the case without capital goods.

for $k = 1, 2$. For the nontradable consumer goods, eq. (61) becomes

$$h^3(p^1(t), p^2(t), \tilde{p}^3(b(t), u(t), l(t), t), \tilde{C}(b(t), u(t), l(t))) - \hat{y}^3(p^1(t), p^2(t), \tilde{p}^3(b(t), u(t), l(t), t), l(t), e(t), t) = 0. \quad (69)$$

In the following, we study the properties of the solution functions $p^3(t) = \tilde{p}^3(\cdot)$, $C(t) = \tilde{C}(\cdot)$, $b(t+1) = \tilde{b}(\cdot)$, $Z(t) = \tilde{Z}(\cdot)$, $W(t) = \tilde{W}(\cdot)$, and $z^k(t) = \tilde{z}^k(\cdot)$, and particularly the vectors of partial derivatives of the solution functions with respect to external prices $p^k(t)$ $(k = 1, 2)$, current bondholdings $b(t)$, the interest factor $R(t)$, current taxes $\Theta(t)$, and factor endowments $l(t)$. For $k = 1, 2, 3$ and $r = 1, 2, 3$, denote the partial $n_r \times n_k$ Slutsky matrices of the demand function $h^k(\cdot)$ by

$$S^{rk}(t) = \frac{\partial h^r(\cdot)}{\partial p^k(t)} + \frac{\partial h^r(\cdot)}{\partial C(t)} h^k(\cdot)',$$

the $n_r \times n_k$ production transformation matrices associated with the Rybczynski function $\hat{y}^r(\cdot)$ by

$$T^{rk}(t) = \frac{\partial \hat{y}^r(\cdot)}{\partial p^k(t)},$$

the trade Slutsky matrices $\hat{S}^{rk}(t)$ by

$$\hat{S}^{rk}(t) = S^{rk}(t) - T^{rk}(t),$$

and the marginal consumption-expenditure coefficients by

$$c^k(t) = \frac{\partial h^k(\cdot)}{\partial C(t)}.$$

Let

$$\Delta_1(t) = V_{CC}(p_c(t), C(t));$$

$$\Delta_2(t) = \beta R(t) E \Re_{bb}(b(t+1), u(t+1), l(t+1));$$

$$\Delta(t) = \Delta_1(t) + \Delta_2(t) = V_{CC}(p_c(t), C(t)) + \beta R(t) E \Re_{bb}(b(t+1), u(t+1), l(t+1));$$

$$D^k(t) = V_{Cp^k}(p_c(t), C(t)) + \Delta_1(t) y^k(t) \quad (k = 1, 2, 3).$$

It may be remarked that by the strict concavity of the indirect utility function and optimal value function, we have $\Delta_1(t) < 0$ and $\Delta_2(t) < 0$.

Note that the optimal debt $b(t+1) = \tilde{b}(b(t), u(t), l(t))$ and the price vector of nontradable goods $\tilde{p}^3(b(t), u(t), l(t))$ are defined implicitly by the first-order condition

$$V_C(p_c(t), \Pi(\cdot) + R(t)b(t) - \Theta(t) - b(t+1)) = R(t)\beta E \Re_b(b(t+1), u(t+1), l(t+1)) \quad (70)$$

and (69). Differentiating (69) and (70), we find that

$$\begin{pmatrix} \hat{S}^{33}(t) & -c^3(t) \\ D^3(t)' & -\Delta(t) \end{pmatrix} \begin{pmatrix} d\tilde{p}^3(\cdot) \\ db(t+1) \end{pmatrix} = -\begin{pmatrix} c^3(t)R(t) \\ \Delta_1(t)R(t) \end{pmatrix} db(t)$$

$$-\begin{pmatrix} c^3(t)b(t) \\ \Delta_1(t)b(t) \end{pmatrix} dR(t) + \begin{pmatrix} c^3(t) \\ \Delta_1(t) \end{pmatrix} d\Theta(t) - \begin{pmatrix} c^3(t)p^7(t)' - \dfrac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)} \\ \Delta_1(t)p^7(t)' \end{pmatrix} dl(t)$$
$$-\begin{pmatrix} \hat{S}^{31}(t) - c^3(t)z^1(t)' \\ D^1(t)' \end{pmatrix} dp^1(t) - \begin{pmatrix} \hat{S}^{32}(t) - c^3(t)z^2(t)' \\ D^2(t)' \end{pmatrix} dp^2(t).$$

From the formula for inverting a partitioned matrix (cf., e.g., Aitken [1, p. 139], we have

$$\begin{pmatrix} \hat{S}^{33}(t) & -c^3(t) \\ D^3(t)' & -\Delta(t) \end{pmatrix}^{-1} = \begin{pmatrix} Q(t)^{-1} & -Q(t)^{-1}c^3(t)\Delta(t)^{-1} \\ -O(t)^{-1}D^3(t)'\hat{S}^{33}(t)^{-1} & O(t)^{-1} \end{pmatrix},$$

where

$$Q(t) = \hat{S}^{33}(t) - c^3(t)\Delta(t)^{-1}D^3(t)'$$

is an $n_3 \times n_3$ matrix and

$$O(t) = -(\Delta(t) - D^3(t)'\hat{S}^{33}(t)^{-1}c^3(t))$$

is a real-valued function.

Thus the vectors of partial derivatives of the solution functions with respect to external prices $p^k(t)$ $(k = 1, 2)$, current bondholdings $b(t)$, the interest factor $R(t)$, and factor endowments $l(t)$ are given by

$$\begin{pmatrix} \dfrac{\partial \tilde{p}^3(\cdot)}{\partial b(t)} \\ \dfrac{\partial b(t+1)}{\partial b(t)} \end{pmatrix} = -\begin{pmatrix} Q(t)^{-1} & -Q(t)^{-1}c^3(t)\Delta(t)^{-1} \\ -O(t)^{-1}D^3(t)'\hat{S}^{33}(t)^{-1} & O(t)^{-1} \end{pmatrix} \begin{pmatrix} c^3(t)R(t) \\ \Delta_1(t)R(t) \end{pmatrix};$$

$$\begin{pmatrix} \dfrac{\partial \tilde{p}^3(\cdot)}{\partial \Theta(t)} \\ \dfrac{\partial b(t+1)}{\partial \Theta(t)} \end{pmatrix} = \begin{pmatrix} Q(t)^{-1} & -Q(t)^{-1}c^3(t)\Delta(t)^{-1} \\ -O(t)^{-1}D^3(t)'\hat{S}^{33}(t)^{-1} & O(t)^{-1} \end{pmatrix} \begin{pmatrix} c^3(t) \\ \Delta_1(t) \end{pmatrix};$$

$$\begin{pmatrix} \dfrac{\partial \tilde{p}^3(\cdot)}{\partial R(t)} \\ \dfrac{\partial b(t+1)}{\partial R(t)} \end{pmatrix} = -\begin{pmatrix} Q(t)^{-1} & -Q(t)^{-1}c^3(t)\Delta(t)^{-1} \\ -O(t)^{-1}D^3(t)'\hat{S}^{33}(t)^{-1} & O(t)^{-1} \end{pmatrix} \begin{pmatrix} c^3(t)b(t) \\ \Delta_1(t)b(t) \end{pmatrix};$$

$$\begin{pmatrix} \dfrac{\partial \tilde{p}^3(\cdot)}{\partial l(t)} \\ \dfrac{\partial b(t+1)}{\partial l(t)} \end{pmatrix} = -\begin{pmatrix} Q(t)^{-1} & -Q(t)^{-1}c^3(t)\Delta(t)^{-1} \\ -O(t)^{-1}D^3(t)'\hat{S}^{33}(t)^{-1} & O(t)^{-1} \end{pmatrix}$$
$$\times \begin{pmatrix} c^3(t)p^7(t)' - \dfrac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)} \\ \Delta_1(t)p^7(t)' \end{pmatrix};$$

$$\begin{pmatrix} \dfrac{\partial \tilde{p}^3(\cdot)}{\partial p^k(t)} \\ \dfrac{\partial b(t+1)}{\partial p^k(t)} \end{pmatrix} = -\begin{pmatrix} Q(t)^{-1} & -Q(t)^{-1}c^3(t)\Delta(t)^{-1} \\ -O(t)^{-1}D^3(t)'\hat{S}^{33}(t)^{-1} & O(t)^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} \hat{S}^{3k}(t) - c^3(t)z^k(t)' \\ D^k(t) \end{pmatrix}.$$

Hence,

$$\frac{\partial b(t+1)}{\partial b(t)} = R(t)O(t)^{-1}[D^3(t)'\hat{S}^{33}(t)^{-1}c^3(t) - \Delta_1(t)]; \tag{71}$$

$$\frac{\partial b(t+1)}{\partial \Theta(t)} = -O(t)^{-1}[D^3(t)'\hat{S}^{33}(t)^{-1}c^3(t) - \Delta_1(t)]; \tag{72}$$

$$\frac{\partial b(t+1)}{\partial R(t)} = b(t)O(t)^{-1}[D^3(t)'\hat{S}^{33}(t)^{-1}c^3(t) - \Delta_1(t)]; \tag{73}$$

$$\frac{\partial b(t+1)}{\partial l(t)} = O(t)^{-1}\left[D^3(t)'\hat{S}^{33}(t)^{-1}\left(c^3(t)p^7(t)' - \frac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)}\right) - \Delta_1(t)p^7(t)'\right]; \tag{74}$$

$$\frac{\partial b(t+1)}{\partial p^k(t)} = O(t)^{-1}[D^3(t)'\hat{S}^{33}(t)^{-1}(\hat{S}^{3k}(t) - c^3(t)z^k(t)') - D^k(t)']; \tag{75}$$

$$\frac{\partial \tilde{p}^3(\cdot)}{\partial b(t)} = -\frac{\Delta_2(t)}{\Delta(t)}R(t)Q(t)^{-1}c^3(t); \tag{76}$$

$$\frac{\partial \tilde{p}^3(\cdot)}{\partial \Theta(t)} = \frac{\Delta_2(t)}{\Delta(t)}Q(t)^{-1}c^3(t); \tag{77}$$

$$\frac{\partial \tilde{p}^3(\cdot)}{\partial R(t)} = -\frac{\Delta_2(t)}{\Delta(t)}b(t)Q(t)^{-1}c^3(t); \tag{78}$$

$$\frac{\partial \tilde{p}^3(\cdot)}{\partial l(t)} = -Q(t)^{-1}\left[\frac{\Delta_2(t)}{\Delta(t)}c^3(t)p^7(t)' - \frac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)}\right]; \tag{79}$$

$$\frac{\partial \tilde{p}^3(\cdot)}{\partial p^k(t)} = -Q(t)^{-1}\left[\hat{S}^{3k}(t) - c^3(t)\left(z^k(t)' + \frac{D^k(t)'}{\Delta}\right)\right]. \tag{80}$$

Thus for next-period bondholdings, if the following conditions are satisfied:[19] (a) the nontradable consumer goods are Hicksian substitutes (i.e., for $k = 1, 2, 3$, $S^{3k}(t) \geqq 0$ except $S^{33}_{ii}(t) < 0$ for all $i \in N_3$) but not inferior; (b) $T^{33}_{ii}(t) > 0$ for all $i \in N_3$, $T^{33}_{ij}(t) \leqq 0$ for $i \neq j$ ($i, j \in N_3$), and $T^{3k}(t) \leqq 0$ for $k = 1, 2$; (c) $V_{C_{p^k}}(\cdot) \leqq 0$ for $k = 1, 2, 3$; (d) the matrix $\hat{S}^{33}(t)$ has a negative dominant diagonal (cf. McKenzie [27]), then

[19] Conditions (a)–(d) are obviously satisfied if preferences and production functions are of the Cobb–Douglas type and there is only one nontradable consumer good.

$c^3(t) \geqq 0$, $\Delta_i(t) < 0$ for $i = 1, 2$, $O(t) > 0$, $D^k(t) < 0$ for $k = 1, 3$, and all elements of $\hat{S}^{33}(t)^{-1}$ are nonpositive. Then, an increase in current-period bondholdings, or in the prices of tradable consumer goods produced by the country, causes next-period bondholdings to increase; an increase in current-period taxes causes next-period bondholdings to decrease; an increase in the current interest factor causes next-period bondholdings to increase if the country is lending or to decrease if the country is borrowing; an increase in current factor endowments causes next-period bondholdings to increase if $\left[c^3(t)p^7(t)' - \dfrac{\partial \bar{y}^3(\cdot)'}{\partial l(t)} \right] \geqq 0$ or to decrease otherwise; and an increase in the external prices of tradable consumer goods not produced in the country causes next-period bondholdings to decrease if $\hat{S}^{32}(t) \leqq c^3(t)z^2(t)'$ and $D^2(t)' \geqq 0$.

For the prices of nontradable consumer goods, if the following conditions are satisfied: (a) the nontradable consumer goods are Hicksian substitutes but not inferior; (b) $T^{33}_{ii}(t) > 0$ for all $i \in N_3$, $T^{33}_{ij}(t) \leqq 0$ for $i \neq j$ $(i, j \in N_3)$, and $T^{3k}(t) \leqq 0$ for $k = 1, 2$; (c) $V_{Cp^k}(\cdot) \leqq 0$ for $k = 1, 2, 3$; (d) $Q_{ij}(t) \geqq 0$ for $i \neq j$ $(i, j \in N_3)$ and the matrix $Q(t)$ has a negative dominant diagonal, then: an increase in current-period bondholdings causes the prices of all nontradable consumer goods to increase; an increase in current-period taxes causes the prices of all nontradable consumer goods to decrease; an increase in the current interest factor causes the prices of all nontradable consumer goods to increase if the country is lending or to decrease if the country is borrowing; an increase in current factor endowments causes the prices of all nontradable consumer goods to increase if $\left[\dfrac{\Delta_2(t)}{\Delta(t)} c^3(t)p^7(t)' - \dfrac{\partial \bar{y}^3(\cdot)'}{\partial l(t)} \right] \geqq 0$ or to decrease otherwise; and an increase in the prices of tradable consumer goods causes the prices of all nontradable consumer goods to increase if $\left(z^k(t) + \dfrac{D^k(t)}{\Delta(t)} \right) \leqq 0$ $(k = 1, 2)$ or to decrease otherwise.

After obtaining the vectors of partial derivatives of $b(t+1)$ and $\tilde{p}^3(\cdot)$, we can easily analyze the responses of the balance-of-payments deficit on current account and the current wealth position defined by (43) and (38) to the current-period bondholdings, current interest factor, current taxes, current factor endowments, and current external prices. From (43) and (38), we have

$$\begin{aligned} \frac{\partial Z(t)}{\partial b(t)} &= R(t) - \frac{\partial b(t+1)}{\partial b(t)} \\ &= -R(t)O(t)^{-1}\Delta_2(t); \end{aligned} \tag{81}$$

$$\begin{aligned} \frac{\partial Z(t)}{\partial \Theta(t)} &= -2 - \frac{\partial b(t+1)}{\partial \Theta(t)} \\ &= -1 + O(t)^{-1}\Delta_2(t); \end{aligned} \tag{82}$$

$$\frac{\partial Z(t)}{\partial R(t)} = b(t) - \frac{\partial b(t+1)}{\partial R(t)}$$
$$= -b(t)O(t)^{-1}\Delta_2(t); \tag{83}$$

$$\frac{\partial Z(t)}{\partial l(t)} = -\frac{\partial b(t+1)}{\partial l(t)}$$
$$= -O(t)^{-1}\left[D^3(t)'\hat{S}^{33}(t)^{-1}\left(c^3(t)p^7(t)' - \frac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)}\right) - \Delta_1(t)p^7(t)'\right]; \tag{84}$$

$$\frac{\partial Z(t)}{\partial p^k(t)} = -\frac{\partial b(t+1)}{\partial p^k(t)}$$
$$= -O(t)^{-1}[D^3(t)'\hat{S}^{33}(t)^{-1}(\hat{S}^{3k}(t) - c^3(t)z^k(t)') - D^k(t)']; \tag{85}$$

$$\frac{\partial W(t)}{\partial b(t)} = R(t) + \hat{y}^3(\cdot)'\frac{\partial \tilde{p}^3(\cdot)}{\partial b(t)}$$
$$= R(t)\left[1 - \frac{\Delta_2(t)}{\Delta(t)}\hat{y}^3(\cdot)'Q(t)^{-1}c^3(t)\right]; \tag{86}$$

$$\frac{\partial W(t)}{\partial \Theta(t)} = \hat{y}^3(\cdot)'\frac{\partial \tilde{p}^3(\cdot)}{\partial \Theta(t)} - 1$$
$$= \frac{\Delta_2(t)}{\Delta(t)}\hat{y}^3(\cdot)'Q(t)^{-1}c^3(t) - 1; \tag{87}$$

$$\frac{\partial W(t)}{\partial R(t)} = b(t) + \hat{y}^3(\cdot)'\frac{\partial \tilde{p}^3(\cdot)}{\partial R(t)}$$
$$= b(t)\left[1 - \frac{\Delta_2(t)}{\Delta(t)}\hat{y}^3(\cdot)'Q(t)^{-1}c^3(t)\right]; \tag{88}$$

$$\frac{\partial W(t)}{\partial l(t)} = p^7(t)' + \hat{y}^3(\cdot)'\frac{\partial \tilde{p}^3(\cdot)}{\partial l(t)}$$
$$= \left[1 - \frac{\Delta_2(t)}{\Delta(t)}\hat{y}^3(\cdot)'Q(t)^{-1}c^3(t)\right]p^7(t)' + \hat{y}^3(\cdot)'Q(t)^{-1}\frac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)}, \tag{89}$$

$$\frac{\partial W(t)}{\partial p^k(t)} = p^k(t)' + \hat{y}^3(\cdot)'\frac{\partial \tilde{p}^3(\cdot)}{\partial p^k(t)}$$
$$= p^k(t)' - Q(t)^{-1}\left[\hat{S}^{3k}(t) - c^3(t)\left(z^k(t)' + \frac{D^k(t)'}{\Delta(t)}\right)\right]. \tag{90}$$

Thus under the assumptions leading to the above conclusions on the derivatives of the solution functions $p^3(t) = \tilde{p}^3(\cdot)$ and $b(t+1) = \tilde{b}(\cdot)$, we have the following results: With regard to the balance of payments on current account $Z(t)$, an increase in current-period bondholdings causes the balance-of-payments deficit on current account to increase; an increase in current-period taxes causes the current-account deficit to decrease; an increase in the current interest factor causes the current-account deficit to increase if the country is lending or to decrease if it is borrowing; an increase in current factor endowments causes the current-account deficit to decrease if $\left[c^3(t)p^7(t)' - \frac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)}\right] \geqq 0$ or increase otherwise; an increase in the prices of tradable consumer goods produced in the country causes the current-account deficit to decrease; and an increase in the prices of tradable consumer goods not produced in the country causes the current-account deficit to increase if $\hat{S}^{32}(t) \leqq c^3(t)z^2(t)'$ and $D^2(t)' \geqq 0$ or to decrease otherwise. The last result of changing the prices of tradable consumer goods not produced in the country is of special interest since it clarifies the theoretical disputes on the so-called Harberger–Laursen–Metzler effect and shows under what conditions on utility and production functions the Harberger–Laursen–Metzler effect holds or does not hold. This again shows that special-purpose models sometimes cannot give a general or satisfactory answer since the answer depends on a very simplified structure.

With regard to the current wealth position $W(t)$, an increase in current-period bondholdings causes current wealth to increase; an increase in current-period taxes causes current wealth to decrease; an increase in the current interest factor causes current wealth to increase if the country is lending or to decrease if it is borrowing; an increase in current factor endowments causes current wealth to increase if $\left[\frac{\Delta_2(t)}{\Delta(t)}c^3(t)p^7(t)' - \frac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)}\right] \geqq 0$ or to decrease if $\hat{y}^3\ (t)Q^{-1}(t)\left[\frac{\Delta_2(t)}{\Delta(t)}c^3(t)p^7(t)' - \frac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)}\right] \geqq p^7(t)'$; an increase in the prices of tradable consumer goods produced in the country causes current wealth to increase if $\left(z^1(t) + \frac{D^1(t)}{\Delta(t)}\right) \leqq 0$; an increase in the prices of tradable consumer goods not produced in the country causes current wealth to increase if $\left(z^2(t) + \frac{D^2(t)}{\Delta(t)}\right) \leqq 0$.

We now investigate effects of current-period bondholdings, current taxes, the current interest factor, current factor endowments, and current external prices on consumption expenditure $C(t)$. Note that the optimal consumption expenditure $C(t) = \tilde{C}(b(t), u(t), l(t))$ and the prices of nontradable consumer goods $\tilde{p}^3(\cdot)$ are defined implicitly by the first-order condition:

$$V_C(p_c(t), C(t)) = R(t)\beta E\Re_b(R(t)b(t) + \Pi(\cdot) - \Theta(t) - C(t), u(t+1), l(t+1)) \quad (91)$$

and (69). Differentiating (69) and (91), we find that

$$\begin{pmatrix} \Xi^{33}(t) & c^3(t) \\ \psi^3(t)' & \Delta(t) \end{pmatrix}\begin{pmatrix} d\tilde{p}^3(\cdot) \\ dC(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \Delta_2(t)R(t) \end{pmatrix} db(t) + \begin{pmatrix} 0 \\ \Delta_2(t)b(t) \end{pmatrix} dR(t)$$
$$- \begin{pmatrix} 0 \\ \Delta_2(t) \end{pmatrix} d\Theta(t) + \begin{pmatrix} \dfrac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)} \\ \Delta_2(t)p^7(t)' \end{pmatrix} dl(t)$$
$$- \begin{pmatrix} \Xi^{31}(t) \\ \psi^1(t)' \end{pmatrix} dp^1(t) - \begin{pmatrix} \Xi^{32}(t) \\ \psi^2(t)' \end{pmatrix} dp^2(t),$$

where

$$\Xi^{rk}(t) = \frac{\partial h^r(\cdot)}{\partial p^k(t)} - \frac{\partial \tilde{y}^r(\cdot)}{\partial p^k(t)} \quad (r, k = 1, 2, 3)$$

and

$$\psi^k(t) = V_{Cp^k}(\cdot) - \Delta_2(t)\tilde{y}^k(\cdot) \quad (k = 1, 2, 3).$$

Solving the above equation, we can obtain the vectors of partial derivatives of the solution function $\tilde{C}(\cdot)$ with respect to current external prices $p^k(t)$ $(k = 1, 2)$, current bondholdings $b(t)$, current taxes $\Theta(t)$, the current interest factor $R(t)$, and current factor endowments $l(t)$:

$$\frac{\partial \tilde{C}(\cdot)}{\partial b(t)} = R(t)\Delta_2(t)\rho(t)^{-1}; \tag{92}$$

$$\frac{\partial \tilde{C}(\cdot)}{\partial \Theta(t)} = -\Delta_2(t)\rho(t)^{-1}; \tag{93}$$

$$\frac{\partial \tilde{C}(\cdot)}{\partial R(t)} = b(t)\Delta_2(t)\rho(t)^{-1}; \tag{94}$$

$$\frac{\partial \tilde{C}(\cdot)}{\partial l(t)} = -\rho(t)^{-1}\left[\psi^3(t)'\Xi^{33}(t)^{-1}\frac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)} - \Delta_2(t)p^7(t)'\right]; \tag{95}$$

$$\frac{\partial \tilde{C}(\cdot)}{\partial p^k(t)} = \rho(t)^{-1}[\psi^3(t)'\Xi^{33}(t)^{-1}\Xi^{3k}(t) - \psi^k(t)'], \tag{96}$$

where

$$\rho(t) = (\Delta(t) - \psi^3(t)'\Xi^{33}(t)^{-1}c^3(t)), \tag{97}$$

which is a real-valued function.

Thus for consumption expenditure if the following conditions are satisfied: (a) the nontradable consumer goods are gross substitutes (i.e., for $k = 1, 2, 3$, $\Xi^{3k}(t) \geqq 0$ except $\Xi^{33}_{ii}(t) < 0$ for all $i \in N_3$) but not inferior; (b) $T^{33}_{ii}(t) > 0$ for all $i \in N_3$, $T^{33}_{ij}(t) \leqq 0$ for $i \neq j$ $(i, j \in N_3)$, and $T^{3k}(t) \leqq 0$ for $k = 1, 2$; (c) $\psi^3(t) \leqq 0$; (d) the matrix $\Xi^{33}(t)$ has a negative dominant diagonal, then an increase in current-period bondholdings causes current consumption expenditure to increase; an increase in current-period taxes causes current consumption expenditure to decrease; an increase in the current interest factor causes current consumption expenditure to increase if the country is lending or to decrease if it is borrowing; an increase in current factor endowments causes current consumption expenditure to increase if $\frac{\partial \tilde{y}^3(\cdot)'}{\partial l(t)} \geqq 0$; an increase in the prices of tradable consumer goods not produced in the country causes current consumption expenditure to decrease if $V_{Cp^2} < 0$; and an increase in the prices of tradable consumer goods produced in the country causes current consumption expenditure to increase if $[\psi^3(t)'\Xi^{33}(t)^{-1}\Xi^{31}(t) - \psi^1(t)'] \leqq 0$ or to decrease otherwise.

Next we consider the responses of net imports to current external prices, the current interest factor, current taxes, current factor endowments, and current bondholdings. Differentiating (68) with respect to $b(t), R(t), l(t)$, and $p^k(t)$, we obtain

$$\frac{\partial \tilde{z}^r(t)}{\partial b(t)} = \Xi^{r3}(t)\frac{\partial \tilde{p}^3(\cdot)}{\partial b(t)} + c^r(t)\frac{\partial \tilde{C}(\cdot)}{\partial b(t)}; \tag{98}$$

$$= R(t)\left[\Delta_2(t)\rho(t)^{-1}c^r(t) - \frac{\Delta_2(t)}{\Delta(t)}\Xi^{r3}(t)Q(t)^{-1}c^3(t)\right]$$

$$\frac{\partial \tilde{z}^r(t)}{\partial \Theta(t)} = \Xi^{r3}(t)\frac{\partial \tilde{p}^3(\cdot)}{\partial \Theta(t)} + c^r(t)\frac{\partial \tilde{C}(t)}{\partial \Theta(t)}; \tag{99}$$

$$= \frac{\Delta_2(t)}{\Delta(t)}\Xi^{r3}(t)Q(t)^{-1}c^3(t) - \Delta_2(t)\rho(t)^{-1}c^r(t)$$

$$\frac{\partial \tilde{z}^r(t)}{\partial R(t)} = \Xi^{r3}(t)\frac{\partial \tilde{p}^3(\cdot)}{\partial R(t)} + c^r(t)\frac{\partial \tilde{C}(\cdot)}{\partial R(t)}; \tag{100}$$

$$= b(t)\left[\Delta_2(t)\rho(t)^{-1}c^r(t) - \frac{\Delta_2(t)}{\Delta(t)}\Xi^{r3}(t)Q(t)^{-1}c^3(t)\right]$$

$$\frac{\partial \tilde{z}^r(t)}{\partial l(t)} = \Xi^{r3}(t)\frac{\partial \tilde{p}^3(\cdot)}{\partial l(t)} + c^r(t)\frac{\partial \tilde{C}(\cdot)}{\partial l(t)} - \frac{\partial \tilde{y}^r(\cdot)}{\partial l(t)}; \tag{101}$$

$$\frac{\partial \tilde{z}^r(t)}{\partial p^k(t)} = \Xi^k(t) + \Xi^{r3}(t)\frac{\partial \tilde{p}^3(\cdot)}{\partial p^k(t)} + c^r(t)\frac{\partial \tilde{C}(\cdot)}{\partial p^k(t)}. \tag{102}$$

Thus under the assumptions leading to the above conclusions on the derivatives of the solution functions $\tilde{p}^3(\cdot)$ and $\tilde{C}(\cdot)$, we have the following results on the current net demand for imports: an increase in current-period bondholdings causes the net-import demand to increase; an increase in current-period taxes causes the net-import demand to decrease; an increase in the current interest factor causes the net-import demand to increase if the country is lending in the current period and to decrease if the country is borrowing in the current period. Without further specification about preferences and technologies, $\partial \tilde{z}^r/\partial p^k(t)$ and $\partial \tilde{z}^r/\partial l(t)$ are ambiguous in sign.

4 Integrability of the net-import demand functions

In this section, we investigate the integrability of the net-import demand function. For simplicity, we assume that there is no government. The main proposition to be proved in this section is that the net-import demand function may be considered as being generated by the maximization of an aggregate trade-utility function $\hat{U}(z^1, z^2)$ subject to the budget constraint

$$p^1(t)\cdot z^1(t) + p^2(t)\cdot z^2(t) = Z(t), \tag{103}$$

where $Z(t) = R(t)b(t) - b(t+1)$ is the deficit in the balance of payments on current account and $b(t+1)$ is the solution given by (56). It is important to stress that only the imports and exports of tradables enter as arguments of this trade-utility function. There are three possible approaches to establishing this proposition as Chipman [7] pointed out. One is a direct set-theoretic approach adopted in Chipman [5], in which a trade-utility function is defined by maximization of the utility function over an appropriate shifted production-possibility set. Another (cf. Chipman [7]) is to compute the Slutsky matrix of the net-import demand function (68) and verify that it is symmetric and negative semi-definite, and then appeal to the results of Hurwicz and Uzawa [18]. A third method, which is the one we adopt here, is to start with an indirect utility function $V(p_c(t), C(t))$ associated with $h^k(\cdot)$ and then define an indirect trade-utility function $\tilde{V}(p^1(t), p^2(t), Z(t))$ by substituting the reduced-form equation in (68) and thence in the arguments of $V(\cdot)$, and showing that the net-import demand function (68) satisfies the Antonelli–Allen–Roy partial differential equations

$$\frac{\partial \tilde{V}(p^1(t), p^2(t), Z(t))}{\partial p_j^k(t)} = -\frac{\partial \tilde{V}(p^1(t), p^2(t), Z(t))}{\partial Z(t)} \tilde{z}_j^k(p^1(t), p^2(t), Z(t)). \tag{104}$$

To prove the proposition, let $(p_c^*(t), Z^*(t))$ be the solution of the minimization problem:

$$\min p_c^*(t)\cdot h(p^1(t), p^2(t), \tilde{p}^3(\cdot), \Pi(\cdot, t) + Z(t)) \tag{105}$$

subject to

$$V(p^1(t), p^2(t), \tilde{p}^3(\cdot), \Pi(\cdot, t) + Z(t)) \geqq V(p^{1*}(t), p^{2*}(t), \tilde{p}^{3*}(\cdot), \Pi^*(\cdot, t) + Z^*(t)), \tag{106}$$

where $\Pi^*(\cdot, t) = \Pi(p^*(t), l(t), e(t), t)$.

Define

$$\begin{aligned} L(p^1(t), p^2(t), \tilde{p}^3(\cdot), Z(t), \lambda) = p_c^*(t)\cdot h(\cdot) - \lambda[&V(p^1(t), p^2(t), \tilde{p}^3(\cdot), \Pi(\cdot, t) + Z(t)) \\ &- V(p^{1*}(t), p^{2*}(t), \tilde{p}^{3*}(\cdot), \Pi^*(\cdot, t) + Z^*(t))], \end{aligned} \tag{107}$$

where $h(\cdot) = (h^1(\cdot), h^2(\cdot), h^3(\cdot))$. The first-order conditions at $(p_c(t), Z(t)) = (p_c^*(t), Z^*(t))$ are

$$\frac{\partial L(\cdot)}{\partial p^k(t)} = p_c^* \cdot \frac{\partial h(\cdot)}{\partial p^k(t)} - \lambda \frac{\partial \tilde{V}(0)}{\partial p^k(t)} \tag{108}$$

$$= \hat{y}^k(\cdot) + \hat{y}^3(\cdot)\frac{\partial \tilde{p}^3(\cdot)}{\partial p^k} - h^k(\cdot) - h^3(\cdot)\frac{\partial \tilde{p}^3(\cdot)}{\partial p^k} - \lambda\frac{\partial \tilde{V}(\cdot)}{\partial p^k(t)}$$

$$= -\tilde{z}^k(p_c^*, Z^*(t)) - \lambda\frac{\partial \tilde{V}(\cdot)}{\partial p^k(t)} = 0$$

and

$$\frac{\partial L(\cdot)}{\partial Z(t)} = p_c^* \cdot \frac{\partial h(\cdot)}{\partial Z(t)} - \lambda\frac{\partial \tilde{V}(\cdot)}{\partial Z(t)}$$

$$= 1 - \lambda\frac{\partial \tilde{V}(\cdot)}{\partial Z(t)} = 0, \tag{109}$$

where

$$\frac{\partial \tilde{V}(\cdot)}{\partial p^k(t)} = \left[\frac{\partial V(\cdot)}{\partial p^k} + \frac{\partial V(\cdot)}{\partial C(t)}\hat{y}^k(\cdot) + \left(\frac{\partial V(\cdot)}{\partial p^3} + \frac{\partial V(\cdot)}{\partial C(t)}\hat{y}^3(\cdot)\right)\frac{\partial \tilde{p}^3(\cdot)}{\partial p^k}\right].$$

Since this is true for any $(p_c^*(t), Z^*(t))$, we have

$$\frac{\partial \tilde{V}(p^1(t), p^2(t), Z(t))}{\partial p_j^k(t)} = -\frac{\partial \tilde{V}(p^1(t), p^2(t), Z(t))}{\partial Z(t)}\tilde{z}_j^k(p^1(t), p^2(t), Z(t)).$$

Therefore the trade-demand function can be generated by an indirect trade-utility function.

Appendix

Proof of Lemma 2. Suppose $\{l^*(t)\}$ is a wealth-maximizing national endowment plan at the price-vector sequence $\{p^*(t)\}$. We define the function

$$g(\tilde{p}, l(0), e) = \Upsilon(\tilde{p}, l(0), e) - \sum_{t=0}^{T}\Phi(t)\left[\Pi(p(t), l^*(t), e(t), t) - \sum_{k=4}^{6} p^k(t)\cdot x^{k*}(t)\right]. \tag{110}$$

Clearly the wealth-maximizing intertemporal national endowment plan $\{l(t)\}$ at price $\{\tilde{p}(t)\}$ will always be at least as better off as the plan $\{l^*(t)\}$ which is the best plan at prices $\{\tilde{p}^*(t)\}$. However, the plan $\{l^*(t)\}$ will be a best plan at price sequence $\{\tilde{p}^*(t)\}$, so the function $g(\tilde{p}, l(0), e)$ reaches a minimum value of 0 at $\{\tilde{p}^*(t)\}$. The assumption on prices implies that this is an interior minimum. The first-order conditions for a minimum then imply that

$$0 = \frac{\partial g(\tilde{p}^*, l(0), e)}{\partial p_j^k(t)} = \frac{\partial \Upsilon(\tilde{p}^*, l(0), e)}{\partial p_j^k(t)} - \Phi(t)\hat{y}_j^k(p^*(t), l^*(t), e(t))$$

for $k = 1, 2, 3$;

$$0 = \frac{\partial g(\tilde{p}^*, l(0), e)}{\partial p_j^4(t)} = \frac{\partial \Upsilon(\tilde{p}^*, l(0), e)}{\partial p_j^4(t)} - \Phi(t)[\hat{y}_j^4(p^*(t), l^*(t), e(t), t) - \hat{x}_j^4(p^*, l(0), e, t)];$$

$$0 = \frac{\partial g(\tilde{p}^*, l(0), e)}{\partial p_j^5(t)} = \frac{\partial \Upsilon(\tilde{p}^*, l(0), e)}{\partial p_j^5(t)} + \Phi(t)\hat{x}_j^5(p^*, l(0), e, t).$$

Since this is true for all $\{p^*(t)\}$, we have

$$\frac{\partial \Upsilon(p, l(0), e)}{\partial p_j^k(t)} = \Phi(t)\hat{y}_j^k(p(t), \tilde{l}(\cdot, t), e(t), t) \quad (k = 1, 2, 3);$$

$$\frac{\partial \Upsilon(p, l(0), e)}{\partial p_j^4(t)} = \Phi(t)[\hat{y}_j^4(p(t), \tilde{l}(\cdot, t), e(t), t) - \hat{x}_j^4(p, l(0), e, t)];$$

$$\frac{\partial \Upsilon(p, l(0), e)}{\partial p_j^5(t)} = -\Phi(t)\hat{x}_j^5(p, l(0), e, t).$$

Q.E.D.

Proof of Lemma 3. We first prove that $\gamma(b, u, l)$ is compact-valued correspondence for each fixed (b, u, l). Solving the difference equation

$$C(t) + b(t+1) = R(t)b(t) + \Gamma(t)$$

forward and using the transversality condition $\Phi(t+1+T)b(t+1+T) = 0$, we obtain

$$\begin{aligned} b(t+1) &= \Phi(t+1)\sum_{s=0}^{T} \Phi(t+1+s)(C(t+1+s) - \Gamma(t+1+s)) \\ &\geqq -\Phi(t+1)\sum_{s=0}^{T} \Phi(t+1+s)\Gamma(t+1+s) \\ &\geqq -\Phi(t+1)\sum_{s=0}^{T}\sum_{k=1}^{4} p^k(t+1)\cdot y^k(t+1) \\ &\equiv -B > -\infty \end{aligned}$$

because $\Sigma_{t=0}^{T}\Sigma_{k=1}^{4} p^k(t)y^k(t)$ is finite by assumption.

Thus $b(t+1)$ is bounded from below. Since

$$W(t) - b(t+1) \geqq C(t) \geqq 0,$$

so $-B \leqq b(t+1) \leqq W(t)$, hence $b(t+1)$ is bounded. Also since $0 \leqq C(t) \leqq W(t) - b(t+1) \leqq W(t) + B$, therefore $C(t)$ is bounded. It is obvious that $\gamma(b, u, l)$ is closed. Thus $\gamma(b, u, l)$ is a compact-valued correspondence. Since we maximize a continuous function of (b', u', l') over the compact-valued correspondence $\gamma(b, u, l)$, therefore it has a solution which is continuous by Berge's Maximum Theorem. The solution is also bounded because $V(p_c, C)$ and $E\mathfrak{R}(b', u', l')$ are bounded. Q.E.D.

Proof of Lemma 4. Since L is a space of bounded continuous functions $\mathfrak{R}$ with norm $\|\mathfrak{R}\|$ defined by (53), to prove Lemma 4, we only need to show that $\mathfrak{R}$ satisfies Blackwell's [2] sufficient conditions for the Contraction Theorem. That is, we only need to show that the operator $\mathcal{T}$ satisfies the following two properties

(1) (monotonicity) $\mathfrak{R}$, $\mathfrak{R}' \in L$ and $\mathfrak{R}(b, u, l) \leqq \mathfrak{R}'(b, u, l)$ for all (b, u, l) implies $\mathcal{T}\mathfrak{R}(b, u, l) \leqq \mathcal{T}\mathfrak{R}'(b, u, l)$ for all (b, u, l);

(2) (discounting) for any $\mathfrak{R} \in L$, all real $a > 0$ and all (b, u, l), $\mathcal{T}(\mathfrak{R} + a)(b, u, l) \leqq \mathcal{T}\mathfrak{R}(b, u, l) + \beta a$ for $\beta \in [0, 1)$

so that $\mathcal{T}$ is a contraction with modulus β.

Indeed, if $\mathfrak{R}(b,u,l) \leqq \mathfrak{R}'(b,u,l)$ for all b,u,l then $\mathscr{T}\mathfrak{R}'(b,u,l)$ is the maximized value of a uniformly higher objective function than $\mathscr{T}\mathfrak{R}(b,u,l)$, so the monotonicity by Hypothesis (1) is obvious.

Since

$$\begin{aligned}\mathscr{T}(\mathfrak{R}+a)(b,u,l) &= \max_{(C,b')\in\gamma(b,u,l)} \{V(p_c,C)+\beta E(\mathfrak{R}(b',u',l')+a)\}\\ &= \max_{(C,b')\in\gamma(b,u,l)} \{V(p_c,C)+\beta E\mathfrak{R}(b',u',l')+a\beta\}\\ &= \mathscr{T}\mathfrak{R}(b,u,l)+a\beta,\end{aligned}$$

Hypothesis (2) holds. Then $\mathscr{T}$ is a contraction with modulus β and thus the conclusions of Lemma 4 follow from the contraction mapping theorem.

Proof of Lemma 5. For any function $\mathfrak{R}\in L$, the fact that $V(p_c,x)$ is a strictly increasing function implies that $\mathscr{T}\mathfrak{R}$ has this property. Since $\mathscr{T}\tilde{\mathfrak{R}}=\tilde{\mathfrak{R}}$, so does $\tilde{\mathfrak{R}}$.

To show that the unique solution $\tilde{\mathfrak{R}}$ of $\mathscr{T}\tilde{\mathfrak{R}}=\tilde{\mathfrak{R}}$ is strictly concave in b, we first show that $\mathscr{T}$ maps the concave function $\mathfrak{R}$ into a strictly concave function. Let $\mathfrak{R}$ be any concave element of L and $b^1 \neq b^2$. Let $\lambda\in(0,1)$ and $b_\lambda=\lambda b^1+(1-\lambda)b^2$. Also let $(C^1,b^{1'})\in\gamma(b^1,u,l)$ attain $\mathscr{T}\mathfrak{R}(b^1,u,l)$ and $(C^2,b^{2'})\in\gamma(b^2,u,l)$ attain $\mathscr{T}\mathfrak{R}(b^2,u,l)$. Then $(C_\lambda,b'_\lambda)=(\lambda C^1+(1-\lambda)C^2,\lambda b^{1'}+(1-\lambda)b^{2'})\in\gamma(b_\lambda,u,l)$. So

$$\begin{aligned}\mathscr{T}\mathfrak{R}(b_\lambda,u,l) &\geqq V(p_c,C_\lambda)+\beta E\mathfrak{R}(b'_\lambda,u',l')\\ &> \lambda V(p_c,C^1)+(1-\lambda)V(p_c,C^2)+\lambda\beta E\mathfrak{R}(b^{1'},u',l')+(1-\lambda)\beta E\mathfrak{R}(b^{2'},u',l')\\ &\geqq \lambda\mathscr{T}\mathfrak{R}(b^1,u)+(1-\lambda)\mathscr{T}\mathfrak{R}(b^2,u,l),\end{aligned}$$

where the first, weak, inequality restates (53), the second, strict, inequality reflects the assumed concavity $\mathfrak{R}$, the strict concavity of $V(p_c,C)$ in C, and the facts that $b^1\neq b^2$ and $\lambda\in(0,1)$, and the final inequality is the definitions of $(C^1,b^{1'})$ and $(C^2,b^{2'})$.

Since $\mathfrak{R}^0\equiv 0$ is clearly concave in b, by the contraction mapping theorem we have $\mathscr{T}^n\mathfrak{R}^0\to\tilde{\mathfrak{R}}$ as n tends to infinity. So we know $\tilde{\mathfrak{R}}$ is strictly concave in b from the above result. Q.E.D.

Proof of Lemma 6. Existence of $\tilde{C}(b,u,l)$ and $\tilde{b}(b,u,l)$ follows from the continuity and compactness properties involved in the proof of Lemma 3, and the fact that $\tilde{C}(b,u,l)$ and $\tilde{b}(b,u,l)$ are functions is a consequence of the strict concavity of $V(p_c,C)$ and $\mathfrak{R}(b,u,l)$. The continuity of $\tilde{C}(\cdot)$ and $\tilde{b}(\cdot)$ follows from the theorem of the maximum specialized to the case when the optimal policy correspondence are functions. Q.E.D.

The proof of Lemma 7 is similar to that of Stokey, Lucas, and Prescott [34]. Before proving the lemma, some preliminary results are needed.

A plan is a sequence of functions $\pi_t\colon R^{tm}\to R$ for $t=0,1,2,\ldots,T$, where $b(t+1)=\pi_t(u(0),u(1),\ldots,u(t))$. A feasible plausible plan π_t is one for which there exists some $C(t)$ such that

$$(C(t),\pi_t(u(0),\ldots,u(t)))\in\gamma(\pi_{t-1}(u(0),\ldots,u(t-1)),u(t))$$

for all possible realizations of the stochastic sequence $\{u(t)\}$ for all $t>1$, $(C(0),b(1))\in\gamma(b(0),u(0))$.

Of particular interest are the feasible stationary Markov plans. They specify date-t actions as a time-invariant function of the date-t state. For our model such a plan is denoted by $\sigma\colon R \times R^m \to R$.

Let $f_\pi\colon R \times R^m \to R$ be the expected return for feasible plan π; that is,

$$f_\pi(b,u,l) = E_\pi\left(\sum_{t=0}^{\infty} \beta^t V(p_c, W - b(t+1))\right),$$

where E_π is the expected return when plan π is adopted, $b = b(0), u = u(0)$.

We define the optimal return function $f(b,u,l)$ to be

$$f(b,u,l) = \sup_\pi f_\pi(b,u,l)$$

given $b = b(0)$ and $u = u(0)$. The supremum is over all plans including nonstationary plans.

It is plausible that the return function for a stationary plan σ satisfies

$$\mathfrak{R}(b,u,l) = V(p_c, W - \sigma(b,u,l)) + \beta E \mathfrak{R}(\sigma(b,u,l), \Psi(u,\varepsilon), l'). \tag{111}$$

We shall show that this intuition is correct for continuous stationary plans.

Let $\mathscr{T}_\sigma$ be defined by the above equation (111) for $\mathfrak{R} \in L$ and σ continuous.

Sublemma 1. $\mathscr{T}_\sigma\colon L \to L$ has exactly one continuous bounded solution $\tilde{\mathfrak{R}}_\sigma \in L$, and for all $\mathfrak{R}^0 \in L$ and $n = 0, 1, \ldots,$

$$\|\mathscr{T}_\sigma^n \mathfrak{R}^0 - \mathfrak{R}_\sigma\| \leqq \alpha^n \|\mathfrak{R}^0 - \mathfrak{R}_\sigma\|.$$

Proof. The proof of this lemma is the same as those of Lemmas 3–4.

Sublemma 2. The unique fixed point $\mathfrak{R}_\sigma$ of the operator $\mathscr{T}_\sigma\colon L \to L$ is f_σ, the return for stationary policy σ.

Proof. Let $\mathfrak{R}^0 \in L$ be the function $\mathfrak{R}^0(b,u,l) \equiv 0$. By definition, for any stationary plan,

$$f_\sigma = \lim_{n \to \infty} \mathscr{T}_\sigma^n \mathfrak{R}^0$$

and by Sublemma 1, this limit is $\mathfrak{R}_\sigma$. Q.E.D.

Now we prove Lemma 7.

Proof of Lemma 7. The function $V(p_c, C)$ is bounded by assumption. Letting this upper bound be B, for any plan π

$$f_\pi \leqq E_\pi \sum_{t=0}^{n-1} V(p_c(t), C(t)) + \sum_{s=n}^{\infty} \beta^s B,$$

where $E_\pi(\cdot)$ denotes the expectation if plan π is adopted. Taking the supremum over (feasible) plans of both sides of the above inequality, we have

$$f \leqq \mathscr{T}^n \mathfrak{R}^0 + \beta^n B/(1-\beta).$$

Using Lemma 4, as the function $\mathfrak{R}^0 \equiv 0 \in L$ for all (b,u,l), the limit of $\mathscr{T}^n \mathfrak{R}^0$ is $\mathfrak{R}$, the unique point of $\mathscr{T}$ in L. This, along with the fact $0 < \beta < 1$, yields $f \leqq \mathfrak{R}$. Since $\mathscr{T}_\sigma \mathfrak{R} = \mathfrak{R}$, by Sublemma 2 the return for the stationary plan $\sigma = \tilde{b}$ given by (56) is $\mathfrak{R}$. As $\mathfrak{R} \geqq f$, the plan $\tilde{b}$ has a return as great as any other plan independent of the initial condition state (b, u, l) and, consequently, the plan $\tilde{b}$ is optimal.

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