Negative Consumption Externalities and Destruction of Resources in Achieving Efficient Allocations

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Abstract

This paper investigates the problem of achieving Pareto efficient allocations in the presence of externalities. In contrast to the conventional wisdom, we show that, even if consumers’ preferences are monotonically increasing in their own consumption, one may have to destroy resources to achieve Pareto efficient allocations when there are negative consumption externalities and some types of production externalities. As such, there is no way to allocate resources efficiently without destroying resources, even under complete information and zero transaction costs. This result is somewhat surprising when wealth is a desired good, and individuals envy each other’s wealth levels, such that a person’s satisfaction decreases as another consumer’s wealth level increases. We provide characterization results on the destruction of resources for pure exchange economies with negative consumption externalities, and we obtain similar results for more general production economies with externalities.

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1 Introduction

Does an increase in social wealth necessarily improve social welfare? More specifically, should resources be completely exhausted to achieve a Pareto efficient allocation in an economy with externalities even though consumers’ preferences are strictly monotonically increasing? The standard textbooks such as Laffant (1988), Varian (1992), Salanie (2000) say yes. However, this paper will show that, when there are negative consumption externalities and some types

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of production externalities, the answer may be different, and one may have to destroy some resources to achieve Pareto efficient allocations. There is neither an institution nor rule, which can mitigate the destruction of resources to achieve efficient allocations even under the assumption of complete information and/or no transaction costs.

Pareto efficiency and externalities are two important concepts in economics. Pareto efficiency is a highly desirable property to strive for when allocating resources. The importance and wide use of Pareto efficiency lies in its ability to offer us a minimal and uncontroversial test, which any social optimal economic outcome should pass. Implicit in every Pareto efficient outcome is that all possible improvements to society have been exhausted. In addition, Pareto efficiency conveniently separates the issue of economic efficiency from the more controversial (and political) questions regarding the ideal distribution of wealth across individuals.

The most important result obtained in general equilibrium theory is the First Fundamental Theorem of Welfare Economics. It is a formal expression of Adam Smith’s claim of the existence of an “invisible hand” working in markets. The first welfare theorem provides a set of conditions under which a market economy will achieve a Pareto optimal outcome. Thus, any inefficiency which arises in a market economy, and hence any role for Pareto-improving market intervention, must be traceable to a violation of at least one assumption in the first welfare theorem.

One typical violation of these assumptions is the existence of externalities in an economy. This violation leads to non-Pareto optimal outcomes and a market failure. Various suggestions for alternative ways to allocate resources which lead to Pareto efficient allocations have been suggested in the economics literature. However, the conventional wisdom in the economic literature, regardless of the method, rule or mechanism, is that resources must be exhausted to reach a Pareto efficient allocation, provided that commodities are divisible and the preferences of individuals are strongly monotone. The destruction of resources in order to achieve Pareto efficient allocations is not necessary and, in fact, counterintuitive. However, the conventional wisdom may not hold when consumers have negative consumption externalities. The story may be quite different in the presence of negative consumption externalities.

This paper investigates the problem of achieving Pareto efficient allocations for both exchange and general production economies with consumption and production externalities. We provide characterization results first for pure exchange economies with negative consumption externalities, and then obtain similar results for more general production economies with externalities.

In contrast to the convenient wisdom, we show that for economies with negative consumption
externalities, even if consumers’ preferences are strictly increasing in their own consumption, one may have to destroy resources when the available level of resources is relatively high to achieve a Pareto efficient allocation. As such, it is not appropriate to fail to take the destruction of resources into account when deriving efficiency conditions. Consequently, our result implies that even if individuals’ preferences are strictly monotone in their own consumption, information is complete, transaction costs are zero, and all regularity conditions are satisfied, there is no way to allocate resources efficiently without destroying some units of the goods which have negative consumption externalities, provided their endowments exceed a threshold.

This result seems counter-intuitive if we regard wealth as a good, and if one person envies another consumer’s consumption level such that the person’s satisfaction decreases as the consumption level of other consumers increases. When the level of social wealth reaches a certain level, one may have to destroy or freely dispose of wealth, if one person envies another’s consumption. Intuitively, in the presence of negative consumptions externalities, goods with externalities have two effects: a positive effect, which benefits the agent who directly consumes the good, and a negative effect, which harms another agent through the channel of the externality. The tradeoff between these two effects determines whether there should be destruction to achieve efficient allocations. If the positive effect dominates the negative, then destruction is not necessary. Otherwise, destruction is necessary.

The remainder of this paper is as follows. Section 2 considers the destruction problem for the simple case of pure exchange economies. We first present a numerical example, which shows the basic results of this paper, then we provide characterization results for general utility functions. Section 3 considers more general production economies with various types of externalities, and we obtain similar results. We present concluding remarks in Section 4.

2 Consumption Externalities and Destruction of Resources

In this section we provide characterization results on the destruction of resources in pure exchange economies with general utility functions and negative externalities. First, we specify pure exchange economies in the presence of externalities, then we illustrate the destruction problem with a numerical example. The example shows that, even though utility functions are strictly monotonic with respect to the consumer’s own consumption and the level of resources is greater than a critical point, one must destroy resources to achieve Pareto optimal allocations.
2.1 Economic Environments with Consumption Externalities

Consider pure exchange economies with consumption externalities. Suppose that there are two goods and two consumers. Consumer $i$’s consumption of the two goods is denoted by a vector $x_i = (x_{i1}, x_{i2})$, $i = 1, 2$, where we adopt the convention that the first subscript denotes agents and the second one denotes goods. Assume that good 1’s consumption exhibits a negative externality, which means that the utility of consumer $i$ is adversely affected by consumer $j$’s consumption on $x_{j1}$, $j \neq i$. Examples of goods with negative consumption externalities are tobacco, loud music and alcohol. Another typical example of a negative consumption externality is when one consumer’s satisfaction decreases as another consumer’s consumption level increases, since the person envies the other’s lifestyle. Consumer $i$’s utility function is then denoted as $u_i(x_{i1}, x_{i2})$ for $i = 1, 2$. Initially, there are $w_1$ units of good 1 available and $w_2$ units of good 2.

An allocation $x \equiv (x_{11}, x_{12}, x_{21}, x_{22})$ is feasible if $x \in \mathbb{R}_{++}^4$, $x_{11} + x_{21} \leq w_1$, and $x_{12} + x_{22} \leq w_2$. An allocation is said to be balanced if $x_{11} + x_{21} = w_1$, and $x_{12} + x_{22} = w_2$. An allocation $x$ is Pareto optimal (efficient) if it is feasible, and another feasible allocation, $x'$, does not exist such that $u_i(x'_{11}, x'_{21}, x'_{22}) \geq u_i(x_{11}, x_{21}, x_{i2})$ for all $i = 1, 2$ and $u_i(x'_{11}, x'_{21}, x'_{22}) > u_i(x_{11}, x_{21}, x_{i2})$ for some $i$. Denote the set of all such allocations by $PO$.

The set of Pareto optimal allocations can be completely characterized by the following lemma.

**Lemma 1** An allocation $x^* = (x^*_{11}, x^*_{12}, x^*_{21}, x^*_{22})$ is Pareto efficient if and only if it solves the following two constrained optimization problems simultaneously, for any $i \in \{1, 2\}, j \in \{1, 2\}, i \neq j$.

\[
\begin{align*}
\max_{x \in \mathbb{R}_{++}^4} & \quad u_i(x_{11}, x_{21}, x_{i2}) \\
\text{s.t.} & \quad x_{11} + x_{21} \leq w_1 \\
& \quad x_{12} + x_{22} \leq w_2 \\
& \quad u_j(x_{11}, x_{21}, x_{j2}) \geq u_j(x^*_{11}, x^*_{21}, x^*_{j2})
\end{align*}
\]

**Proof.** The proof can be found in Varian(1992, P330).

Thus, the process of finding Pareto optimal allocations can be reduced to solving the above constrained optimization problem.

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1 Here, we implicitly assume the consumption sets of both consumers are open sets, in order to apply the Kuhn-Tucker theorem easily.
2.2 A Numerical Example

To illustrate the dilemma of the necessity of destroying resources to achieve Pareto optimal allocations in the presence of negative consumption externalities, let us consider the following numerical example. Suppose consumers’ preferences are given by the following specific utility function:

\[ u_i(x_{11}, x_{21}, x_{12}) = \sqrt{x_{11}x_{12}} - x_{j1}, \quad i \in \{1, 2\}, j \in \{1, 2\}, j \neq i \]

By Lemma 1, the Pareto efficient allocations are completely characterized by the following two problems:

\[
\begin{align*}
(P1) & \quad \max_{x \in R_4^+} \sqrt{x_{11}x_{12}} - x_{21} \\
& \quad \text{s.t. } x_{11} + x_{21} \leq w_1 \quad x_{12} + x_{22} \leq w_2 \quad \sqrt{x_{21}x_{22}} - x_{11} \geq \sqrt{x_{11}^*x_{12}^*} - x_{11}^*
\end{align*}
\]

and

\[
\begin{align*}
(P2) & \quad \max_{x \in R_4^+} \sqrt{x_{21}x_{22}} - x_{11} \\
& \quad \text{s.t. } x_{11} + x_{21} \leq w_1 \quad x_{12} + x_{22} \leq w_2 \quad \sqrt{x_{11}x_{12}} - x_{21} \geq \sqrt{x_{11}^*x_{12}^*} - x_{21}^*
\end{align*}
\]

Since the objective function and constraints are continuously differentiable and concave on \( R_4^+ \) and the Slater’s condition is satisfied, the Pareto efficient points are completely characterized by the FOCs of (P1) and (P2). In particular, if the Kuhn-Tucker multiplier of \( \sqrt{x_{11}x_{12}} - x_{21} \geq \sqrt{x_{11}^*x_{12}^*} - x_{21}^* \) in (P2) is positive, then we can only consider the FOCs of (P2), which is the case in this problem and will be shown soon.

Define the Lagrangian function as:

\[
L = \sqrt{x_{21}x_{22}} - x_{11} + \lambda_1 (w_1 - x_{11} - x_{21}) + \lambda_2 (w_2 - x_{12} - x_{22}) + \mu \left( \sqrt{x_{11}x_{12}} - x_{21} - \sqrt{x_{11}^*x_{12}^*} + x_{21}^* \right).
\]

\(^2\)Of course, in this example, the utility function itself does not require an open consumption set. However, if we use the closed first quadrant as consumption set, the result would not change significantly, which will be seen in Remark 1 below.

\(^3\)Slater’s condition states that there is an point \( \hat{x} \in R_4^+ \) such that all constraints hold with strict inequality.
We can obtain the following FOCs:

\[ x_{11} : -1 - \lambda_1 + \frac{\mu}{2} \sqrt{\frac{x_{12}}{x_{11}}} = 0, \]  
(1)

\[ x_{12} : -\lambda_2 + \frac{\mu}{2} \sqrt{\frac{x_{11}}{x_{12}}} = 0, \]  
(2)

\[ x_{21} : \frac{1}{2} \sqrt{\frac{x_{22}}{x_{21}}} - \lambda_1 - \mu = 0, \]  
(3)

\[ x_{22} : \frac{1}{2} \sqrt{\frac{x_{21}}{x_{22}}} - \lambda_2 = 0, \]  
(4)

\[ \lambda_1 : w_1 - x_{11} - x_{21} \geq 0, \lambda_1 \geq 0, \lambda_1 (w_1 - x_{11} - x_{21}) = 0, \]  
(5)

\[ \lambda_2 : w_2 - x_{12} - x_{22} \geq 0, \lambda_2 \geq 0, \lambda_2 (w_2 - x_{12} - x_{22}) = 0, \]  
(6)

\[ \mu : \sqrt{x_{11}x_{12} - x_{21}} = \sqrt{x_{11}x_{12} - x_{21}} + x_{21} \geq 0, \mu \geq 0, \mu \left( \sqrt{x_{11}x_{12} - x_{21}} - \sqrt{x_{21}^2 + x_{21}^2} \right) = 0 \]  
(7)

By solving (2) and (4) for \( \mu \), we have \( \mu = \sqrt{x_{12}x_{21}} > 0 \). Thus, any allocation \( x^* \in R_+^4 \) satisfying (1)-(7), will also satisfy the FOCs of problem (P1) with different multipliers. Therefore, the Pareto optimal set of allocations is fully characterized by (1)-(6), since (7) is trivial when evaluated at the solutions.

By (4), we have \( \lambda_2 > 0 \) and thus, in (6), we obtain the following restriction:

\[ x_{12} + x_{22} = w_2. \]  
(8)

Using (1)-(4), we can cancel out \( x \) and \( \lambda_2 \), which leads to,

\[ (1 - \mu) \left[ \lambda_1 (1 + \mu) + 1 + \mu + \mu^2 \right] = 0 \]

which implies that \( \mu = 1 \).

Using the fact that \( \mu = 1 \) and \( \lambda_2 > 0 \), from (1) and (2) we have

\[ \lambda_1 = \frac{1}{4\lambda_2} - 1. \]  
(9)

By (2), (4), \( \mu = 1 \), and \( \lambda_2 > 0 \), we have

\[ x_{12} = \frac{1}{4\lambda_2^2} x_{11} \]  
(10)

\[ x_{22} = \frac{1}{4\lambda_2^2} x_{21} \]  
(11)

Summing up (10) and (11) and using (8), we have

\[ \lambda_2 = \frac{1}{2} \sqrt{\frac{x_{11} + x_{21}}{w_2}}, \]  
(12)

which implies that,
By substituting (12) into (9), we have

\[ \lambda_1 = \sqrt{w_2} - 1 \]

which will be used to determine the critical level of the endowment of \( w_1 \) beyond which there will be destruction. Since \( \lambda_1 \geq 0 \) at equilibrium, there are two cases to consider.

**Case 1.** \( \lambda_1 > 0 \). In this case, we must have \( x_{11} + x_{21} < w_2/4 \) by (15), and thus by (5), we have

\[ x_{11} + x_{21} = w_1. \]

Therefore if \( w_1 < w_2/4 \), there is no destruction, and by (13) and (14)

\[ x_{12} = \frac{x_{11}w_2}{w_1}, \]

\[ x_{22} = \frac{x_{21}w_2}{w_1}. \]

**Case 2.** \( \lambda_1 = 0 \). Then, by (15), we must have \( x_{11} + x_{21} = w_2/4 \) which is true for any \( w_1 \geq w_2/4 \).

Thus, when \( w_1 > w_2/4 \), there is always destruction of good 1, which is equal to \( w_1 - w_2/4 \). When \( w_1 = w_2/4 \), there is no destruction even though \( \lambda_1 = 0 \). Thus, the critical level for the endowment of \( w_1 \) is equal to \( w_2/4 \).

Finally, by (13) and (14) and by \( x_{11} + x_{21} = w_2/4 \), we have

\[ x_{12} = 4x_{11} \]

\[ x_{22} = 4x_{21}, \]

and \( \lambda_2 = 1/4 \).

Summarizing the above discussion, we have the following proposition.

**Proposition 1** For a pure exchange economy with the above specific utility functions, it is necessary to destroy some of endowment \( w_1 \) if and only if \( w_1 > w_2/4 \). Specially,

1. When the endowment \( w_1 > w_2/4 \), there is destruction in the endowment of \( w_1 \) in achieving Pareto efficient allocations. Furthermore, the amount of destruction is equal to \( w_1 - w_2/4 \), and the set of Pareto optimal allocations is characterized by

\[ PO = \{ x \in \mathbb{R}_+^4 : x_{12} = 4x_{11}, x_{22} = 4x_{21}, x_{12} + x_{22} = w_2, w_2/4 = x_{11} + x_{21} < w_1 \} \]
(2) When \( w_1 \leq \frac{w_2}{4} \), destruction is unnecessary to achieve any Pareto efficient allocation. Furthermore, the set of Pareto optimal allocations is characterized by

\[
PO = \left\{ x \in R_+^4 : \frac{2x_{11}}{w_1}, x_{22} = \frac{2x_{21}}{w_1}, x_{12} + x_{22} = w_2, x_{11} + x_{21} = w_1 \right\}.
\]

**Remark 1** If we allow boundary points, i.e., \( x \in R_+^4 \), then the two statements in Proposition 1 would change to

(1) When the endowment level of \( w_1 > \frac{w_2}{4} \), there is destruction in the endowment of \( w_1 \) in achieving Pareto efficient allocations, except for the case when a single agent consumes all amounts of both goods, and the other consumes nothing. Furthermore, the set of Pareto optimal allocations is characterized by

\[
PO = \left\{ x \in R_+^4 : x_{12} = 4x_{11}, x_{22} = 4x_{21}, x_{12} + x_{22} = w_2, w_2/4 = x_{11} + x_{21} < w_1 \right\}
\]

\[
\cup \left\{ x_{11} = x_{12} = 0, x_{22} = w_2, w_2/4 \leq x_{21} \leq w_1 \right\}
\]

\[
\cup \left\{ x_{21} = x_{22} = 0, x_{12} = w_2, w_2/4 \leq x_{11} \leq w_1 \right\}
\]

(2) When \( w_1 \leq \frac{w_2}{4} \), destruction is unnecessary to achieve any Pareto efficient allocation. The set of Pareto optimal allocations is characterized by

\[
PO = \left\{ x \in R_+^4 : \frac{x_{11}w_2}{w_1}, x_{22} = \frac{x_{21}w_2}{w_1}, x_{12} + x_{22} = w_2, x_{11} + x_{21} = w_1 \right\}
\]

The proof of the case which involves boundary Pareto efficient allocations is much more complicated. We leave the interested reader to the appendix for a proof.

**Remark 2** From the above example, we observe that for an economy with negative consumption externalities, in Pareto efficient allocations, whether there is destruction for the goods with negative externalities depends on the characteristics (that is, preference and endowment) of the economy. Roughly speaking, given preferences and \( w_2 \), when the total endowment of \( w_1 \) reaches a certain level, attaining the Pareto efficient set requires destruction. In our numerical example discussed above, the two effects of consumption on good 1 are obvious. The positive effect is realized by diminishing marginal utility, and the negative effect takes on a linear form. Thus, as the endowment \( w_1 \) becomes larger, the positive effect diminishes and the negative effect remains constant. The tradeoff between these two effects will always favor the negative effect as the total endowment of \( w_1 \) increases. This also accounts for the existence of the critical level for the total endowment \( w_1 \), which is equal to \( w_2/4 \) in this example.
2.3 Characterizations on Destruction of Resources

Now we consider pure exchange economies with general utility functions and provide characterization results on the destruction of resources in achieving Pareto efficient allocations.

We assume that utility functions $u_i(x_{11}, x_{21}, x_{i2})$ are continuously differentiable, quasi-concave and differentiably increasing, i.e., $x_{il} > 0$ for $l = 1, 2$. We further assume that the Slater condition is satisfied, and the gradient of $u_i(\cdot)$ is nonzero at Pareto efficient allocations. Thus Pareto efficient allocations $x^*$ can be completely determined by the FOCs of the following problem.

$$\begin{align*}
\max_{x \in \mathbb{R}^4_+} & \quad u_2(x_{11}, x_{21}, x_{22}) \\
\text{s.t.} & \quad x_{11} + x_{21} \leq w_1 \\
& \quad x_{12} + x_{22} \leq w_2 \\
& \quad u_1(x_{11}, x_{21}, x_{12}) \geq u_1(x_{11}^*, x_{21}^*, x_{12}^*)
\end{align*}$$

The first order conditions are

$$
\begin{align*}
x_{11} : & \quad \frac{\partial u_2}{\partial x_{11}} - \lambda_1 + \mu \frac{\partial u_1}{\partial x_{11}} = 0 \quad (21) \\
x_{12} : & \quad -\lambda_2 + \mu \frac{\partial u_1}{\partial x_{12}} = 0 \quad (22) \\
x_{21} : & \quad \frac{\partial u_2}{\partial x_{21}} - \lambda_1 + \mu \frac{\partial u_1}{\partial x_{21}} = 0 \quad (23) \\
x_{22} : & \quad \frac{\partial u_2}{\partial x_{22}} - \lambda_2 = 0 \quad (24) \\
\lambda_1 : & \quad w_1 - x_{11} - x_{21} \geq 0, \lambda_1 \geq 0, \lambda_1 (w_1 - x_{11} - x_{21}) = 0 \quad (25) \\
\lambda_2 : & \quad w_2 - x_{12} - x_{22} \geq 0, \lambda_2 \geq 0, \lambda_2 (w_2 - x_{12} - x_{22}) = 0 \quad (26) \\
\mu : & \quad u_1 - u_1^* \geq 0, \mu \geq 0, \mu (u_1 - u_1^*) = 0 \quad (27)
\end{align*}
$$

By (24), $\lambda_2 = \frac{\partial u_2}{\partial x_{22}} > 0$, and thus by (26),

$$x_{12} + x_{22} = w_2 \quad (28)$$

which means there is never destruction of the good which does not exhibit a negative externality. Also, by (22) and (24), we have

$$\mu = \frac{\partial u_2}{\partial x_{22}} \frac{\partial u_1}{\partial x_{12}} \quad (29)$$
Then, by (21) and (22), we have
\[
\frac{\lambda_1}{\lambda_2} = \left[ \frac{\partial u_1}{\partial x_{11}} + \frac{\partial u_2}{\partial x_{21}} \right]
\] (30)
and by (23) and (24), we have
\[
\frac{\lambda_1}{\lambda_2} = \left[ \frac{\partial u_2}{\partial x_{21}} + \frac{\partial u_1}{\partial x_{11}} \right]
\] (31)

Thus, by (30) and (31), we obtain the conventional marginal equality condition given in standard textbooks such as Varian (1992, p. 438):
\[
\frac{\partial u_1}{\partial x_{11}} + \frac{\partial u_2}{\partial x_{11}} = \frac{\partial u_3}{\partial x_{21}} + \frac{\partial u_4}{\partial x_{21}}
\] (32)
which expresses the equality of the social marginal rates of substitution for the two consumers at Pareto efficient points. From the above marginal equality condition, we know that, in order to evaluate the relevant marginal rates of substitution for the optimality conditions, we must take into account both the direct and indirect effects of consumption activities in the presence of externalities. That is, to achieve Pareto optimality, when one consumer increases the consumption of good 1, not only does the consumer’s consumption of good 2 need to change, the other consumer’s consumption of good 2 must also be changed. Therefore the social marginal rate of substitution of good 1 for good 2 by consumer \(i\) equals \(\frac{\partial u_i}{\partial x_{11}} + \frac{\partial u_i}{\partial x_{11}}\).

Solving (21) and (23) for \(\mu\) and \(\lambda_1\), we have
\[
\mu = \frac{\partial u_2}{\partial x_{21}} - \frac{\partial u_3}{\partial x_{11}} > 0 \quad (33)
\]
and
\[
\lambda_1 = \frac{\partial u_1, \partial u_2}{\partial x_{11}} - \frac{\partial u_1, \partial u_2}{\partial x_{21}}. \quad (34)
\]
When the consumption externality is positive, from (30) or (31), we can easily see that \(\lambda_1\) is always positive since \(\lambda_2 = \frac{\partial u_2}{\partial x_{22}} > 0\). Also, when no externality or a one-sided externality exists, by either (30) or (31), \(\lambda_1\) is positive. Thus, the marginal equality condition (32) and the balanced conditions, completely determine all Pareto efficient allocations for these cases. However, when there are negative externalities for both consumers, the Kuhn-Tucker multiplier \(\lambda_1\) directly given by (34) or indirectly given by (30) or (31) is the sum of a negative and positive term, and thus the sign of \(\lambda_1\) may be indeterminate. However, some textbooks such as Varian

\[^4\text{Only one consumer imposes an externality on another consumer.}\]
(1992, 438), claim that the marginal equality condition, (32), and the balanced conditions may not guarantee finding Pareto efficient allocations correctly.

To guarantee an allocation is Pareto efficient in the presence of negative externalities, we must require \( \lambda_1 \geq 0 \) at efficient points, which in turn requires that social marginal rates of substitution be nonnegative, that is,

\[
\frac{\partial u_1}{\partial x_{11}} + \frac{\partial u_2}{\partial x_{21}} = \frac{\partial u_2}{\partial x_{22}} + \frac{\partial u_1}{\partial x_{12}} \geq 0,
\]

or equivalently requires both (32) and

\[
\frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}} \geq \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}}
\]

for all Pareto efficient points.

We can interpret the term in the left-hand side of (36), \( \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}} \), as the total marginal benefit of consuming good 1, and the term in the right-hand side, \( \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}} \), as the total marginal cost of consuming good 1 because the negative externality hurts the consumers. To consume the goods efficiently, a necessary condition is that the total marginal benefit of consuming good 1 should not be less than the total marginal cost of consuming good 1.

Thus, the following conditions

\[
\begin{align*}
\frac{\partial u_1}{\partial x_{11}} + \frac{\partial u_2}{\partial x_{21}} &= \frac{\partial u_2}{\partial x_{22}} + \frac{\partial u_1}{\partial x_{12}} \geq 0, \\
x_{12} + x_{22} &= w_2 \\
x_{11} + x_{21} &\leq w_1 \\
\left(\frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}} - \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}}\right) (w_1 - x_{11} - x_{21}) &= 0
\end{align*}
\]

constitute a system (PO) from which all Pareto efficient allocations can be obtained.

Note that, when \( \frac{\partial u_1}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}} > \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}} \), or equivalently \( \frac{\partial u_1}{\partial x_{11}} + \frac{\partial u_2}{\partial x_{21}} = \frac{\partial u_2}{\partial x_{22}} + \frac{\partial u_1}{\partial x_{12}} > 0 \), \( \lambda_1 > 0 \) and thus the last two conditions in the above system (PO) reduce to \( x_{11} + x_{21} = w_1 \). In this case, there is no destruction. Substituting \( x_{11} + x_{21} = w_1 \) and \( x_{12} + x_{22} = w_2 \) into the marginal equality condition (32), it would give us a relationship between \( x_{11} \) and \( x_{12} \), which exactly defines the Pareto efficient allocations.

When the total marginal benefit equals the total marginal cost:

\[
\frac{\partial u_1}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}} = \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}},
\]
then
\[
\frac{\partial u_1}{\partial x_{11}} + \frac{\partial u_2}{\partial x_{12}} = \frac{\partial u_2}{\partial x_{21}} + \frac{\partial u_1}{\partial x_{12}} = 0
\]  
(38)

and thus \( \lambda_1 = 0 \). In this case, when \( x_{11} + x_{21} \leq w_1 \), the necessity of destruction is indeterminant. However, even when destruction is necessary, we can still determine the set of Pareto efficient allocations by using \( x_{12} + x_{22} = w_2 \) and the zero social marginal equality conditions (38). Indeed, after substituting \( x_{12} + x_{22} = w_2 \) into (38), we can solve for \( x_{11} \) in terms of \( x_{12} \).

When \( \frac{\partial u_1}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}} < \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}} \) for any allocations that satisfy \( x_{11} + x_{21} = w_1 \), \( x_{12} + x_{22} = w_2 \), and the marginal equality condition (32), the social marginal rates of substitution must be negative. Hence, the allocation will not be Pareto efficient. In this case, there must be a destruction for good 1 for Pareto efficiency, and a Pareto efficient allocation satisfies (38).

Summarizing, we have the following proposition that provides two categories of sufficiency conditions for characterizing whether or not there should be destruction of endowment \( w_x \) in achieving Pareto efficient allocations.

**Proposition 2** For 2×2 pure exchange economies, suppose that utility functions \( u_i(x_{11}, x_{21}, x_{12}) \) are continuously differentiable, quasi-concave, and \( \frac{\partial u_l(x_{11}, x_{21}, x_{12})}{\partial x_{il}} > 0 \) for \( l = 1, 2 \).

1. If the social marginal rates of substitution are positive at a Pareto efficient allocation \( x^* \), then there is no destruction of \( w_1 \) in achieving Pareto efficient allocation \( x^* \).

2. If the social marginal rates of substitution are negative for any allocation \( (x_1, x_2) \) satisfying \( x_{11} + x_{21} = w_1 \), \( x_{12} + x_{22} = w_2 \), and the marginal equality condition (32), then there is destruction of \( w_1 \) in achieving any Pareto efficient allocation \( x^* \). That is, \( x_{11}^* + x_{21}^* < w_1 \) and \( x^* \) is determined by \( x_{12}^* + x_{22}^* = w_2 \) and (38).

Thus, from the above proposition, we know that a sufficient condition for destruction is that for any allocation \( (x_{11}, x_{12}, x_{21}, x_{22}) \),

\[
\begin{align*}
\frac{\partial u_1}{\partial x_{11}} + \frac{\partial u_2}{\partial x_{12}} &= \frac{\partial u_2}{\partial x_{21}} + \frac{\partial u_1}{\partial x_{12}} \\
x_{11} + x_{21} &= w_1 \\
x_{12} + x_{22} &= w_2
\end{align*}
\] 

\[ \Rightarrow \frac{\partial u_1}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}} < \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}}. \]  
(39)

\(^5\)As we discussed above, this is true if the consumption externality is positive, or there is no externality or only one side externality.
A sufficient condition for all Pareto efficient allocations \((x_{11}, x_{12}, x_{21}, x_{22})\) for no destruction is,

\[
\begin{align*}
\frac{\partial u_1}{\partial x_{11}} & + \frac{\partial u_1}{\partial x_{12}} + \frac{\partial u_2}{\partial x_{21}} & + \frac{\partial u_2}{\partial x_{22}} \\
x_{11} + x_{21} & \leq w_1 \\
x_{12} + x_{22} & = w_2
\end{align*}
\Rightarrow \frac{\partial u_1}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}} > \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}}
\] (40)

**Remark 3** Let us revisit our numerical example in section 2.2 using the above sufficient conditions.

By the marginal equality condition (32), we have

\[
\left(\frac{x_{12}}{x_{11}} + 1\right)^2 = \left(\frac{x_{22}}{x_{21}} + 1\right)^2
\] (41)

and thus

\[
\frac{x_{12}}{x_{11}} = \frac{x_{22}}{x_{21}}
\] (42)

Let \(x_{11} + x_{21} \equiv x_1\). Substituting \(x_{11} + x_{21} = x_1\) and \(x_{12} + x_{22} = w_2\) into (42), we have

\[
\frac{x_{12}}{x_{11}} = \frac{w_2}{x_1}
\] (43)

Then, by (42) and (43), we have

\[
\frac{\partial u_1}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}} = \frac{1}{4} \sqrt{\frac{x_{12}}{x_{11}}} \sqrt{\frac{x_{22}}{x_{21}}} = \frac{x_{12}}{4x_{11}} = \frac{w_2}{4x_1}
\] (44)

and

\[
\frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}} = 1.
\] (45)

Thus, \(x_1 = w_2/4\) is the critical point that makes \(\frac{\partial u_1}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}} - \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}} = 0\), or equivalently \(\frac{\partial u_1}{\partial x_{11}} + \frac{\partial u_2}{\partial x_{12}} + \frac{\partial u_1}{\partial x_{21}} + \frac{\partial u_2}{\partial x_{22}} = 0\). Hence, if \(w_1 > \frac{w_2}{4}\), then \(\frac{\partial u_1}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}} - \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}} < 0\), and thus, by (39), there is destruction in any Pareto efficient allocation. If \(w_1 < \frac{w_2}{4}\), then \(\frac{\partial u_1}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}} - \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}} > 0\), and, by (40), no Pareto optimal allocation requires destruction. Finally, when \(w_1 = \frac{w_2}{4}\), any allocation that satisfies the marginal equality condition (32) and the balanced conditions \(x_{11} + x_{21} = w_1\) and \(x_{12} + x_{22} = w_2\) also satisfies (36), and thus it is a Pareto efficient allocation with no destruction.

Now, let us set out the sufficient conditions for destruction and non-destruction in detail. Note that if the sufficient condition for non-destruction holds, then \(x_{11} + x_{21} = w_1\) would also hold as an implication. Thus, to apply both the sufficient conditions for non-destruction and destruction, one should use the following three conditions
If an allocation also satisfies the condition \( \frac{\partial u_1}{\partial x_{21}} \frac{\partial u_2}{\partial x_{11}} \geq \frac{\partial u_1}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}} \), then the allocation is Pareto efficient, and further there is no destruction. Otherwise, it is not Pareto efficient. If this is true for all such non-Pareto allocations, all Pareto efficient condition must hold with destruction.

Note that, since \( \frac{\partial u_1}{\partial x_{11}} \) and \( \frac{\partial u_2}{\partial x_{21}} \) represent marginal benefit, they are usually diminishing in consumption in good 1. Since \( \frac{\partial u_1}{\partial x_{21}} \) and \( \frac{\partial u_2}{\partial x_{11}} \) are in the form of a marginal cost, their absolute values would be typically increasing in the consumption of good 1. Hence, when total endowment \( w_1 \) is small, the social marginal benefit would exceed the social marginal cost so that there is no destruction. As the total endowment of \( w_1 \) increases, the social marginal cost will ultimately outweigh the marginal social benefit, which results in the destruction of the endowment of \( w_1 \).

Alternatively, we can get the same result by using social marginal rates of substitution. When utility functions are quasi-concave, marginal rates of substitution are diminishing. Therefore, in the presence of negative consumption externalities, social marginal rates of substitution may become negative when the consumption of good 1 becomes sufficiently large. When this occurs, it is better to destroy some resources for good 1. As the destruction of good 1 increases, which will, in turn, decrease the consumption of good 1, social marginal rates of substitution will increase. Eventually they will become nonnegative.

3 Destruction Involving Production Externalities

In this section we consider the destruction of resources to achieve Pareto efficient allocations for production economies with both consumption and production externalities. We will first consider production economies with two goods, two consumers and two firms, and then consider production economies with three goods (one input good and two consumption goods), two consumers, and two firms.

3.1 Destruction for \( 2 \times 2 \times 2 \) Production Economies

There are two goods, two consumers, and two firms. Each firm produces only one good by using another good. We assume firm \( j \) produces good \( j \). There are various externalities in this economy. As in the previous section, the good 1’s consumption of one consumer would affect the utility level of the other consumer; the production of good \( j \) would have externality on
the production of good \( l \); the consumption of good 1 would affect the amount of good 2 in production; and the production of good 2 would influence the happiness of both consumers. A classic example is one in which both production processes produce pollution, which decreases the air quality.

We will continue to use \( x \) to denote the consumers’ consumption of goods. We will use \( y \) to denote outputs of firms, \( v \) to denote the input used by firms, and \( w = (w_1, w_2) \) will represent the endowment vector. When double subscripts are used, the first one is used to index individuals (consumers or firms) and the second is used to index goods. For example, \( x_{ij} \) means the amount of good \( j \) consumed by consumer \( i \) and \( v_{jl} \) means the amount of good \( l \) used by firm \( j \) when producing good \( j \).

Let the utility functions \( u_i (x_{11}, x_{21}, x_{12}, y_1, y_2) \) be defined on \( R^5_{++} \). We assume the utility functions are continuously differentiable, quasi-concave, and differentiably increasing in their own consumption, i.e., \( \frac{\partial u_i (x_{11}, x_{21}, x_{12}, y_1, y_2)}{\partial x_{ii}} > 0 \) for \( i = 1, 2 \). Since we mainly study the destruction issue in the presence of negative consumption externalities, we also assume \( \frac{\partial u_i (x_{11}, x_{21}, x_{12}, y_1, y_2)}{\partial x_{j1}} \leq 0 \) for \( j \neq i \). Production functions \( y_1 = y_1 (v_{12}, v_{21}) \) and \( y_2 = y_2 (v_{12}, v_{21}, x_1) \) are defined on \( R^2_+ \) and \( R^3_{++} \), which are continuously differentiable, and concave, where \( x_1 = x_{11} + x_{21} \). Thus, we only require a negative consumption externality, while the production externality and cross externality between production and consumption may be negative or positive, because we believe the consumption externality is essential for destructing resources.

The endowments for goods are \( w_1 \geq 0 \) and \( w_2 \geq 0 \). We assume that at least one of them is strictly positive, depending on the preferences and technologies, in order to make this economy meaningful.

In this economy, an allocation \((x, y, v) \equiv (x_{11}, x_{12}, x_{21}, x_{22}, y_1, y_2, v_{12}, v_{21})\) is feasible if \((x, y, v) \in R^8_+, y_1 = y_1 (v_{12}, v_{21}), y_2 = y_2 (v_{12}, v_{21}, x_1), x_{11} + x_{21} + v_{21} \leq y_1 + w_1, \) and \( x_{12} + x_{22} + v_{12} \leq y_2 + w_2 \). An allocation \((x, y, v)\) is balanced if \( x_{11} + x_{21} + v_{21} = y_1 + w_1 \), and \( x_{12} + x_{22} + v_{12} = y_2 + w_2 \). An allocation \((x, y, v)\) is Pareto efficient if it is feasible and there does not exist another feasible allocation \((x', y', v')\) such that \( u_i (x'_{11}, x'_{21}, x'_{12}, y'_{2}) \geq u_i (x_{11}, x_{21}, x_{12}, y_2) \) for all \( i = 1, 2 \) and \( u_i (x'_{11}, x'_{21}, x'_{12}, y'_{2}) > u_i (x_{11}, x_{21}, x_{12}, y_2) \) for some \( i \).

Under the conditions imposed on utility functions and production functions, an allocation \((x^*_{11}, x^*_{12}, x^*_{21}, x^*_{22}, v^*_{12}, v^*_{21}) \)\(^6\) is Pareto efficient if and only if it solves the following program

\[
\max_{(x, v) \in R^6_{++}} u_2 (x_{11}, x_{21}, x_{22}, y_2)
\]

\(^6\)The output \( y^*_2 \) can be found by production function accordingly. So, for the sake of simplicity in calculation, we drop the output variables in allocation.
Equation (55) and (56) are the social marginal rates of substitution for both consumers, corrected and by (48) and (49)

and keeping in mind the fact that $y_1 = y_1 \left( v_{12}, v_{21} \right), y_2 = y_2 \left( v_{12}, v_{21}, x_{11} + x_{21} \right)$, we can find the following FOCs,

$$x_{11} : \frac{\partial u_2}{\partial x_{11}} + \frac{\partial u_2}{\partial y_2} \frac{\partial y_2}{\partial x_{11}} - \lambda_1 + \lambda_2 \frac{\partial y_2}{\partial x_{11}} + \mu \left( \frac{\partial u_1}{\partial x_{11}} + \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial x_{11}} \right) = 0 \quad (46)$$

$$x_{12} : -\lambda_2 + \mu \frac{\partial u_1}{\partial x_{12}} = 0 \quad (47)$$

$$x_{21} : \frac{\partial u_2}{\partial x_{21}} + \frac{\partial u_2}{\partial y_2} \frac{\partial y_2}{\partial x_{21}} - \lambda_1 + \lambda_2 \frac{\partial y_2}{\partial x_{21}} + \mu \left( \frac{\partial u_1}{\partial x_{21}} + \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial x_{21}} \right) = 0 \quad (48)$$

$$x_{22} : \frac{\partial u_2}{\partial x_{22}} - \lambda_2 = 0 \quad (49)$$

$$v_{12} : \frac{\partial u_2}{\partial y_2} \frac{\partial y_2}{\partial v_{12}} + \lambda_1 \frac{\partial y_1}{\partial v_{12}} + \lambda_2 \left( \frac{\partial y_2}{\partial v_{12}} - 1 \right) + \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial v_{12}} = 0 \quad (50)$$

$$v_{21} : \frac{\partial u_2}{\partial y_2} \frac{\partial y_2}{\partial v_{21}} + \lambda_1 \left( \frac{\partial y_1}{\partial v_{21}} - 1 \right) + \lambda_2 \frac{\partial y_2}{\partial v_{21}} + \mu \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial v_{21}} = 0 \quad (51)$$

$$\lambda_1 : \lambda_1 \geq 0, x_{11} + x_{21} + v_{21} \leq y_1 + w_1, \lambda_1 (y_1 + w_1 - x_{11} - x_{21} - v_{21}) = 0 \quad (52)$$

$$\lambda_2 : \lambda_2 \geq 0, x_{12} + x_{22} + v_{12} \leq y_2 + w_2, \lambda_2 (y_2 + w_2 - x_{12} - x_{22} - v_{12}) = 0 \quad (53)$$

$$\mu : \mu \geq 0, u_1 \geq u_1^*, \mu (u_1 - u_1^*) = 0 \quad (54)$$

By (47) and (49), we still have $\lambda_2 = \frac{\partial u_2}{\partial x_{22}} > 0$ and $\mu = \frac{\partial u_2}{\partial x_{22}}$.

Then, by (46) and (47)

$$\frac{\lambda_1}{\lambda_2} = \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial x_{11}} + \frac{\partial u_1}{\partial x_{12}} + \frac{\partial u_2}{\partial y_2} \frac{\partial y_2}{\partial x_{21}} + \frac{\partial y_2}{\partial x_{11}} + \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial x_{21}} \equiv SMRS_1 \quad (55)$$

and by (48) and (49)

$$\frac{\lambda_1}{\lambda_2} = \frac{\partial u_2}{\partial x_{21}} + \frac{\partial u_2}{\partial x_{22}} + \frac{\partial u_2}{\partial x_{12}} + \frac{\partial y_2}{\partial x_{21}} + \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial x_{21}} \equiv SMRS_2. \quad (56)$$

Equation (55) and (56) are the social marginal rates of substitution for both consumers, corrected by the various external effects.
Also, by (50) and (51), we have
\[ \lambda_1 \lambda_2 = 1 - \frac{\partial y_2}{\partial v_{12}} = \frac{\partial y_2}{\partial y_{12}} - \frac{\partial y_2}{\partial y_{12}} \frac{\partial y_2}{\partial v_{12}} = \text{SMRT}_1, \] (57)
\[ \lambda_1 \lambda_2 = 1 - \frac{\partial y_2}{\partial v_{12}} = \text{SMRT}_2 \] (58)
which are the social marginal rates of transformation for firm 1 and firm 2, respectively.

Thus, combining (55), (56), (57), and (58), we obtain the marginal equality conditions
\[ \lambda_1 \lambda_2 = \text{SMRS}_1 = \text{SMRS}_2 = \text{SMRT}_1 = \text{SMRT}_2. \] (59)
which express the equality of the social marginal rates of substitution and the social marginal rates of transformation for all consumers and producers, and which are the necessary conditions for an allocation to be Pareto efficient.\(^7\) Again, in order these conditions to guarantee Pareto efficient allocations, we need the social marginal rates of substitution and the social marginal rates of transformation are not only equal but also nonnegative, i.e.,
\[ \text{SMRS}_1 = \text{SMRS}_2 = \text{SMRT}_1 = \text{SMRT}_2 \geq 0. \] (60)
and further there is no destruction for good 1 when they are positive, or equivalently \( \lambda_1 > 0 \).

We can compactly summarize our discussion above in the following proposition.

**Proposition 3** For \( 2 \times 2 \times 2 \) production economies, suppose that utility functions \( u_i(x_{11}, x_{21}, x_{i2}, y_2) \) are continuously differentiable, quasi-concave,
\[ \frac{\partial u_i(x_{11}, x_{21}, x_{i2}, y_2)}{\partial x_{il}} > 0 \text{ for } l = 1, 2, \text{ and } \frac{\partial u_i(x_{11}, x_{21}, x_{i2}, y_2)}{\partial x_{j1}} \leq 0 \text{ for } j \neq i. \] Suppose production functions are continuously differentiable and concave. Then we have the following statements.

(1) If the social marginal rates of substitution and social marginal rate of transformation are positive at a Pareto efficient allocation \( (x^*, y^*, v^*) \), then there is no destruction of resources in achieving Pareto efficient allocation \( (x^*, y^*, v^*) \).

(2) If the social marginal rates of substitution and social marginal rate of transformation are negative for any allocation \( (x_1, x_2, y_1, y_2, v_1, v_2) \) satisfying \( x_{11} + x_{21} + v_{21} = w_1 + y_1, x_{12} + x_{22} + v_{12} = w_2 + y_2, \) and the marginal equality conditions, then there is destruction of resources in achieving Pareto efficient allocation \( (x^*, y^*, v^*) \).

\(^7\)We assume the marginal rates of transformation for each firm are well-defined in Pareto efficient allocations. In other words, we assume \( (1 - \frac{\partial y_1}{\partial v_{21}}) \frac{\partial y_1}{\partial v_{12}} \neq 0 \) when evaluated at efficient allocations.
To have some explicit results, let us consider a special case where there is no cross externality between consumption and production, that is, the cross derivatives $\frac{\partial u_{i}}{\partial y_{2}} = 0$ and $\frac{\partial y_{2}}{\partial u_{i}} = 0$ for all $i = 1, 2$. Then the first four FOCs are the same as those in section 2, and thus by (21)-(24), we have

$$\frac{\lambda_1}{\lambda_2} = \frac{\partial u_1}{\partial x_{11}} + \frac{\partial u_2}{\partial x_{12}} = \frac{\partial u_2}{\partial x_{21}} + \frac{\partial u_1}{\partial x_{12}}. \quad (61)$$

The conditions (50) and (51) reduce to

$$\lambda_1 \frac{\partial y_1}{\partial v_{12}} + \lambda_2 \left( \frac{\partial y_2}{\partial v_{12}} - 1 \right) = 0 \quad (62)$$

$$\lambda_1 \left( \frac{\partial y_1}{\partial v_{21}} - 1 \right) + \lambda_2 \frac{\partial y_2}{\partial v_{21}} = 0 \quad (63)$$

and thus

$$\frac{\lambda_1}{\lambda_2} = 1 - \frac{\partial y_2}{\partial v_{12}} = 1 - \frac{\partial y_1}{\partial v_{21}}. \quad (64)$$

Therefore, by (61) and (64), we obtain the marginal equality conditions for production economies with the above simplified consumption and production externalities:

$$\frac{\lambda_1}{\lambda_2} = \frac{\partial y_1}{\partial v_{12}} + \frac{\partial y_2}{\partial v_{12}} = \frac{\partial y_2}{\partial v_{21}} + \frac{\partial y_1}{\partial v_{21}} = 1 - \frac{\partial y_2}{\partial v_{21}} = 1 - \frac{\partial y_1}{\partial v_{12}}. \quad (65)$$

Now when $\frac{\partial y_1}{\partial v_{21}} < 1$, $\frac{\partial y_2}{\partial v_{12}} < 1$, $\frac{\partial y_2}{\partial v_{21}} > 0$, and $\frac{\partial y_1}{\partial v_{12}} > 0$, the social marginal rates of transformation are positive, and thus $\lambda_1 > 0$ because $\lambda_2 = \frac{\partial y_2}{\partial x_{22}} > 0$. Therefore, the marginal equality conditions (65) and the two resource balanced conditions fully characterize the set of Pareto efficient allocations. In this case, there is no destruction of resources in achieving Pareto efficient allocations. Thus, negative production externalities together with positive marginal product of inputs eliminate the problem of destruction of resources in the presence of negative consumption externalities. Formally, we have the following corollary.

**Corollary 1** For $2 \times 2 \times 2$ production economies, suppose utility functions $u_i (x_{11}, x_{21}, x_{i2})$ are continuously differentiable, quasi-concave, and differentiable increasing in their own consumption, and production functions $y_1 = y_1 (v_{12}, v_{21})$ and $y_2 = y_2 (v_{12}, v_{21})$ are continuously differentiable and concave. Furthermore, suppose production externalities are negative and marginal product of inputs are positive. Then, there is no destruction in achieving efficient allocations.

While negative production externalities offset the problem of destruction of resources in the presence of negative consumption externalities, positive production externalities may not. The following example shows that, when production externalities are positive and consumption
externalities are negative, one may have to destroy resources in order to achieve Pareto efficient allocations.

Example 1 Suppose the utility functions are the same as in the numerical example in Section 2

\[ u_i(x_{11}, x_{21}, x_{i2}) = \sqrt{x_{i1}x_{i2}} - x_{j1}, \quad j \neq i, \]

and the endowments are \((w_1, w_2) = (1, 10)\). By Proposition 1 in Section 2, we know that, when there is no production, there is no destruction since \(w_1 = 1 < w_2/4 = 2.5\). However, when a production with positive externalities are allowed, one may have to destroy some resources. To see this, let production functions be given by

\[
y_1 = 2\sqrt{v_{21}} + 2v_{21} \\
y_2 = 2\sqrt{v_{21}} - v_{21} + 2\sqrt{v_{12}},
\]

Since \(\frac{\partial y_2}{\partial v_{21}} = 1 - \frac{1}{\sqrt{v_{21}}} < 0\) and \(1 - \frac{\partial y_2}{\partial v_{12}} = \sqrt{v_{12}} - 1 < 0\) for all \(0 < v_{12} < 1\) and \(0 < v_{21} < 1\), the social marginal rates of transformation are negative when inputs are less than one. Thus, to have Pareto efficient allocations, we must require \(v_{12} \geq 1\) and \(v_{21} \geq 1\).

Since consumers’ supply for good 2, \(w_2 + y_2 - v_{12} = 10 + 2\sqrt{v_{21}} - v_{21} + 2\sqrt{v_{12}} - v_{12}\) is maximized at \(v_{21} = 1\) and \(v_{12} = 1\), which is equal to 12, the critical point for destruction \((w_2 + y_2 - v_{12})/4\) is also maximized at \(v_{21} = 1\) and \(v_{12} = 1\), which is equal to 3. Also, because the resource constraint for good 2 is always binding at Pareto efficient allocations, \(v_{12} > 1\) and \(v_{21} > 1\) cannot be Pareto efficient inputs. We now show \((v_{12}, v_{21}) = (1, 1)\) is the only input vector that results in Pareto efficient allocations. Indeed, when \((v_{12}, v_{21}) = (1, 1)\), \(y_1 = 4\) and \(y_2 = 3\), and the feasible conditions become \(x_{11} + x_{21} \leq 4\) and \(x_{12} + x_{22} \leq 12\). Since the critical level for the destruction of good 1 is \(12/4 = 3 < 4\), and thus, applying Proposition 1 again, we need to destroy one unit of good 1 for achieving Pareto efficient allocations, and thus the set of Pareto efficient allocations is given by

\[
PO = \{(x, v, y) \in R^8_{++} : \quad x_{12} = 4x_{11}, \ x_{22} = 4x_{21}, \ x_{12} + x_{22} = 12, \ x_{11} + x_{21} = 3, \ v_{21} = 1, \ v_{12} = 1, \ y_1 = 4, \ y_2 = 3.\}
\]

3.2 Destruction for 3 × 2 × 2 Production Economies

From Corollary 1, it seems that introducing production may solve the destruction problem as long as production externalities are negative. However, this may not be true in general. We will
show that when production and consumption both have negative externalities, the destruction of resources in order to achieve Pareto efficient allocations is still necessary.

Consider production economies with three goods (two consumption goods and one input), two consumers and two firms. Each consumer consumes both consumption goods. Firm \( j \) produces consumption good \( j \). There are no initial consumption goods available. However, firms can produce both consumption goods from common raw materials, or natural resources, which we denote as \( r \). The initial resource endowment is \( w_r \), where \( w_r > 0 \). The other specifications are identical to those in the preceding subsection. Namely, the consumption of good 1 by one consumer negatively affects the other consumer, as well as the production of both consumption goods. We specify the preferences and technologies as follows,

\[
  u_i = u_i (x_{11}, x_{21}, x_{22})
\]

\[
  y_j = y_j (r_1, r_2)
\]

where \( \frac{\partial u_i (x_{11}, x_{21}, x_{22})}{\partial x_{il}} > 0 \) (l=1, 2), \( \frac{\partial u_i (x_{11}, x_{21}, x_{22})}{\partial x_{j1}} \leq 0 \), \( \frac{\partial y_j}{\partial r_j} > 0 \), and \( \frac{\partial y_j}{\partial r_l} \leq 0 \) for \( i \neq j \), \( j \neq l \).

In this economy, an allocation \((x, y, r) \equiv (x_{11}, x_{12}, x_{21}, x_{22}, y_1, y_2, r_1, r_2)\) is feasible if \((x, y, r) \in \mathbb{R}_+^8\), \( y_1 = y_1 (r_1, r_2) \), \( y_2 = y_2 (r_1, r_2) \), \( x_{11} + x_{21} \leq y_1 \), \( x_{12} + x_{22} \leq y_2 \) and \( r_1 + r_2 \leq w_r \). An allocation \((x, y, r)\) is balanced if the feasibility conditions hold with equality. Pareto efficiency can be similarly defined.

Then, under some regularity conditions, convex preference and technology for example, we can define the Pareto optimal allocations by solving the following program,

\[
  \text{Max} \quad u_2(x_{11}, x_{21}, x_{22})
\]

s.t. \( x_{11} + x_{21} \leq y_1 (r_1, r_2) \)
\( x_{12} + x_{22} \leq y_2 (r_1, r_2) \)
\( r_1 + r_2 \leq w_r \)
\( u_1(x_{11}, x_{21}, x_{12}) \geq u_1(x_{11}^*, x_{21}^*, x_{12}^*) \)

In this model, we want to characterize when there is destruction for consumption goods or the raw material resource.

The Lagrangian is:

\[
  L = u_2(x_{11}, x_{21}, x_{22}) + \lambda_1 [y_1 (r_1, r_2) - x_{11} - x_{21}] + \lambda_2 [y_2 (r_1, r_2) - x_{12} - x_{22}]
\]

\[
  + \delta (w_r - r_1 - r_2) + \mu [u_1(x_{11}, x_{21}, x_{12}) - u_1(x_{11}^*, x_{21}^*, x_{12}^*)]
\]
The signs of Kuhn-Tucker multipliers ($\lambda_1$, $\lambda_2$ and $\delta$) are still critical for judging the existence of destruction.

Then the first four FOCs are the same as those in Section 2 so that we have

\[
\frac{\lambda_1}{\lambda_2} = \frac{\partial u_2}{\partial x_{11}} - \lambda_1 + \mu \frac{\partial u_2}{\partial x_{12}} = 0
\]  
(66)

\[
x_{12} : -\lambda_2 + \mu \frac{\partial u_1}{\partial x_{12}} = 0
\]  
(67)

\[
x_{21} : \frac{\partial u_2}{\partial x_{21}} - \lambda_1 + \mu \frac{\partial u_1}{\partial x_{21}} = 0
\]  
(68)

\[
x_{22} : \frac{\partial u_2}{\partial x_{22}} - \lambda_2 = 0
\]  
(69)

By (70) and (71)

\[
\frac{\lambda_1}{\lambda_2} = \frac{\partial y_2}{\partial r_2} - \lambda_2 \frac{\partial y_2}{\partial r_1} - \delta = 0
\]  
(77)

Thus, equalizing (66) and (77) yields the marginal equality conditions:

\[
\frac{\partial u_2}{\partial x_{11}} + \frac{\partial u_2}{\partial x_{12}} = \frac{\partial u_2}{\partial x_{21}} + \frac{\partial u_2}{\partial x_{22}} = \frac{\partial y_2}{\partial r_2} - \lambda_2 \frac{\partial y_2}{\partial r_1} - \delta
\]  
(78)

which is positive. Thus, there are no destructions for consumptions goods 1 and 2 so that

\[
x_{11} + x_{21} = y_1
\]  
(79)

and

\[
x_{12} + x_{22} = y_2
\]  
(80)

Solving (70) and (71) for $\delta$, we have

\[
\delta = \frac{\partial u_2}{\partial x_{22}} \frac{\partial y_2}{\partial r_2} - \lambda_2 \frac{\partial y_2}{\partial r_1} - \delta
\]  
(81)
The sign of \( \delta \) is indeterminate and depends on the magnitude of \( \frac{\partial u_1}{\partial r_1} \frac{\partial y_2}{\partial r_2} \) and \( \frac{\partial u_2}{\partial r_2} \frac{\partial y_1}{\partial r_1} \). We will call these two terms the total marginal benefit and total marginal cost, respectively. Now, the marginal equality conditions (78), the two balanced consumption conditions, along with (81) and (74) constitute the system which determines all Pareto efficient allocations,

\[
\begin{align*}
\frac{\partial u_1}{\partial x_{11}} + \frac{\partial u_2}{\partial x_{12}} &= \frac{\partial u_2}{\partial x_{21}} + \frac{\partial u_1}{\partial x_{22}} = \frac{\partial u_2}{\partial x_{11}} - \frac{\partial u_1}{\partial x_{22}} \\
x_{11} + x_{21} &= y_1 (r_1, r_2) \\
x_{12} + x_{22} &= y_2 (r_1, r_2) \\
r_1 + r_2 &\leq w_r \\
\left( \frac{\partial u_1}{\partial r_2} - \frac{\partial u_1}{\partial r_1} \right) (w_r - r_1 - r_2) &= 0
\end{align*}
\]

(PPO)

Then, by the same discussion as in Section 2, we have the following proposition.

**Proposition 4** For the production economies specified in this subsection, let \((x^*, y^*, r^*)\) be a Pareto efficient allocation. Then we have the following results

1. There is never destruction of consumption goods.
2. Suppose \( \frac{\partial u_1}{\partial r_2} > \frac{\partial u_2}{\partial r_2} \) at \((x^*, r^*)\). Then, there is also no destruction of \( w_r \) in achieving the Pareto efficient allocation \((x^*, r^*)\).
3. For any allocation \((x_{11}, x_{12}, x_{21}, x_{22}, r_1, r_2)\), suppose that \( r_1 + r_2 = w_r, x_{11} + x_{21} = y_1 (r_1, r_2), x_{12} + x_{22} = y_2 (r_1, r_2), \) and that the marginal equality conditions (78) hold. Then \( \frac{\partial u_1}{\partial r_1} \frac{\partial y_2}{\partial r_2} < \frac{\partial u_2}{\partial r_2} \frac{\partial y_1}{\partial r_1} \). In this case there is destruction of \( w_r \) in achieving the Pareto efficient allocation \((x^*, y^*, r^*)\).

**Remark 4** If we assume the production of good 1 has a negative externality with respect to the satisfaction of consumers, that is \( u_i = u_i (x_{11}, x_{21}, x_{i2}, y_1) \), where \( \frac{\partial u_i}{\partial y_1} < 0 \), then we can still guarantee \( \lambda_1 > 0 \) so that there is no destruction of good 1. To see this, under this specification, (70) and (71) would change to

\[
\begin{align*}
r_1 : \quad &\frac{\partial u_2}{\partial y_1} \frac{\partial y_1}{\partial r_1} + \lambda_1 \frac{\partial y_1}{\partial r_1} + \lambda_2 \frac{\partial y_2}{\partial r_1} - \delta + \mu \frac{\partial u_1}{\partial y_1} \frac{\partial y_1}{\partial r_1} = 0 \\
r_2 : \quad &\frac{\partial u_2}{\partial y_1} \frac{\partial y_1}{\partial r_2} + \lambda_1 \frac{\partial y_1}{\partial r_2} + \lambda_2 \frac{\partial y_2}{\partial r_2} - \delta + \mu \frac{\partial u_1}{\partial y_1} \frac{\partial y_1}{\partial r_2} = 0
\end{align*}
\]

Then \( \lambda_1 = -\frac{\partial u_2}{\partial y_1} - \mu \frac{\partial u_1}{\partial y_1} + \lambda_2 \left( \frac{\partial y_2}{\partial r_2} \frac{\partial y_1}{\partial r_1} \right) > 0 \). However, if we assume \( u_i = u_i (x_{11}, x_{21}, x_{i2}, y_2) \), where \( \frac{\partial u_i}{\partial y_2} < 0 \), we cannot get this result.
4 Conclusion

We have considered the problem of achieving Pareto efficient allocations in the presence of various externalities. We provided the full set of conditions, which characterize Pareto efficient allocations for both exchange economies and production economies with externalities. We provided the specific conditions for determining whether or not there should be destruction for those goods with negative externalities. Roughly speaking, when the preferences are well-behaved, in this case monotone and convex, large endowments require destruction, and small endowments require no destruction. This result occurs because, in Pareto efficient allocations, social marginal rates of substitutions may diminish from positive to negative when the consumption good with negative externalities increases.

The conclusions are somewhat surprising. In contrast to the conventional wisdom, we showed that, even with preferences which are monotonically increasing in own consumption, negative consumption externalities, and some types of production externalities, one may have to destroy a certain amount of resources in order to achieve Pareto efficient allocations. Furthermore, even if information is complete and there are no transaction costs, there is no way to allocate resources efficiently without destroying some resources. Thus, all the existing alternative solutions cannot solve the market failure without destroying resources in the presence of externalities considered in this paper. These solutions must be appropriately modified.

To end the paper, we would mention some possible future research on negative consumption externalities and destruction of resources in achieving Pareto efficient allocations. This paper has ignored the incentive issue of implementing Pareto efficient allocations with destruction. It is an open question how to design an incentive mechanism that implements Pareto efficient allocations when utility functions and productions are unknown to the designer. However, some technics developed in Tian (2003, 2004) may be used to give such a mechanism.
Appendix

In this appendix, we will determine the boundary Pareto efficient allocations for our numerical example in section 2. We take boundary Pareto efficient allocations to mean a Pareto efficient point \( x^* = (x^*_{11}, x^*_{12}, x^*_{21}, x^*_{22}) \) to characterize the situation with at least one of the four coordinates is equal to zero. We will derive our results using a number of claims.

Claim 1 For a boundary Pareto optimal allocation \( x^* = (x^*_{11}, x^*_{12}, x^*_{21}, x^*_{22}) \), if \( x^*_{i1} = 0 \) or \( x^*_{i2} = 0 \), then \( x^*_{i1} = x^*_{i2} = 0 \) and \( x^*_{j1} > 0, x^*_{j2} = w_2 \) for any \( i \neq j \).

Proof. Suppose \( x^*_{i1} = 0 \). We first show \( x^*_{j1} > 0 \). Suppose not, then \( x^*_{j1} = 0 \). By recalling the form of utility functions, \( u^*_{i1} = u^*_{j1} = 0 \). Then the allocation \( x'_{i1} = x'_{j1} = \varepsilon, x'_{i2} = x'_{j2} = 4\varepsilon \) will be Pareto superior, since \( u'_{i1} = u'_{j1} = \varepsilon > 0 \) for any arbitrarily small positive number \( \varepsilon \).

Second, we show \( x^*_{i2} = 0 \). Suppose \( x^*_{j2} > 0 \). Then, \( x^*_{j2} < w_2 \) by the resource constraint. So, \( u^*_{i1} = -x^*_{j1}, u^*_{j1} = \sqrt{x^*_{j1}x^*_{j2}} < \sqrt{x^*_{j1}w_2} \) by \( x^*_{j1} > 0 \). Hence, we find another superior allocation by assigning all of good 2 to consumer \( j \).

Finally, we show \( x^*_{j2} = w_2 \). If \( x^*_{j2} < w_2 \), we can find the allocation \( x'_{i1} = x'_{i2} = 0, x'_{j1} = x^*_{j1}, x'_{j2} = w_2 \) superior to \( x^* \).

As for the case \( x^*_{i2} = 0 \), a similar argument applies. ■

Claim 1 helps us to shrink the potential boundary Pareto optimal points to a rather small set. We need only to check which of such points \( (x^*_{i1} = x^*_{i2} = 0, 0 < x^*_{j1} \leq w_1, x^*_{j2} = w_2) \) are Pareto Optimal.

Claim 2 An allocation \( x^* \) such that \( x^*_{i1} = x^*_{i2} = 0, 0 < x^*_{j1} \leq w_1, x^*_{j2} = w_2 \) is Pareto optimal if and only if it solves the following problem

\[
\begin{aligned}
\max_{x \in \mathbb{R}_+^4} & \quad \sqrt{x_{j1}x_{j2}} - x_{i1} \\
\text{s.t.} & \quad x_{11} + x_{21} \leq w_1 \\
& \quad x_{12} + x_{22} \leq w_2 \\
& \quad \sqrt{x_{i1}x_{i2}} - x_{j1} \geq -x^*_{j1}
\end{aligned}
\]

(Q1)

Proof. The only if part follows directly from Lemma 1.

As for the if part, by Lemma 1, we need only to show that if \( x^* \) solves (Q1), then it also solves
An allocation $x$ such that $\sqrt{x_j^1 x_j^2} - x_j - x_j^*$ is Pareto Optimal allocation if and only if for any interior Pareto Optimal points $\sqrt{x_j^1 x_j^2} - x_i \geq \sqrt{x_j^1 x_j^2}$ solves (Q1). Suppose not. Then there is another allocation $x'$ such that $\sqrt{x_j^1 x_j^2} - x_{i1} > \sqrt{x_j^1 x_j^2} - x_i$. We know $x_j^*$ should be positive since $x_j^* > 0$. Thus, if $x_i > 0$, then we can subtract $x_i$ a little bit and increase $x_j^*$ the same amount, resulting in $\sqrt{x_j^1 x_j^2} - x_{i1} > \sqrt{x_j^1 x_j^2} - x_i^* > \sqrt{x_j^1 x_j^2} - x_i$. So, it is contradicted with the fact $x^*$ solves (Q1). If $x_i = 0$, then $x_{i1} > x_i$. So, $\sqrt{x_j^1 x_j^2} - x_{i1} > \sqrt{x_j^1 x_j^2} - x_i^* \geq \sqrt{x_j^1 x_j^2} - x_i$. Thus, Claim 2 allows us only consider problem (Q1) in order to check boundary Pareto optimal points.

**Claim 3** An allocation $x^*$ satisfying $x_{i1}^* = x_{i2}^* = 0$, $0 < x_{j1}^* \leq w_1$, $x_{j2}^* = w_2$ is a boundary Pareto Optimal allocation if and only if for any interior Pareto Optimal points $\tilde{x}$, at least one of the following inequalities holds

\[
\begin{align*}
\sqrt{x_j^1 x_j^2} - x_{i1} &\leq \sqrt{x_j^1 w_2} \\
\sqrt{x_{i1} x_{i2}} - \tilde{x}_{j1} &< -x_{j1}^* 
\end{align*}
\]  

(82) (83)

Before we give a formal proof of Claim 3, this claim itself needs further interpretation. By Claim 2, after taking contrapositive, Claim 3 says that the allocation $x^*$ satisfying $x_{i1}^* = x_{i2}^* = 0$, $0 < x_{j1}^* \leq w_1$, $x_{j2}^* = w_2$ does not solve problem (Q1) if and only if there is some interior Pareto Optimal allocation $\tilde{x}$, such that

\[
\begin{align*}
\sqrt{x_j^1 x_j^2} - \tilde{x}_{i1} &> \sqrt{x_j^1 w_2} \text{ and } \sqrt{x_{i1} x_{i2}} - \tilde{x}_{j1} \geq -x_{j1}^* 
\end{align*}
\]  

(84)

Since any interior Pareto Optimal allocation satisfies the resource constraint conditions, then (84) is equivalent to say $x^*$ does not solve problem (Q1) if and only if there is some interior Pareto Optimal allocation $\tilde{x}$ does better than $x^*$ in (Q1).

The “if” part in the above sentence is obvious. How about the “only if” part? Now, we would prove the “only if” part in the following claim.
Claim 4  If an allocation $x^*$ satisfying $x^*_{i_1} = x^*_{i_2} = 0$, $0 < x^*_{j_1} \leq w_1$, $x^*_{j_2} = w_2$ does not solve problem (Q1), then there is some interior Pareto Optimal allocation $\tilde{x}$ does better than $x^*$ in (Q1).

Proof. First, we show that any boundary allocation $x'$ can never do better than $x^*$ in (Q1), that is, if such an allocation satisfies the constraint, then it fails to attain a higher objective function value. We finish the discussion case by case.

Case 1 $x'_{i_1} = 0$. By constraint $\sqrt{x'_{i_1}x'_{i_2}} = x'_{j_1} = -x'_{j_1} \geq -x^*_{j_1}$, we know $x'_{j_1} \leq x^*_{j_1}$. Then, $\sqrt{x'_{j_1}x'_{j_2} - x'_{i_1}} = \sqrt{x'_{j_1}x'_{j_2}} \leq \sqrt{x^*_j x^*_j w_2}$.

Case 2 $x'_{j_1} = 0$. Similarly, $x'_{j_1} \leq x^*_{j_1}$. Then, $\sqrt{x'_{j_1}x'_{j_2} - x'_{i_1}} = -x'_{i_1} \leq 0 < \sqrt{x^*_j x^*_j w_2}$.

Case 3 $x'_{i_1} = 0$. By the definition of Pareto superiority and the fact any allocation satisfies the resource constraint, $x$ will do better than $x^*$ in problem (Q1). Note that by the argument of the first part, $\tilde{x}$ never be a boundary point. The proof is completed. ■

Claim 5 The set of boundary Pareto efficient allocations is given by

(1) when $w_1 > w_2/4$

$$PO_B = \{x_{i_1} = x_{i_2} = 0, x_{j_2} = w_2, w_2/4 \leq x_{j_1} \leq w_1\}$$

$$\cup \{x_{j_1} = x_{j_2} = 0, x_{i_2} = w_2, w_2/4 \leq x_{i_1} \leq w_1\}$$

(2) when $w_1 \leq w_2/4$

$$PO_B = \{x_{i_1} = x_{i_2} = 0, x_{i_2} = w_2, x_{j_1} = w_1\}$$

$$\cup \{x_{j_1} = x_{j_2} = 0, x_{i_2} = w_2, x_{i_1} = w_1\}$$

Proof. By Claim 3, an allocation $x^*$ satisfying $x^*_{i_1} = x^*_{i_2} = 0$, $0 < x^*_{j_1} \leq w_1$, $x^*_{j_2} = w_2$ is a boundary Pareto Optimal allocation if and only if for any interior Pareto Optimal allocation $\tilde{x}$, at least one of the following inequalities holds

$$\sqrt{x_{j_1} x_{j_2} - \tilde{x}_{i_1}} \leq \sqrt{x^*_j w_2}$$

$$\sqrt{x_{i_1} x_{i_2} - \tilde{x}_{j_1}} < -x^*_j$$
Our task is to pin down the range of \( x_{j1}^* \). According to Proposition 1, we discuss in two cases.

**Case 1** \( w_1 > w_2/4 \)

By Proposition 1, any interior Pareto Optimal points \( \tilde{x} \) satisfies \( \tilde{x}_{j1} = \frac{1}{4} \tilde{x}_{j2}, \tilde{x}_{i2} = w_2 - \tilde{x}_{j2}, \tilde{x}_{i1} = \frac{1}{4} (w_2 - \tilde{x}_{j2}) \). Thus, we need a positive \( x_{j1}^* \) which has the property that for all \( 0 < \tilde{x}_{j2} < w_2 \), at least one of the following inequalities holds

\[
\frac{3}{4} \tilde{x}_{j2} - \frac{w_2}{4} \leq \sqrt{x_{j1}^* w_2} \tag{85}
\]
\[
\frac{w_2}{2} - \frac{3}{4} \tilde{x}_{j2} < -x_{j1}^* \tag{86}
\]

For \( \tilde{x}_{j2} \in (0, w_2/3] \), inequality (85) holds. So, no restriction imposed by \( \tilde{x}_{j2} \) in this range. Thus, we only consider the range \( \tilde{x}_{j2} \in (w_2/3, w_2) \). We require for any \( \tilde{x}_{j2} \in (w_2/3, w_2) \), either \( x_{j1}^* \geq \frac{9}{4 \sqrt{w_2}} (\tilde{x}_{j2} - \frac{w_2}{3})^2 \) or \( 0 < x_{j1}^* < \frac{3}{4} \tilde{x}_{j2} - \frac{w_3}{2} \). So, \( x_{j1}^* \) must falls into the interval \([w_2/4, w_1]\).

**Case 2** \( w_1 \leq w_2/4 \)

By Proposition 1, \( \tilde{x}_{j1} = \frac{w_1}{w_2} \tilde{x}_{j2}, \tilde{x}_{i2} = w_2 - \tilde{x}_{j2}, \tilde{x}_{i1} = \frac{w_1}{w_2} (w_2 - \tilde{x}_{j2}) \). Thus, (85) and (86) become

\[
\left( \sqrt{\frac{w_1}{w_2} + \frac{w_1}{w_2}} \right) \tilde{x}_{j2} - w_1 \leq \sqrt{x_{j1}^* w_2} \tag{87}
\]
\[
\sqrt{w_1 w_2} - \left( \sqrt{\frac{w_1}{w_2} + \frac{w_1}{w_2}} \right) \tilde{x}_{j2} < -x_{j1}^* \tag{88}
\]

By the same argument, we can find \( x_{j1}^* = w_1 \) in this case.
References


