Existence of Nash Equilibria in Games with Arbitrary Strategy Spaces and Preferences: A Full Characterization

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Abstract

This paper provides a complete solution to the existence of equilibria in games with any number of players that may be finite, infinite, or even uncountable; arbitrary strategy spaces that may be discrete, continuum, non-compact or non-convex; payoffs (resp. preferences) that may be discontinuous or do not have any form of quasi-concavity (resp. nontotal, nontransitive, discontinuous, nonconvex, or nonmonotonic). We establish a single condition, recursive diagonal transfer continuity for aggregate payoffs or recursive weak transfer quasi-continuity for individuals’ preferences, which is necessary and further, under compactness of strategy space, sufficient for the existence of Nash equilibria. Also, we introduce a non-fixed point theoretic proof of equilibrium existence.

Keywords: full characterization, Nash equilibrium; discontinuous games, non-ordered preferences, arbitrary strategy space, recursive transfer continuity.

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1 Introduction

The notion of Nash equilibrium is probably one of the most important solution concepts in economics in general and game theory in particular, which has wide applications in almost all areas of economics and in business and other social sciences. The classical existence theorems on Nash equilibrium (e.g. in Nash (1950, 1951), Debreu (1952), Glicksberg (1952), Nikaido and Isoda (1955)) typically assume continuity and quasiconcavity for the payoff functions, in addition to convexity and compactness of strategy spaces. However, in many important economic models, such as those in Bertrand (1883), Hotelling (1929), Milgrom and Weber (1985), Dasgupta and Maskin (1986), and Jackson (2009), payoffs are discontinuous and/or non-quasiconcave, and strategy spaces are nonconvex and/or noncompact.


However, most the existing results only provide sufficient conditions for the existence of equilibrium, and no full characterization has been given yet for general games. In order to apply a fixed-point theorem (say, Brouwer, Kakutani, Tarski’s fixed point theorem, or KKM lemma, etc.), they all need to assume some forms of quasiconcavity\(^1\) (or transitivity/monotonicity) and continuity of payoffs, in addition to compactness and convexity of strategy space.\(^2\) As such, the intrinsic nature of equilibrium has not been fully understood yet. Why does or does not a game have an equilibrium? Are continuity and quasiconcavity both essential to the existence of equilibrium? If so, can continuity and quasiconcavity be combined into one single condition? One can easily find simple examples of economic games that have or do not have an equilibrium, but none of them can be used to reveal the existence/non-existence of equilibria in these games.

There is no complete answer to these questions till recently. By strengthening Reny’s existence theorem, McLennan, Monteiro, and Tourky (2011) characterize the existence of Nash equilibrium and established a condition, called multiply restrictional security, which is necessary and sufficient

\(^1\)For mixed strategy Nash equilibrium, quasiconcavity is automatically satisfied since the mixed extension has linear payoff functions. Thus only some form of continuity matters for the existence of mixed strategy Nash equilibrium.

\(^2\)Thus, convexity assumption excludes the possibility of considering discrete games and requires the topological space be somewhat restrictive.
for the existence of Nash equilibrium in compact games. A key idea in Reny (1999), McLennan, Monteiro, and Tourky (2011), and Barelli and Soza (2009) is “securing a payoff” at a strategy profile for the other players by playing a pure strategy that insures that payoff when the profile of others’ strategies is near the given profile. Multiply Restrictional security refines Reny’s conditions in two ways: (1) weakening better reply security at a point by allowing several securing strategies to be used, and (2) allowing subcorrespondences of better reply correspondence by restricting the agents to subsets of strategy space.

It is worth noticing that “securing a payoff” is the basic nature of transfer continuity that was first introduced by Tian (1992a), Tian and Zhou (1992), Baye, Tian, and Zhou (1993) and Tian and Zhou (1995) to study preference maximization and the existence of equilibrium. In fact, payoff security and better-reply security introduced by Reny (1999) and their extensions by many others actually fall in the forms of transfer continuity. Indeed, as Prokopovych (2011) shows in Lemma 1 and 2, payoff security is equivalent to the transfer lower semicontinuity introduced in Tian (1992a), better-reply security is equivalent to the transfer reciprocal upper semicontinuity in payoff secure games, respectively.

In this paper we provide a full characterization for the existence of equilibrium in general games with any number of players that may be finite, infinite, or even uncountable; arbitrary strategy spaces that may be discrete, continuum, non-compact or non-convex; payoffs (resp. preferences) that may be discontinuous or do not have any form of quasi-concavity (resp. nontotal, nontransitive, discontinuous, nonconvex, or nonmonotonic). In particular, the strategy space under consideration is a general topological space that may not be metrizable, locally convex, Hausdorff, or even not regular. We introduce the notions of recursive transfer continuities, specifically recursive diagonal transfer continuity for aggregate payoffs and recursive weak transfer quasi-continuity for individuals’ preferences, respectively.

It is shown that the single condition, recursive diagonal transfer continuity (or recursive weak transfer quasi-continuity) is necessary, and further, under compactness of strategy space, sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs (or preferences). We also provide a complete solution for the case of any arbitrary strategy space that may be noncompact. We show that recursive diagonal transfer continuity (or recursive weak transfer quasi-continuity) with respect to a compact set is necessary and sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary (possibly noncompact or open) strategy spaces and general preferences. Recursive diagonal transfer continuity (or recursive weak transfer quasi-continuity) defined on respective spaces also permits full characterization

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3 Barelli and Soza (2009) also provide a necessary and sufficient condition for the existence of Nash equilibrium. But, as McLennan, Monteiro, and Tourky (2011) pointed out, it adopts many techniques from an earlier version of their paper.

4 Thus, one cannot say that recursive diagonal transfer continuity (or recursive weak transfer quasi-continuity) is equivalent to Nash equilibrium.
of symmetric pure strategy Nash equilibria in games with general strategy spaces and payoffs.

Our results not only help us understand what kind of games can have or cannot have equilibria, but also be used to develop new sufficient conditions for the existence of equilibrium. Indeed, Scalzo (2010) adopts many techniques and results in the paper to provide sufficient conditions for the existence of Pareto efficient Nash equilibria in discontinuous games. In the paper, we also provide some new classes of sufficient conditions for the existence of equilibrium without imposing any form of quasiconcavity.

The notion of recursive transfer continuity extends usual transfer continuity from single transfer to allow recursive (sequential) transfers so that it turns out to be a necessary and sufficient condition for the existence of equilibrium in compact games.\(^5\) As such, quasiconcavity/monotonicity of payoffs (or convexity of preferences) is unnecessary for characterizing the existence of Nash equilibria. Roughly speaking, an upsetting relation \(\succ\) is recursively diagonal transfer continuous if, whenever \(x\) is not an equilibrium, there exists a starting transfer point \(y^0\) and a neighborhood of \(x\), all of which are upset by any \(z\) that recursively upsets \(y^0\).\(^6\) When the number of such securing strategy profiles is \(m\), \(\succ\) is then called \(m\)-recursive diagonal transfer continuity. Then diagonal transfer continuity, introduced by Baye, Tian, and Zhou (1993) implies 1-recursive diagonal transfer continuity, and weak transfer quasi-continuity introduced by Nessah and Tian (2008) implies 1-recursive weak transfer quasi-continuity (by letting \(y^0 = x\)), respectively. Since they are in form of single transfer, diagonal transfer continuity or weak transfer quasi-continuity is neither necessary nor sufficient, and thus some form of quasiconcavity such as (strong) diagonal transfer quasiconcavity is needed for the existence of equilibrium as studied in Baye, Tian, and Zhou (1993) and Nessah and Tian (2008).

The relation of recursive transfer continuities and direct transfer continuities is somewhat like that of the weak axiom of revealed preference (WARP) and strong axiom of revealed preference (SARP). Directly revealing a preference by WARP is not enough to fully reveal individuals’ preferences, and then one may resort to indirectly revealing a preference by SARP to fully reveal an individual rational behavior. Similarly, diagonal transfer continuity or better-reply security alone is not enough to guarantee the existence of Nash equilibrium, one then may need to use a notion of recursive transfer continuity to fully characterize the existence of equilibrium.

The basic idea why recursive diagonal transfer continuity of an upsetting relation \(\succ\) ensures the existence of pure strategy Nash equilibrium in a compact game can be roughly described as follows. When a game fails to have a pure strategy Nash equilibrium on a compact strategy space \(X\), by recursive diagonal transfer continuity, for every \(x\), there is a starting transfer strategy profile

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\(^5\)The multiply restrictional security introduced by McLennan, Monteiro, and Tourky (2011) also has a similar feature: it extends better reply condition from single transfer to allow multiple transfers, which enables them characterizing the existence of Nash equilibrium.

\(^6\)We say a strategy profile \(z\) recursively upsets a strategy profile \(y^0\) if there exists a finite set of strategy profiles \(\{y^1, \ldots, y^{m-1}, z\}\) such that \(z\) upsets \(y^{m-1}\), \(y^{m-1}\) upsets \(y^{m-1}\), and so on till finally \(y^1\) upsets \(y^0\).
Then there are finite strategy profiles \(\{x^1, x^2, \ldots, x^n\}\) whose neighborhoods cover \(X\). Thus, all of the points in a neighborhood, say \(V_{x^1}\), will be upset by a corresponding deviation profile \(z^1\), which means \(z^1\) cannot be in \(V_{x^1}\). If it is in some other neighborhood, say, \(V_{x^2}\), then it can be shown that \(z^2\) will upset all strategy profiles in the union of \(V_{x^1}\) and \(V_{x^2}\) so that \(z^2\) is not in the union of \(V_{x^1}\) and \(V_{x^2}\). Suppose \(z^2 \in V_{x^3}\). Then we can similarly show that \(z^3\) is not in the union of \(V_{x^1}\), \(V_{x^2}\), and \(V_{x^3}\). With this process going on, we can finally show that \(z^n \notin V_{x^1} \cup V_{x^2} \cup \ldots \cup V_{x^n}\), which means \(z^n\) will not be in the strategy space \(X\), resulting in a contradiction.

The basic transfer method is systematically developed in Tian (1992a, 1993), Tian and Zhou (1992, 1995), Zhou and Tian (1992), and Baye, Tian, and Zhou (1993) for studying the maximization of binary relations that may be nontotal or nontransitive and the existence of equilibrium in games that may have discontinuous or nonquasiconcave payoffs. These papers, especially Zhou and Tian (1992), develop three types of transfers: transfer continuities, transfer convexities, and transfer transitivities to study the maximization of binary relations and the existence of equilibrium in games with discontinuous and/or nonquasiconcave payoffs. Various notions of transfer continuities, transfer convexities and transfer transitivities provide complete solutions to the question of the existence of maximal elements for complete preorders and interval orders (cf. Tian (1993) and Tian and Zhou (1995)). Incorporating recursive transfers into various transfer continuities can enable us to obtain full characterization results for many other solution problems.

The remainder of the paper is organized as follows. Section 2 provides basic notation and definitions, and analyze the intrinsic nature of Nash equilibrium. Section 3 fully characterizes the existence of pure strategy Nash equilibrium by using aggregate payoffs and individuals’ payoffs or preferences, respectively. We also provide sufficient conditions for recursive transfer continuities. Section 4 extends the full characterization results to symmetric pure strategy Nash equilibrium. Concluding remarks are offered in Section 5.

2 Preliminaries: Nash Equilibrium and Its Intrinsic Nature

2.1 Notion and Definitions

Let \(I\) be the set of players that can be finite, infinite or even uncountable. Suppose that each player \(i\)'s strategy set \(X_i\) is a nonempty subset of a general topological space \(E_i\) that may not be metrizable, locally convex, Hausdorff, or even not regular. Denote by \(X = \prod_{i \in I} X_i\) the set of strategy profiles. For each player \(i \in I\), denote by \(-i\) all other players rather than player \(i\). Also denote by \(X_{-i} = \prod_{j \neq i} X_j\) the Cartesian product of the sets of strategies of players \(-i\). Without loss of generality, suppose player \(i\)'s preference relation is given by the weak preference \(\succeq_i\) defined on
2.2 The Essence of Equilibrium and Why the Existing Results are Only Sufficient

A game $G = (X_i, \succ_i)_{i \in I}$ is simply a family of ordered tuples $(X_i, \succ_i)$.

When $\succ_i$ can be represented by a payoff function $u_i : X \to \mathbb{R}$, the game $G = (X_i, u_i)_{i \in I}$ is a special case of $G = (X_i, \succ_i)_{i \in I}$.

A strategy profile $x^* \in X$ is a pure strategy Nash equilibrium of a game $G$ if,

$$x^* \succ_i (y_i, x_{-i}) \forall i \in I, \forall y_i \in X_i.$$

A game $G = (X_i, \succ_i)_{i \in I}$ is compact, convex, and the weakly upper contour set $U = \{y \in X : y \succ x\}$ of $\succ$ is closed for all $x \in X$, respectively. A game $G = (X_i, u_i)_{i \in I}$ is quasiconcave if, for every $i \in I, X_i$ is convex and the payoff function $u_i$ is quasiconcave in $x_i$.

2.2 The Essence of Equilibrium and Why the Existing Results are Only Sufficient

Before proceeding to the notions of recursive transfer continuities, we first analyze the intrinsic nature of Nash equilibrium, and why the conventional continuity is unnecessarily strong and most the existing results provide only sufficient but not necessary conditions.

In doing so, we define an “upsetting” (irreflexive) binary relation, denoted by $\succ$ as follows:

$$y \succ x \text{ iff } \exists i \in I \text{ s.t. } (y_i, x_{-i}) \succ_i x. \quad (1)$$

In this case, we say strategy profile $y$ upsets strategy profile $x$.

When individuals’ preferences $\succ_i$ can be represented by numerical payoff functions $u_i$ and the number of players is finite, define the aggregator function, $U : X \times X \to \mathbb{R}$ by

$$U(y, x) = \sum_{i \in I} u_i(y_i, x_{-i}), \quad \forall (x, y) \in X \times X, \quad (2)$$

which refers to the aggregate payoff across individuals where for every player $i$ assuming she or he deviates to $y_i$ given that all other players follow the strategy profile $x$. The “upsetting” relation $\succ$ is then defined as:

$$y \succ x \text{ iff } U(y, x) > U(x, x). \quad (3)$$

The results obtained for weak preferences $\succ_i$ can be also used to get the results for strict preferences $\succ_i$. Indeed, from $\succ_i$, we can define a weak preferences $\succeq_i$ defined on $X \times X$ as follows: $y \succeq_i x$ if and only if $\neg x \succ_i y$. The preference $\succeq_i$ defined in such a way is called the completion of $\succ_i$. A preference $\succeq_i$ is said to be complete if, for any $x, y \in X$, either $x \succeq_i y$ or $y \succeq_i x$. A preference $\succeq_i$ is said to be total if, for any $x, y \in X, x \neq y$ implies $x \succeq_i y$, or $y \succeq_i x$.

$I$ can be a countably infinite set. In this case, one may define $U$ according to $U(y, x) = \sum_{i \in I} \frac{1}{n} u_i(y_i, x_{-i})$. This is a more general formulation.
Then, one can easily see that a strategy profile \( x^* \in X \) is a pure strategy Nash equilibrium if and only if there does not exist any strategy \( y \in X \) that upsets \( x^* \).

When \( x \in X \) is not a pure strategy Nash equilibrium, then there exists a strategy profile \( y \in X \) such that \( y \succ x \). To ensure the existence of an equilibrium, it usually requires all strategies in a neighborhood \( \mathcal{V}_x \) of \( x \) be upset by some strategy profile \( z \in X \), denoted by \( z \succ \mathcal{V}_x \), i.e., \( z \succ x' \) for all \( x' \in \mathcal{V}_x \). The topological structure of the conventional continuity surely secures this upsetting relation locally at \( x \) by \( y \), i.e., there always exists a neighborhood \( \mathcal{V}_x \) of \( x \) such that \( y \succ \mathcal{V}_x \). As such, no transfers (say, from \( y \) to \( z \)) or switchings (from player \( i \) to \( j \)) are needed for securing this upsetting relation locally at \( x \). However, when \( u_i \) is not continuous, such a topological relation between the upsetting point \( y \) and the neighborhood \( \mathcal{V}_x \) may no longer be true, i.e., we may not have \( y \succ \mathcal{V}_x \). But, if \( y \) can be transferred to \( z \) so that \( z \succ \mathcal{V}_x \), then the upsetting relation \( \succ \) can be secured locally at \( x \). This naturally leads to the following notion of transfer continuity, which is a weak notion of continuity.

**Definition 2.1** The upsetting relation \( \succ \) is transfer continuous if, whenever \( y \succ x \) for \( x, y \in X \), there exists an other deviation strategy profile \( z \in X \) and a neighborhood \( \mathcal{V}_x \subset X \) of \( x \) such that \( z \succ x' \) for all \( x' \in \mathcal{V}_x \).

It is clear that “\( y \succ x \) for \( x, y \in X \)” implies “\( x \in X \) is not an equilibrium”. We will use these terms interchangeably. The above definition in turn immediately reduces to the notion of diagonal transfer continuity introduced by Baye, Tian, and Zhou (1993) for aggregator function.

**Definition 2.2** A game \( G = (X_i, u_i)_{i \in I} \) is diagonally transfer continuous if, whenever \( U(y, x) > U(x, x) \) for \( x, y \in X \), there exists another deviation strategy profile \( z \in X \) and a neighborhood \( \mathcal{V}_x \subset X \) of \( x \) such that \( U(z, x') > U(x', x') \) for all \( x' \in \mathcal{V}_x \).

Also, note that, to secure “upsetting” relation locally at \( x \) by \( z \) \( z \succ \mathcal{V}_x \), it is unnecessary to have \( u_i(z_i, \mathcal{V}_{x, -i}) > u_i(\mathcal{V}_x) \) for all players \( i \), but is enough for just one player. Diagonal transfer continuity in Baye, Tian, and Zhou (1993), better-reply security in Reny (1999), weak transfer continuity in Nessah and Tian (2008), for instance, weaken the conventional continuity along this line.

Moreover, to secure “upsetting” relation locally at \( x \) by \( z \), it is unnecessary to just fix one player so that \( (z_i, \mathcal{V}_{x, -i}) \succ_i \mathcal{V}_x \), but players can be switched to secure this upsetting relation locally at \( x \). If for every \( x' \in \mathcal{V}_x \), there exists a player \( i \) such that \( (z_i, x'_{-i}) \succ_i x' \), all is done here. In other words, we can secure this “upsetting” relation locally by possibly switching players for every strategy in a neighborhood. This exactly comes up with the notion of weak transfer quasi-continuity introduced by Nessah and Tian (2008).

**Definition 2.3** A game \( G = (X_i, u_i)_{i \in I} \) is said to be weakly transfer quasi-continuous if, whenever \( x \in X \) is not an equilibrium, there exists a strategy profile \( y \in X \) and a neighborhood \( \mathcal{V}_x \) of \( x \) such that for every \( x' \in \mathcal{V}_x \), there exists a player \( i \) such that \( u_i(y_i, x'_{-i}) > u_i(x') \).
Note that, while the weak transfer quasi-continuity and lower single-deviation property in Reny (2009) explicitly exhibit such switchings, the notion of diagonal transfer continuity for aggregator function $U$ internalizes (implicitly allows) the switchings. Such implicit switchings have an advantage that it may become easy to check “upsetting” relations, especially for “complementary discontinuities”, by which a downward jump in one player’s payoff can always be accompanied by an upward jump in another player’s payoff (cf. Maskin and Dasgupta (1986) and Simon (1987)).

Although this “upsetting” relation is ensured, it may not still be sufficient for the existence of equilibrium unless imposing some forms of quasiconcavity, transitivity, or monotonicity. This is why, to make this upsetting relation sufficient for the existence of equilibrium, most the existing results impose additional conditions such as convexity of strategy spaces and (weak forms of) quasiconcavity of payoffs or transitivity/monotonicity of payoffs in order to use a fixed point theorem. But, such kind of combined conditions are only sufficient but not necessary. As mentioned in the introduction, to apply a fixed-point theorem to prove the existence of equilibrium, they all need to assume some forms of quasiconcavity (or transitivity/monotonicity) and continuity of payoffs, in addition to compactness and convexity of strategy spaces.

As such, a single upsetting transfer along may not be enough to guarantee the existence of an equilibrium unless imposing some forms of quasiconcavity, transitivity, or monotonicity. This is why, to make this upsetting relation sufficient for the existence of equilibrium, most the existing results impose additional conditions such as convexity of strategy spaces and (weak forms of) quasiconcavity of payoffs or transitivity/monotonicity of payoffs in order to use a fixed point theorem. But, such kind of combined conditions are only sufficient but not necessary. As mentioned in the introduction, to apply a fixed-point theorem to prove the existence of equilibrium, they all need to assume some forms of quasiconcavity (or transitivity/monotonicity) and continuity of payoffs, in addition to compactness and convexity of strategy spaces.

As such, a single upsetting transfer along may not be enough to guarantee the existence of an equilibrium, some recursive (sequential multiple) upsetting transfers starting from a strategy profile $y^0$ may be needed for the existence of equilibrium without imposing quasiconcavity or monotonicity condition. This is what we study in this paper. With such recursively upsetting relations, we are able to allow not only sequential transfers, but also the switchings of players in any stage of upsetting transfers and for different strategy profiles in a neighborhood. As a result, such a weak notion of transfer continuity may become both necessary and sufficient for the existence of Nash equilibria for compact games. Indeed, we will show that the above intuition and insights turn to be correct. It is the two requirements—securing upsetting relation and recursive transfers that characterize the existence or nonexistence of an equilibrium.

From the above discussion, one may see that the reasons why recursive transfer continuity is necessary and sufficient for the existence of equilibrium in general games come down to its four features: (1) it enables us to transfer to a strategy $z$ that secures upsetting relations locally; (2) it allows for recursive upsetting relations; (3) it permits us to make the switchings (transfers) among players for any upsettings of a particular transfer; (4) the starting point for making a recursive upsetting chain can be any point, say, it can be an equilibrium point to the necessity of the condition or the nonexistence of such a starting point if a game does not possess an equilibrium.
3 Full Characterization of Pure Strategy Nash Equilibria

In this section we provide a complete solution to the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs/preferences by providing necessary and sufficient conditions. We will provide two classes of necessary and sufficient conditions. One is based on aggregate payoffs and the other is based on individuals’ payoffs/preferences. We provide a number of simple examples of games that cannot use the existing theorems to argue the existence/non-existence of equilibria, but our results can do.

3.1 Full Characterization By the Aggregate Payoffs

In this subsection we assume that individuals’ preferences can be represented by payoff functions. We consider a mapping of individual payoffs into the aggregator function \( U : X \times X \to \mathbb{R} \), and then provide a necessary and sufficient condition on the aggregator function for the existence of pure strategy Nash equilibrium. The aggregator function approach is pioneered by Nikaido and Isoda (1955), and is also used by Baye, Tian, and Zhou (1993). Dasgupta and Maskin (1986) also use a similar approach to prove the existence of mixed strategy Nash equilibrium in games with discontinuous payoff functions. An advantage of this approach is that it internalizes the switchings among players so that checking “upsetting” relations are easier due to the complementarity that a game generally has.

**Definition 3.1 (Recursive Upsetting)** A strategy profile \( y^0 \in X \) is said to be **recursively upset** by \( z \in X \) if there exists a finite set of deviation strategy profiles \( \{y^1, y^2, \ldots, y^{m-1}, z\} \) such that \( U(y^1, y^0) > U(y^0, y^0), U(y^2, y^1) > U(y^1, y^1), \ldots, U(z, y^{m-1}) > U(y^{m-1}, y^{m-1}) \).

We say that a strategy profile \( y^0 \in X \) is **m-recursively upset** by \( z \in X \) if the number of such deviation strategy profiles is \( m \). For convenience, we say \( y^0 \) is **directly upset** by \( z \) when \( m = 1 \), and **indirectly upset** by \( z \) when \( m > 1 \). Recursive upsetting says that a strategy profile \( y^0 \) can be directly or indirectly upset by a strategy profile \( z \) through sequential deviation strategy profiles \( \{y^1, y^2, \ldots, y^{m-1}\} \) in a recursive way that \( y^0 \) is upset by \( y^1 \), \( y^1 \) is upset by \( y^2 \), \ldots, and \( y^{m-1} \) is upset by \( z \).

**Definition 3.2 (Recursive Diagonal Transfer Continuity)** A game \( G = (X_i, u_i)_{i \in I} \) is said to be **recursively diagonal transfer continuous** if, whenever \( U(y, x) > U(x, x) \) for \( x, y \in X \), there exists a strategy profile \( y^0 \in X \) (possibly \( y^0 = x \)) and a neighborhood \( V_x \) of \( x \) such that \( U(z, V_x) > U(V_x, V_x) \) for any \( z \) that recursively upsets \( y^0 \).

In the definition of recursive diagonal transfer continuity, \( x \) is transferred to \( y^0 \) that could be any point in \( X \). Recursive diagonal transfer continuity implies that, whenever \( x \) is not an equilibrium, there exists a starting point \( y^0 \) such that any recursive upsetting chain \( \{y^0, y^1, y^2, \ldots, y^m\} \)
disproves the possibility of an equilibrium in a sufficiently small neighborhood of \( x \), i.e., all points in the neighborhood are upset by all securing strategy profiles that directly or indirectly upset \( y^0 \). This implies that, if the game is not recursively diagonal transfer continuous, then there is a nonequilibrium strategy profile \( x \) such that for every \( y^0 \in X \) and every neighborhood \( \mathcal{V}_x \) of \( x \), some deviation strategy profiles in the neighborhood cannot be upset by a securing strategy profile \( z \) that directly or indirectly upsets \( y^0 \).

**Remark 3.1** Under recursive diagonal transfer continuity, when \( U(z, y^{m-1}) > U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}) \ldots, U(0^1, y^0) > U(y^0, y^0) \), we have not only \( U(z, y^0) > U(z, y^0) \), but also \( U(y^{m-1}, y^0) > U(y^{m-1}, y^0) \ldots, U(0^1, y^0) > U(0^1, y^0) \). That is, any chain of securing strategy profiles \( \{y^1, y^2, \ldots, y^{m-j}\} \) obtained by truncating a recursive upsetting chain \( \{y^1, y^2, \ldots, y^{m-1}, z\} \) is also a recursive upsetting chain, including \( y^1 \).

Similarly, we can define \( m \)-recursive diagonal transfer continuity. A game \( G = (X_i, u_i)_{i \in I} \) is \( m \)-recursively diagonal transfer continuous if the phrase “for any \( z \) that recursively upsets \( y^0 \)” in the above definition is replaced by “for any \( z \) that \( m \)-recursively upsets \( y^0 \)”. Thus, a game \( G = (X_i, u_i)_{i \in I} \) is recursively diagonal transfer continuous if it is \( m \)-recursively diagonal transfer continuous on \( X \) for all \( m = 1, 2, \ldots \).

**Remark 3.2** It is clear that diagonal transfer continuity implies 1-recursive diagonal transfer continuity by letting \( y^0 = x \), but the converse may not be true since \( x \) possibly cannot be selected as \( y^0 \). Thus, diagonal transfer continuity (thus continuity) is in general stronger than 1-recursive diagonal transfer continuity. Also, recursive diagonal transfer continuity neither implies nor is implied by continuity for games with two or more players.\(^9\) This point becomes clear when one sees recursive diagonal transfer continuity is a necessary and sufficient condition for the existence of pure strategy Nash equilibrium while continuity of the aggregate payoff function is neither a necessary nor sufficient condition for the existence of pure strategy Nash equilibrium.

Now we are ready to state our main results on the existence of pure strategy Nash equilibrium in games. We first show that recursive diagonal transfer continuity is a necessary condition for any game to possess a pure strategy Nash equilibrium.

**Theorem 3.1 (Necessity Theorem)** *If a game \( G = (X_i, u_i)_{i \in I} \) possesses a pure strategy Nash equilibrium, it must be recursively diagonal transfer continuous on \( X \).*

**Proof.** First, note that, if \( x^* \in X \) is a pure strategy Nash equilibrium of a game \( G \), we must have \( U(y, x^*) \leq U(x^*, x^*) \) for all \( y \in X \), which is obtained by summing up
\[
u_i(y_i, x^*_{-i}) \leq u_i(x^*) \quad \forall \; y_i \in X_i,
\] (4)

\(^9\)In one-player games recursive diagonal transfer continuity is equivalent to the player’s utility function possessing a maximum on a compact set, and consequently it implies transfer weak upper continuity introduced in Tian and Zhou (1995), which is weaker than continuity.
for all players.

Let \( x^* \) be a pure strategy Nash equilibrium and \( U(y, x) > U(x, x) \) for \( x, y \in X \). Let \( y^0 = x^* \) and \( \mathcal{V}_x \) be a neighborhood of \( x \). Since \( U(y, x^*) \leq U(x^*, x^*) \) for all \( y \in X \), it is impossible to find any securing strategy profile \( y^1 \) such that \( U(y^1, y^0) > U(y^0, y^0) \) (then of course it is impossible to find any sequence of strategy profiles \( \{y^1, y^2, \ldots, y^m\} \) such that \( U(y^1, y^0) > U(y^0, y^0), U(y^2, y^1) > U(y^1, y^1), \ldots, U(y^m, y^{m-1}) > U(y^{m-1}, y^{m-1}) \)). Hence, the recursive diagonal transfer continuity holds trivially.

The Necessity Theorem, Theorem 3.1, can help us understand why a game may not have an equilibrium. Example 3.1 below shows that a game may not be recursively diagonal transfer continuous, and then, by Theorem 3.1, the games do not possess any pure strategy Nash equilibrium.

**Example 3.1 (Karlin)** Consider games of “timing” or “silent duel”, which have been studied by Karlin (1959), Owen (1968), Jones (1980), and Dasgupta and Maskin (1986). These are symmetric two-person zero-sum games on the unit square so that \( n = 2, X_1 = X_2 = [0, 1] \), and \( U(x, x) = 0 \) for all \( x \in X \). The version called the “silent duel” has player I’s payoff function of the form:

\[
U_1(x_1, x_2) = \begin{cases} 
   x_1 - x_2 + x_1x_2, & \text{if } x_1 < x_2 \\
   0, & \text{if } x_1 = x_2 \\
   x_1 - x_2 - x_1x_2, & \text{if } x_1 > x_2
\end{cases}
\]

We show the game is not recursively diagonally transfer continuous. To see this, consider \( x = (x_1, x_2) = (1, 1) \).

We then cannot find any \( y^0 \in X \) and neighborhood \( \mathcal{V}_x \) of \( x \) such that \( U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x) \) for every deviation profile \( z \) that recursively upsets \( y^0 \). To show this, four cases need to be considered.

**Case 1.** \( y_1^0 < 1 \) and \( y_2^0 < 1 \). Then, for any neighborhood \( \mathcal{V}_{(1,1)} \) of \( (1, 1) \), choose strategy profiles \( z \in X \) and \( x' \in \mathcal{V}_{(1,1)} \) such that \( y_1^0 < z_1 < x_1' \) and \( y_2^0 < z_2 < x_2' \). Since \( u_1(y_1, y_2) \) and \( u_2(y_1, y_2) \) are increasing in \( y_1 \) and \( y_2 \), respectively, we have \( u_1(z_1, y_2^0) - u_1(y_1^0, y_2^0) > 0 \) and \( u_2(y_1^0, z_2) - u_2(y_1^0, y_2^0) = u_1(y_1^0, y_2^0) - u_1(y_1^0, z_2) > 0 \), \( u_1(z_1, x_2') - u_1(x_1', x_2') < 0 \) and \( u_2(x_1', z_2) - u_2(x_1', x_2') = u_1(x_1', x_2') - u_1(x_1', z_2) < 0 \). Thus, we have \( U(z, y^0) > U(y^0, y^0) \) but \( U(z, x') < U(x', x') \).

**Case 2.** \( y_1^0 = 1 \) and \( y_2^0 < 1 \). Then, for any neighborhood \( \mathcal{V}_{(1,1)} \) of \( (1, 1) \), choose strategy profiles \( z \in X \) and \( x' \in \mathcal{V}_{(1,1)} \) such that \( y_1^0 = z_1 = x_1' \) and \( y_2^0 < z_2 < x_2' \). Then, by the monotonicity of \( u_1(y_1, y_2) \) and \( u_2(y_1, y_2) = -u_1(y_1, y_2) \), we have \( U(z, y^0) > U(y^0, y^0) \) but \( U(z, x') < U(x', x') \).

**Case 3.** \( y_1^0 < 1 \) and \( y_2^0 = 1 \). Then, for any neighborhood \( \mathcal{V}_{(1,1)} \) of \( (1, 1) \), choose strategy profiles \( z \in X \) and \( x' \in \mathcal{V}_{(1,1)} \) such that \( y_1^0 < z_1 < x_1' \) and \( y_2^0 = z_2 = x_2' \). Then, by similar reasoning, we have \( U(z, y^0) > U(y^0, y^0) \) but \( U(z, x') < U(x', x') \).
Case 4. \( y_1^0 = 1 \) and \( y_2^0 = 1 \). Then, for any neighborhood \( \mathcal{V}_{(1,1)} \) of \((1,1)\), choose strategy profiles \( z \in X \) and \( x' \in \mathcal{V}_{(1,1)} \) such that \( 1/2 < z_1 < x_1' \) and \( 1/2 = z_2 = x_2' \). We then have
\[
 u_1(z_1, y_2^0) - u_1(y_1^0, y_2^0) = 2z_1 - 1 > 0 \quad \text{and} \quad u_2(y_1^0, z_2) - u_2(y_1^0, y_2^0) = u_1(y_1^0, y_2^0) - u_1(y_1^0, z_2) = 2z_2 - 1 = 0, \quad u_1(z_1, z_2') - u_1(x_1', x_2') < 0 \quad \text{and} \quad u_2(x_1', z_2) - u_2(x_1', x_2') = u_1(x_1', x_2') - u_1(x_1', z_2) = 0,
\]
and consequently, \( U(z, y_0) > U(y_0, y_0) \) but \( U(z, x') < U(x', x') \).

Thus, we cannot find any \( y_0 \in X \) and any neighborhood \( \mathcal{V}_0 \) of \((1,1)\) such that \( U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x) \) for every deviation profile \( z \) that recursively upsets \( y_0 \). Hence, the game is not recursively diagonal transfer continuous on \( X \), and therefore, by Theorem 3.1, there is no pure strategy Nash equilibrium on \( X \).

The following example shows that, although the recursive diagonal transfer continuity is a necessary condition for the existence of a pure strategy Nash equilibrium, but it is a not sufficient condition for the existence of a pure strategy Nash equilibrium. As such, recursive diagonal transfer continuity cannot be regarded as the same as Nash equilibrium.

**Example 3.2** Consider the following game with \( X_1 = X_2 = (0, 1) \) and the payoff functions given by
\[
 u_i(x_1, x_2) = x_i \quad i = 1, 2.
\]

The game clearly does not possess a pure strategy Nash equilibrium. However, it is recursively diagonal transfer continuous on \( X \).

Indeed, for any two strategy profiles \( x, y \in X \) with \( U(y, x) > U(x, x) \), choose \( \epsilon > 0 \) such that \( (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \subset X \). Let \( y_0 = (x_1 + \epsilon, x_2 + \epsilon) \in X \) and \( \mathcal{V}_x \subseteq (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \). Note that the \( U(y, x) = y_1 + y_2 \). Then, for any finite set of deviation strategy profiles \( \{y^1, y^2, \ldots, y^{m-1}, z\} \) with \( U(y^1, y^0) > U(y^0, y^1), U(y^2, y^1) > U(y^1, y^1), \ldots, U(z, y^{m-1}) > U(y^{m-1}, y^{m-1}) \), i.e., \( z_1 + z_2 > y_1^{m-1} + y_2^{m-1} > \ldots > y_1^0 + y_2^0 \), we have \( U(z, x') = z_1 + z_2 > y_1^0 + y_2^0 > x_1' + x_2' \) for all \( x' \in \mathcal{V}_x \). Thus, \( U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x) \), which means the game is recursively diagonal transfer continuous on \( X \).

Although the recursive diagonal transfer continuity is a necessary but not sufficient condition for the existence of a pure strategy Nash equilibrium for a general strategy space, the next theorem, however, shows that, when the strategy space \( X \) of a game is compact, recursive diagonal transfer continuity turns out to be sufficient for the existence of pure strategy Nash equilibrium.

**Theorem 3.2 (Sufficiency Theorem)** Suppose the strategy space \( X \) of a game \( G = (X_i, u_i)_{i \in I} \) is compact. Then, if the game is recursively diagonal transfer continuous on \( X \), it possesses a pure strategy Nash equilibrium.
Consider a variation of “timing” or “silent duel” games, in Example 3.1. The version, called the “noisy duel”, has player l’s payoff function of the form: 

$$U(x) \text{ noting that } \{y^1, \ldots, y^{m-1}, z\} \text{ with } U(z, y^{m-1}) > U(y^{m-1}, y^{m-1}), U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}), \ldots, U(y^1, y^0) > U(y^0, y^0) \text{ for } m \geq 1,$$

we have 

$$U(z, y), U(z, x) \text{ for } (y, x) \in \mathcal{V} \times \mathcal{V}.$$ 

Since there is no equilibrium by the contrapositive hypothesis, $$y^0$$ is not an equilibrium and thus, by recursive diagonal transfer continuity, such a sequence of recursive securing strategy profiles \{y^1, \ldots, y^{m-1}, z\} exists for some $$m \geq 1$$.

Since $$X$$ is compact and $$X \subseteq \bigcup_{x \in X} \mathcal{V}_x$$, there is a finite set \{x^1, \ldots, x^L\} such that $$X \subseteq \bigcup_{i=1}^L \mathcal{V}_{x^i}$$. For each of such $$x^i$$, the corresponding initial deviation profile is denoted by $$y^{0i}$$ so that 

$$U(z, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i}) \text{ whenever } y^{0i} \text{ is recursively upset by } z^i.$$

Since there is no equilibrium, for each of such $$y^{0i}$$, there exists $$z^i$$ such that 

$$U(z^i, y^{0i}) > U(y^{0i}, y^{0i}), \text{ and then, by 1-recursive diagonal transfer continuity, we have } U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i}).$$ 

Now consider the set of securing strategy profiles \{z^1, \ldots, z^L\}. Then, $$z^i \not\in \mathcal{V}_{x^i},$$ otherwise, by 

$$U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i}),$$ 

we will have 

$$U(z^i, z^i) > U(z^i, z^i),$$ 

a contradiction. So we must have $$z^1 \not\in \mathcal{V}_{x^1}.$$

Without loss of generality, we suppose $$z^1 \in \mathcal{V}_{x^2}$$. Since 

$$U(z^2, z^1) > U(z^1, z^1) \text{ by noting that } z^1 \in \mathcal{V}_{x^2} \text{ and } U(z^1, y^{01}) > U(y^{01}, y^{01}),$$ 

then, by 2-recursive diagonal transfer continuity, we have 

$$U(z^2, \mathcal{V}_{x^1}) > U(\mathcal{V}_{x^1}, \mathcal{V}_{x^1}).$$ 

Also, 

$$U(z^2, \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^2}, \mathcal{V}_{x^2}).$$ 

Thus 

$$U(z^2, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}),$$ 

and consequently 

$$z^2 \not\in \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}.$$ 

Again, without loss of generality, we suppose $$z^2 \in \mathcal{V}_{x^3}$$. Since 

$$U(z^3, z^2) > U(z^2, z^2) \text{ by noting that } z^2 \in \mathcal{V}_{x^3}, U(z^2, z^1) > U(z^1, z^1), \text{ and } U(z^1, y^{01}) > U(y^{01}, y^{01}),$$ 

by 3-recursive diagonal transfer continuity, we have 

$$U(z^3, \mathcal{V}_{x^1}) > U(\mathcal{V}_{x^1}, \mathcal{V}_{x^1}).$$ 

Also, 

$$U(z^3, z^2) > U(z^2, z^2) \text{ and } U(z^2, y^{02}) > U(y^{02}, y^{02}),$$ 

by 2-recursive diagonal transfer continuity, we have 

$$U(z^3, \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^2}, \mathcal{V}_{x^2}).$$ 

Thus, 

$$U(z^3, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}) > U(\mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}),$$ 

and consequently 

$$z^3 \not\in \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}.$$ 

With this process going on, we can show that 

$$z^k \not\in \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \ldots, \cup \mathcal{V}_{x^k}, \text{ i.e., } z^k \text{ is not in the union of } \mathcal{V}_{x^1}, \mathcal{V}_{x^2}, \ldots, \mathcal{V}_{x^k} \text{ for } k = 1, 2, \ldots, L. \text{ In particular, for } k = L, \text{ we have } z^L \not\in \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \ldots \cup \mathcal{V}_{x^L} \text{ and so } z^L \not\in X \subseteq \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \ldots \cup \mathcal{V}_{x^L}, \text{ a contradiction.} \blacksquare$$

**Example 3.3** Consider a variation of “timing” or “silent duel” games, in Example 3.1. The version, called the “noisy duel”, has player 1’s payoff function of the form:

$$u_1(x_1, x_2) = \begin{cases} 2x_1 - 1, & \text{if } x_1 < x_2 \\ 0, & \text{if } x_1 = x_2 \\ 1 - 2x_2, & \text{if } x_1 > x_2 \end{cases}$$
In this game, the payoff function $u_i(x_1, x_2)$ is not diagonally transfer continuous for $i = 1$. Therefore, theorems in Baye, Tian, and Zhou (1993) and Reny (1999) are not applicable.

However, the game has a pure strategy Nash equilibrium. To see this, suppose $U(y, x) > U(x, x)$ for $x = (x_1, x_2) \in X$ and $y = (y_1, y_2) \in X$. Let $y^0 = (1/2, 1/2)$ and $V_x$ be a neighborhood of $x$. Since $U(y, y^0) \leq U(y^0, y^0)$ for all $y \in X$, it is impossible to find any securing strategy profile $y^1$ such that $U(y^1, y^0) > U(y^0, y^0)$. Hence, the recursive diagonal transfer continuity holds. Therefore, by Theorem 3.2, this game has a pure strategy Nash equilibrium.

**Example 3.4** Consider the two-player game with the following payoff functions defined on $[0, 1] \times [0, 1]$ studied by Barelli and Soza (2009).

$$u_i(x_i, x_{-i}) = \begin{cases} 
0 & \text{if } x_i \in (0, 1) \\
1 & \text{if } x_i = 0 \text{ and } x_{-i} \in \mathbb{Q} \\
1 & \text{if } x_i = 1 \text{ and } x_{-i} \notin \mathbb{Q} \\
0 & \text{otherwise}
\end{cases}$$

where $\mathbb{Q} = \{x \in [0, 1] : x \text{ is a rational number}\}$.

This game is convex, compact, bounded and quasiconcave, but it is not weakly transfer quasi-continuous, and consequently, it is not diagonally transfer continuous nor better-reply secure either.\(^{10}\)

To see the game is not weakly transfer quasi-continuous, consider the nonequilibrium $x = (1, 1)$. We then cannot find any $y \in X$ and any neighborhood $V_{(1,1)}$ of $(1,1)$ such that for every $x' \in V_x$, there is a player $i$ with $u_i(y_1, x'_{-i}) > u_i(x')$. We show this by considering two cases.

Case 1. $y_2 \neq 0$. Then, for any neighborhood $V_{(1,1)}$ of $(1,1)$, choosing $x' \in V_x$ with $x'_1 = 1$ and $x'_2 \notin \mathbb{Q}$, we have $u_1(y_1, x'_2) \leq u_1(x'_1, x'_2) = 1$ and $u_2(x'_1, y_2) = u_2(x'_1, x'_2) = 0$.

Case 2. $y_2 = 0$. When $y_1 \neq 0$, choosing $x' \in V_x$ with $x'_2 = 1$ and $x'_1 \notin \mathbb{Q}$, we have $u_1(y_1, x'_2) = u_1(x'_1, x'_2) = 0$ and $u_2(x'_1, y_2) \leq u_2(x'_1, x'_2) = 1$. When $y_1 = 0$, choosing $x' \in V_x$ with $x'_1 \notin \mathbb{Q}$ and $x'_2 \notin \mathbb{Q}$, we have $u_1(y_1, x'_2) = u_1(x'_1, x'_2) = 0$ and $u_2(x'_1, y_2) = u_2(x'_1, x'_2) = 0$.

Thus, the game is not weakly transfer quasi-continuous, and the existence theorems in Nessah and Tian (2008) cannot be applied.

However, it is recursively diagonal transfer continuous. Indeed, suppose $U(y, x) > U(x, x)$ for $x = (x_1, x_2) \in X$ and $y = (y_1, y_2) \in X$. Let $y^0 = (0, 0)$ and $V_x$ be a neighborhood of $x$. Since $U(y, y^0) \leq U(y^0, y^0)$ for all $y \in X$, it is impossible to find any securing strategy profile $y^1$ such that $U(y^1, y^0) > U(y^0, y^0)$. Hence, the recursive diagonal transfer continuity holds. Therefore, by Theorem 3.2, this game has a pure strategy Nash equilibrium.

\(^{10}\)A game $G = (X_i, u_i)_{i \in I}$ is better-reply secure if, whenever $(x^*, u^*) \in \bar{\Gamma}$, $x^*$ is not an equilibrium implies that there is some player $i$, $\pi \in X_i$, $\epsilon > 0$, and an open neighborhood $V_{x_{-i}}$ of $x_{-i}$ such that $u_i(\pi, y_{-i}) > u_i^* + \epsilon$ for all $y_{-i} \in V_{x_{-i}}$, where $\bar{\Gamma}$ is the closure of the graph $\Gamma = \{(x, u) \in X \times \mathbb{R}^n : u_i(x) = u_i, \forall i \in I\}$.  

The above Sufficiency Theorem assumes that the strategy space of a game is compact. This may still be a restrictive assumption since strategy space of a game may not be closed or bounded. For instance, it is well known that Walrasian mechanism can be regarded as a generalized game. However, when preferences are strictly monotone, excess demand functions are not well defined for zero prices. In this case, we cannot use Theorem 3.2 to fully characterize the existence of competitive equilibrium.

In the following we show that the compactness of strategy space in Theorem 3.2 can also be relaxed. To do so, we first introduce the following stronger version of recursive diagonal transfer continuity.

**Definition 3.3** Let $B$ be a subset of $X$. A game $G = (X_i, u_i)_{i \in I}$ is said to be recursively diagonal transfer continuous on $X$ with respect to $B$ if, whenever $x$ is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of $x$ such that (1) whenever $y^0$ is upset by a strategy profile in $X \setminus B$, it is upset by a strategy profile in $B$ and (2) $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ for any finite subset of securing strategy profiles $\{y^1, \ldots, y^m\} \subset B$ with $y^m = z$ and $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1}), U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}), \ldots, U(y^1, y^0) > U(y^0, y^0)$ for $m \geq 1$.

Condition (1) in the above definition ensures that if a strategy profile $x$ is not an equilibrium for the game $G = (X_i, u_i)_{i \in I}$, it must not be an equilibrium when the strategy space is constrained to be $B$.

Note that, while $\{y^1, \ldots, y^m\}$ are required to be in $B$, $y^0$ is not necessarily in $B$ but can be any point in $X$. Also, when $B = X$, recursive diagonal transfer continuity on $X$ with respect to $B$ reduces to recursive diagonal transfer continuity on $X$. We then have the following theorem that generalizes Theorem 3.2 by relaxing compactness of games.

**Theorem 3.3 (Full Characterization Theorem)** A game $G = (X_i, u_i)_{i \in I}$ possesses a pure strategy Nash equilibrium if and only if there exists a compact set $B \subseteq X$ such that it is recursively diagonal transfer continuous on $X$ with respect to $B$.

**Proof.** Sufficiency ($\Rightarrow$). The proof of sufficiency is essentially the same as that of Theorem 3.2 and we just outline the proof here. To show the existence of a pure strategy Nash equilibrium on $X$, it suffices to show that the game possesses a pure strategy Nash equilibrium $x^*$ in $B$ if it is recursively diagonal transfer continuous on $X$ with respect to $B$. Suppose, by way of contradiction, that there is no pure strategy Nash equilibrium in $B$. Then, since the game $G$ is recursively diagonal transfer continuous on $X$ with respect to $B$, for each $x \in B$, there exists $y^0$ and a neighborhood $\mathcal{V}_x$ such that (1) whenever $y^0$ is upset by a strategy profile in $X \setminus B$,
Another way to deal with the existence of pure strategy Nash equilibrium of a game is to extend (rather than restrict) a non-compact strategy space that may be discrete, continuum, non-convex or non-compact and an arbitrary payoff function that may be discontinuous or nonquasiconcave.

Thus, Theorem 3.3 fully characterizes the existence of equilibrium in games with an arbitrary strategy space that may be discrete, continuum, non-convex or non-compact and an arbitrary payoff function that may be discontinuous or nonquasiconcave.

**Remark 3.3** Another way to deal with the existence of pure strategy Nash equilibrium of a game with noncompact strategy space is to extend (rather than restrict) a noncompact $X$ to another compact set $\bar{X}$ (by compactification of topological space) and extend $U$ to $\bar{U}$ on $\bar{X}$. Then a necessary and sufficient condition for the existence of Nash equilibrium is that $\bar{U}$ is recursively diagonal transfer continuous on $\bar{X}$.

**Example 3.5** Consider again Example 3.2. We know the game is recursively diagonal transfer continuous on $X$. However, there does not exist any compact set $B \subset X$ such that the game is recursively diagonal transfer continuous on $X$ with respect to $B$. To see this, choose $x \in X \setminus B$ such that $U(x, x) > U(y', x)$ for all $y' \in B$. Also, for any $y > x$, we have $U(y, x) > U(x, x)$. We then cannot find any strategy profile $y^0 \in X$ and a neighborhood $V_x$ of $x$ such that such that (1) whenever $y^0$ is upset by a strategy profile in $X \setminus B$, it is upset by a strategy profile in $B$ and (2) $U(z, V_x) > U(V_x, V_x)$ for every deviation profile $z$ that is upset directly or indirectly by $y^0$. We show this by considering three cases.

\[\text{Example 3.5}\] Thanks Adam Wong for pointing out this to me.
Case 1. \( y^0 \in X \setminus B \). Then, \( y^0 \) is upset by a strategy profile \( y' \in X \setminus B \) with \( y' > y^0 \), but it cannot be upset by any strategy profile in \( B \).

Case 2. \( y^0 \in B \) such that \( y^0 \) is the maximum of \( u_1(y') + u_2(y') \) on \( B \). Then, \( y^0 \) is upset by a strategy profile \( y' \in X \setminus B \) with \( y' > y^0 \), but it cannot be upset by any other strategy profile in \( B \).

Case 3. \( y^0 \in B \) such that \( y^0 \) is not the maximum of \( u_1(y') + u_2(y') \) on \( B \). Let \( z \) be the maximum of \( u_1(y') + u_2(y') \) on \( B \). Then \( U(z, y^0) > U(y^0, y^0) \) but \( U(z, x') < U(x', x') \) when \( x' \) is sufficiently close to \( x \).

Thus, we cannot find any strategy profile \( y^0 \in X \) and a neighborhood \( \mathcal{V}_x \) of \( x \) such that such that (1) whenever \( y^0 \) is upset by a strategy profile in \( X \setminus B \), it is upset by a strategy profile in \( B \) and (2) \( U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x) \) for every deviation profile \( z \) that recursively upsets \( y^0 \). Hence, the game is not recursively diagonal transfer continuous on \( X \) with respect to \( B \). It of course, by Theorem 3.3, there is no pure strategy Nash equilibrium on \( X \).

On the other hand, the following example shows that, although the strategy space is an open unit interval, highly discontinuous and nonquasiconcave, we can use Theorem 3.3 to argue the existence of equilibrium.

**Example 3.6** Consider a game with \( n = 2 \), \( X_1 = X_2 = (0, 1) \) that is an open unit interval set, and the payoff functions are defined by

\[
u_i(x_1, x_2) = \begin{cases} 
1 & \text{if } (x_1, x_2) \in \mathbb{Q} \times \mathbb{Q} \\
0 & \text{otherwise}
\end{cases} \quad i = 1, 2,
\]

where \( \mathbb{Q} = \{x \in (0, 1) : x \text{ is a rational number} \} \).

Then the game is not compact, nor quasiconcave. It is not weakly transfer quasi-continuous either (so, as shown in Nessah and Tian (2008), it is not diagonally transfer continuous, better-reply secure, or weakly transfer continuous either). To see this, consider any nonequilibrium \( x \) that consists of irrational numbers. Then, for any neighborhood \( \mathcal{V}_x \) of \( x \), choosing \( x' \in \mathcal{V}_x \) with \( x'_1 \in \mathbb{Q} \) and \( x'_2 \in \mathbb{Q} \), we have \( u_1(y_1, x'_2) \leq u_1(x'_1, x'_2) = 1 \) and \( u_2(x'_1, y_2) \leq u_2(x'_1, x'_2) = 1 \) for any \( y \in X \). So the game is not weakly transfer quasi-continuous. Thus, there is no existing theorem that can be applied.

However, it is recursively diagonal transfer continuous on \( X \). Indeed, suppose \( U(y, x) > U(x, x) \) for \( x = (x_1, x_2) \in X \) and \( y = (y_1, y_2) \in X \). Let \( y^0 \) be any vector with rational numbers, \( B = \{y^0\} \), and \( \mathcal{V}_x \) be a neighborhood of \( x \). Since \( U(y, y^0) \leq U(y^0, y^0) \) for all \( y \in X \), it is impossible to find any securing strategy profile \( y^1 \) such that \( U(y^1, y^0) > U(y^0, y^0) \). Hence, the game is recursively diagonal transfer continuous on \( X \) with respect to \( B \). Therefore, by Theorem 3.3, this game has a pure strategy Nash equilibrium. In fact, the set of pure strategy Nash equilibria consists of all rational numbers on \((0, 1)\).
In general, the weaker the conditions in an existence theorem, the harder it is to verify whether the conditions are satisfied in a particular game. For this reason it is useful to provide sufficient conditions for recursive diagonal transfer transfer continuity. Indeed, our main characterization results help us not only understand what is possible for a game to have or not have a pure strategy Nash equilibrium, but also develop new sufficient conditions for the existence of pure strategy Nash equilibrium. It is well known that the convexity of preferences can be substituted for transitivity of preferences in maximizing preferences of individuals. The following results show that this is true also for economic games.

**Definition 3.4** (Deviation Transitivity) \( G = (X_i, u_i)_{i \in I} \) is said to be deviational transitive if \( U(y^2, y^1) > U(y^1, y^1) \) and \( U(y^1, y^0) > U(y^0, y^0) \) imply that \( U(y^2, y^0) > U(y^0, y^0) \). That is, the upsetting dominance relation is transitive.

We then we have the following result without assuming the convexity of strategy space and imposing any form of quasiconcavity.

**Proposition 3.1** Suppose \( G = (X_i, u_i)_{i \in I} \) is compact and deviational transitive. Then, there exists a pure strategy Nash equilibrium point if and only if \( G \) is 1-recursively diagonal transfer continuous.

**Proof.** We only need to show that, when \( G \) is deviational transitive, 1-recursive diagonal transfer continuity implies \( m \)-recursive diagonal transfer continuity for \( m \geq 1 \). Suppose \( x \) is not an equilibrium. Then, by 1-recursive diagonal transfer continuity, there exists a strategy profile \( y^0 \in X \) and a neighborhood \( V_x \) of \( x \) such that \( U(z, V_x) > U(V_x, V_x) \) whenever \( U(z, y^0) > U(y^0, y^0) \) for any \( z \in X \).

Now, for any sequence of deviation profiles \( \{y^1, \ldots, y^{m-1}, y^m\} \), if \( U(y^m, y^{m-1}) > U(y^{m-1}, y^{m-1}) \), \( U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}) \), \( \ldots \), \( U(y^1, y^0) > U(y^0, y^0) \), we then have \( U(y^m, y^0) > U(y^0, y^0) \) by deviation transitivity of \( U \), and thus by 1-recursive diagonal transfer continuity, \( U(y^m, V_x) > U(V_x, V_x) \). Since \( m \) is arbitrary, \( G \) is recursively diagonal transfer continuous. \( \blacksquare \)

Baye, Tian, and Zhou (1993) show that a game possesses a pure strategy Nash equilibrium if it is compact, convex, diagonally transfer continuous, and diagonally transfer quasi-concave.\(^{13}\) Since diagonal transfer continuity implies 1-recursive diagonal transfer continuity, we immediately have the following corollary.

**Corollary 3.1** Suppose \( G = (X_i, u_i)_{i \in I} \) is compact, deviational transitive, and diagonally transfer continuous. Then, there exists a pure strategy Nash equilibrium.

\(^{13}\)A game \( G = (X_i, u_i)_{i \in I} \) is diagonally transfer quasiconcave in \( y \) if, for any finite subset \( Y^m = \{y^1, \ldots, y^m\} \subset X \), there exists a corresponding finite subset \( X^m = \{x^1, \ldots, x^m\} \subset X \) such that for any subset \( \{x^1, x^2, \ldots, x^s\} \subset X^m, 1 \leq s \leq m \), and any \( x \in \text{co}\{x^1, x^2, \ldots, x^s\} \), we have \( \min_{1 \leq l \leq s} U(y^{k_l}, x) \leq U(x, x) \).
The above corollary is a new result that uses diagonal transfer continuity as a weak notion of continuity condition, and assumes neither the convexity of strategy space nor any form of quasi-concavity.

### 3.2 Full Characterization By Individuals’ Preferences

The aggregator function approach adopted in the previous subsection captures the idea of using multiple (finite) securing strategies in a way of transferring upsetting relation from one strategy or agent to another strategy or agent so that the upsetting relations can be preserved locally. This approach has a number of advantages such as, it implicitly allows for, or internalizes, the switchings of players in an upsetting relation so that the proof is relatively simpler, and it is relatively easy to check the upsetting relations due to the complementarity that secures payoffs. The aggregator function approach, however, also has a number of disadvantages. First, we need to assume that the preferences of each player can be represented by a payoff function. Secondly, we need to assume that the number of players is either finite or countable. Thirdly, it is a cardinal approach, but not an ordinal approach. While monotonic transformations preserve individuals’ upsetting relations unchanged, it may not be true after aggregation, i.e., with a mapping by the aggregator function, a deviation strategy profile $y$ may no longer upset a strategy profile $x$ after some monotonic transformation although $y$ upsets $x$ before the monotonic transformation. Fourthly, the aggregator function approach only reveals the total upsetting relations, it is less clear about individuals’ strategic interactions, and thus it lacks a more natural game theoretical analysis.

Nevertheless, the method developed in this paper does not necessarily need to define the aggregator function $U$. What matters is the concept of upsetting and multiple (finite) securing strategies. We can also get the full characterization results in terms of individuals’ payoffs or preferences. The individual preference approach overcomes all the shortcomings above and has the following advantages: (1) Preferences may not be represented by a payoff function; (2) the set of players can be arbitrary, (3) monotonic transformations preserve individuals’ upsetting relations, and (4) the analysis reveals more clear individuals’ strategic interactions.

**Definition 3.5** A game $G = (X_i, \succ_i)_{i \in I}$ is said to be *recursively weakly transfer quasi-continuous* if, whenever $x \in X$ is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of $x$ such that for every $x' \in \mathcal{V}_x$ and every finite set of deviation strategy profiles \{\(y^1, y^2, \ldots, y^{m-1}, z\)\} with $(y^1_{i_1}, y^0_{-i_1}) \succ_{i_1} y^0$ for some $i_1 \in I$, $(y^2_{i_2}, y^1_{-i_2}) \succ_{i_2} y^1$ for some $i_2 \in I$, \ldots, $(z_{i_m}, y^{m-1}) \succ_{i_m} y^{m-1}$ for some $i_m \in I$, there exists player $i \in I$ such that $(z_i, x'_{-i}) \succ_i x'$.

Note that, the notion of recursive weak transfer quasi-continuity allows the switchings (transfers) among players in the process of recursive upsetting transfers and at every point in the neighborhood $\mathcal{V}_x$. 

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Similarly, we can define the notions of $m$-recursive weak transfer quasi-continuity and recursive weak transfer quasi-continuity on $X$ with respect to $B$ for $B \subset X$. Note that, if a game is weakly transfer quasi-continuous, it is 1-recursively weakly transfer continuous by letting $y^0 = x$. But, the converse may not be true. Thus, weak transfer quasi-continuity is stronger than 1-recursive weak transfer quasi-continuity.

Now it is worth noticing what is the relationship between recursive weak transfer quasi-continuity defined above and the recursive diagonal transfer continuity of a game defined in the previous subsection. This relationship may help us to simplify the proof, as it will be seen below. By using the upsetting binary relation $\succ$ defined in the previous section, we can define recursive transfer continuity accordingly.

**Definition 3.6** The “upsetting” relation $\succ$ is said to be recursively transfer continuous if, whenever $x \in X$ is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $V_x$ of $x$ such that $z \succ V_x$ for any $z$ that recursively upsets $y^0$.

**Lemma 3.1** Let $\succ$ be the upsetting relation defined by (1). We then have

1. A game $G = (X_i, \succ_i)_{i \in I}$ is recursively weak transfer quasi-continuous on $X$ if and only if $\succ$ is recursively transfer continuous on $X$.

2. A game $G = (X_i, \succ_i)_{i \in I}$ is weakly transfer quasi-continuous on $X$ if and only if the upsetting relation $\succ$ is transfer continuous on $X$.

The proof is straightforward, and thus it is omitted here.

By Lemma 3.1, we then have the following result that fully characterizes the existence of pure strategy Nash equilibrium in qualitative games with arbitrary compact strategy spaces and general preferences.

**Theorem 3.4** Recursive weak transfer quasi-continuity is necessary, and further under compactness of $X$, sufficient for a game $G = (X_i, \succ_i)_{i \in I}$ to have a pure strategy Nash equilibrium on $X$.

Recursive transfer continuity is needed for guaranteeing the existence of equilibrium. Recently, Reny (2009) provides an example in which the game is weakly transfer quasi-continuous, and thus it is 1-recursively weakly transfer quasi-continuous,\(^{14}\) but it does not possess a pure strategy Nash equilibrium.

Similarly, the compactness of strategy space in Theorem 3.4 can also be removed and we have the following theorem.

\(^{14}\)It is also convex, compact, bounded, and quasiconcave.
Theorem 3.5 (Full Characterization Theorem) A game $G = (X_i, \succ_i)_{i \in I}$ possesses a pure strategy Nash equilibrium if and only if there exists a compact set $B \subseteq X$ such that it is recursively weakly transfer quasi-continuous on $X$ with respect to $B$.

Theorem 3.5 thus fully characterizes the existence of equilibrium in games with an arbitrary strategy space that may be discrete, continuum, non-convex or non-compact and preferences that may not be represented by a payoff function, nontotal/nontransitive, non-convex or discontinuous.

Similarly, we can provide some new sufficient conditions by using deviation transitivity.

Definition 3.7 (Deviation Transfer Transitivity) $G = (X_i, \succ_i)_{i \in I}$ is said to be deviation transfer transitive if for $y^0, y^1, y^2 \in X$, $y^2 \succ_{i_2} y^1$ for some $i_2 \in I$ and $y^1 \succ_{i_1} y^0$ for some $i_1 \in I$ imply that $y^2 \succ_{i'} y^0$ for some $i' \in I$.

Note that, like the notion of recursive weak transfer quasi-continuity, deviation transfer transitivity allows the transfers among players for each transfer in the recursive upsetting transfers.

Nessah and Tian (2008) show that a game possesses a pure strategy Nash equilibrium if it is compact, convex, weakly transfer quasi-continuous, and strongly diagonal transfer quasi-concave. Similar to Proposition 3.1, replacing strong diagonal transfer quasiconcavity by deviation transfer transitivity, we have the following result without assuming the convexity and convexity of strategy space and imposing any form of quasiconcavity.

Proposition 3.2 Suppose $G = (X_i, \succ_i)_{i \in I}$ is compact and deviational transfer transitive. Then, there exists a pure strategy Nash equilibrium point if and only if $G$ is 1-recursively weakly transfer quasi-continuous.

We now provide some sufficient conditions for a game $G = (X_i, u_i)_{i \in I}$ to be deviational transfer transitive and 1-recursively transfer quasi-continuous.

Definition 3.8 $u_i : X_1 \times X_2 \rightarrow \mathbb{R}$ is diagonally monotonic if (1) $X_1 = X_2 \subset \mathbb{R}$ and (2) for each $x_{-i} \in X_{-i}$, $u_i(x_i, x_{-i})$ is either non-increasing in $x_i$ for all $x_i < x_{-i}$ and $x_i \geq x_{-i}$ or nondecreasing in $x_i$ for all $x_i \leq x_{-i}$ and $x_i > x_{-i}$.

Note that diagonal monotonicity of $u_i$ allows discontinuity at $x_i = x_{-i}$. It is clear that $u_i$ is diagonally monotonic if it is monotonic. A game $G = (X_i, u_i)_{i \in I}$ is diagonally monotonic if $u_i : X_1 \times X_2 \rightarrow \mathbb{R}$ is diagonally monotonic for $i = 1, 2$.

Lemma 3.2 If a game $G = (X_i, u_i)_{i \in I}$ is diagonally monotonic, it is deviational transitive.

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15 A game $G = (X_i, u_i)_{i \in I}$ is said to be strongly diagonal transfer quasiconcave if for any finite subset $\{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $\{x^1, ..., x^m\} \subset X$ such that for any subset $\{x^{k_1}, x^{k_2}, ..., x^{k_s}\} \subset X^m$, $1 \leq s \leq m$, and any $x \in \text{co}\{x^{k_1}, x^{k_2}, ..., x^{k_s}\}$, there exists $y^h \in \{y^{k_1}, ..., y^{k_s}\}$ satisfying $u_i(y^h_{-i}, x_{-i}) \leq u_i(x)$ for all $i \in I$. 

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**Proof.** We only need to show the case where $u_i$ is nondecreasing in $x_i$. The proof of the case where $u_i$ is non-increasing in $x_i$ is similar.

We need to show that $u_i(z_i, y_{-i}) > u_i(y)$ and $u_i(y, x_{-i}) > u_i(x)$ imply $u_i(z_i, x_{-i}) > u_i(x)$. Indeed, by monotonicity of $u_i$, we have $z_i > y_i$ when $u_i(z_i, y_{-i}) > u_i(y)$ and $y_i > x_i$ when $u_i(y, x_{-i}) > u_i(x)$, and thus we have $z_i > y_i > x_i$. Then, by monotonicity of $u_i$, we have $u_i(z_i, x_{-i}) \geq u_i(y, x_{-i}) > u_i(x, x_{-i})$, which means $\psi$ is deviational transitive.

Notice that better-reply security and its strength by many others\textsuperscript{16} imply weak transfer quasi-continuity which in turn implies 1-recursive weak transfer quasi-continuity.\textsuperscript{17} Also, payoff security and better-reply are closely related to transfer lower continuity defined in Tian (1992a) and the transfer reciprocal upper semicontinuity. Indeed, as shown in Lemmas 1 and 2 in Prokopovych (2011), a game is payoff secure if and only if it is transfer lower semicontinuous introduced in Tian (1992a), and better-reply security is equivalent to the transfer reciprocal upper semicontinuity in payoff secure games.

**Definition 3.9** Let $X$ and $Y$ be two topological spaces. A function $\phi: X \times Y \to \mathbb{R}$ is $\gamma$-transfer lower semi continuous in $y$ if for all $x \in X$ and $y \in Y$, $\phi(x, y) > \gamma$ implies that there exists some point $x' \in X$ and some neighborhood $\mathcal{N}(y)$ of $y$ such that $\phi(x', z) > \gamma$ for all $z \in \mathcal{N}(y)$.

A game $G = (X_i, u_i)_{i \in I}$ is transfer lower semicontinuous if each $u_i$ is $\gamma$-transfer lower semicontinuous in $x_{-i}$ for every $\gamma \in \mathbb{R}$.

The following notion is due to Prokopovych (2011).

**Definition 3.10** A game $G = (X_i, u_i)_{i \in I}$ is said to be **transfer reciprocal upper semicontinuous** if whenever $(x, \alpha) \in Fr\Gamma$ and $x$ is not an equilibrium, there are a player $i$ and $\bar{x}_i \in X_i$ such that $u_i(\bar{x}_i, x_{-i}) > \alpha_i$.

Then, by Proposition 3.2 and Lemma 3.2, we have the following corollary.

**Corollary 3.2** Suppose $G = (X_i, \succ_i)_{i \in I}$ is compact, and either deviational transfer transitive or diagonally monotonic. Then, $G = (X_i, \succ_i)_{i \in I}$ possesses a pure strategy Nash equilibrium if any of the following conditions is satisfied:

(a) It is weakly transfer quasi-continuous.

(b) It is better-reply secure.

(c) It is transfer lower semicontinuous and transfer reciprocal upper semicontinuous.

\textsuperscript{16}Reny (1999) shows that a game $G = (X_i, u_i)_{i \in I}$ is better-reply secure if it is payoff secure and reciprocally upper semi-continuous. Bagh and Jofre (2006) further show that $G = (X_i, u_i)_{i \in I}$ is better-reply secure if it is payoff secure and weakly reciprocal upper semi-continuous.

\textsuperscript{17}The proofs of these implications can be found in Nessah and Tian (2008).
(d) It is transfer lower semicontinuous and reciprocal upper semi-continuous.

(e) It is transfer lower semicontinuous and weakly reciprocal upper semi-continuous.

Again, these are new results that use weak notions of continuity conditions and assumes neither the convexity and convexity of strategy space nor any form of quasiconcavity.

**Example 3.7** (Baye and Kovenock; Baye, Tian, and Zhou) Consider the two-player quasi-symmetric game studied by Baye and Kovenock (1993), and Baye, Tian, and Zhou (1993). Two duopolists have zero costs and set prices \((p_1, p_2)\) on \(Z = [0, T] \times [0, T]\). The payoff functions are (for \(0 < c < T\)):

\[
u_i(p_1, p_2) = \begin{cases} p_i & \text{if } p_i \leq p_{-i} \\ p_i - c & \text{otherwise} \end{cases}.
\]

One can interpret the game as a duopoly in which each firm has committed to pay brand loyal consumers a penalty of \(c\) if the other firm beats its price.\(^\text{18}\) These payoffs are neither quasiconcave nor continuous. However, the game is diagonally monotonic and weakly transfer quasi-continuous, and thus it is 1-recursively diagonal transfer continuous. Thus, by Corollary 4.1, this game possesses a symmetric pure strategy equilibrium.

### 4 Full Characterization of Symmetric Pure Strategy Nash Equilibria

The techniques developed in the previous section can be used to fully characterize the existence of symmetric pure strategy Nash equilibrium. Throughout this section, we assume that the strategy spaces for all players are the same. As such, let \(X_0 = X_1 = \ldots = X_n\). If in addition, \(u_1(y, x, \ldots, x) = u_2(x, y, x, \ldots, x) = \ldots, u_n(x, \ldots, x, y)\) for all \(x, y \in X\), we say that \(G = (X_i, u_i)_{i \in I}\) is a quasi-symmetric game.

**Definition 4.1** A Nash equilibrium \((x_1^*, \ldots, x_n^*)\) of a game \(G\) is said to be symmetric if \(x_1^* = \ldots = x_n^*\).

For convenience, we denote, for each player \(i\), and for all \(x, y \in X_0\), \(u_i(x, \ldots, y, \ldots, x)\) the function \(u_i\) evaluated at the strategy in which player \(i\) chooses \(y\) and all others choose \(x\).

Define a quasi-symmetric function \(\psi : X_0 \times X_0 \to \mathbb{R}\) by

\[
\psi(y, x) = u_i(x, \ldots, y, \ldots, x).
\]

Since \(G\) is quasi-symmetric, \(x^*\) is a symmetric pure strategy Nash equilibrium if and only if \(\psi(y, x^*) \leq \psi(x^*, x^*)\) for all \(y \in X_i\).

\(^\text{18}\)See Baye and Kovenock (1993) for an alternative formulation with both brand loyal and price conscious consumers, whereby a firm commits to pay a penalty if it does not provide the best price in the market.
**Definition 4.2** \( \psi : X_0 \times X_0 \to \mathbb{R} \) is said to be *recursively diagonal transfer continuous* if, whenever \( x \in X \) is not equilibrium, there exists a strategy profile \( y^0 \in X \) (possibly \( y^0 = x \)) and a neighborhood \( \mathcal{V}_x \) of \( x \) such that \( \psi(z, \mathcal{V}_x) > \psi(\mathcal{V}_x, \mathcal{V}_x) \) for any \( z \) that recursively upsets \( y^0 \).

We then have the following theorem.

**Theorem 4.1** Suppose a game \( G = (X_i, u_i)_{i \in I} \) is quasi-symmetric and compact. Then it possesses a symmetric pure strategy Nash equilibrium if and only if \( \psi : X_0 \times X_0 \to \mathbb{R} \) is recursively diagonal transfer continuous on \( X \).

**Proof.** The proof is the same as that of Theorems 3.1 and 3.2 provided \( U \) is replaced by \( \psi \), thus it is omitted here. ■

Theorem 4.1 strictly generalizes all the existing results on the existence of symmetric pure strategy Nash equilibrium such as those in Reny (1999).

Similar to Theorem 3.3, we can get a full characterization result on the existence of symmetric pure strategy Nash equilibrium for arbitrary (possibly) noncompact strategy space.

**Example 4.1 (Hendricks and Wilson)** Consider the concession quasi-symmetric game between two players studied by Hendricks and Wilson (1983), Simon (1987), and Reny (1999). The players must choose a time \( x_1, x_2 \in [0, 1] \) to quit the game. The player who quits last wins, although conditional on winning, quitting earlier is preferred. If both players quit at the same time, the unit prize is divided evenly between them. Then payoffs are:

\[
 u_i(x_1, x_2) = \begin{cases} 
 -x_i, & \text{if } x_i < x_{-i} \\
 1/2 - x_i, & \text{if } x_i = x_{-i} \\
 1 - x_i, & \text{if } x_i > x_{-i} 
\end{cases}
\]

Note that the payoffs are not quasiconcave (nor are they quasiconcave along the diagonal of the unit square) although \( U \) is diagonally transfer continuous by Proposition 2.(e) in Baye, Tian, and Zhou (1993). We now show that \( \psi \) is not recursively diagonal transfer continuous, and thus the game does not possess a symmetric pure strategy equilibrium. To see this, consider \( x = 0 \). It is clear that \( \psi(y, x) = u_i(y, 0) > u_i(0, 0) \) implies that \( 0 < y < 1/2 \). We then cannot find any \( y^0 \in X_0 \) and neighborhood \( \mathcal{V}_x \) of \( x \) such that \( \psi(z, x') > \psi(x', x') \) for every deviation profile \( z \) that is upset by \( y^0 \) for all \( x' \in \mathcal{V}_x \). We show this by considering two cases.

Case 1. \( y^0 \neq 0 \). Then, for any neighborhood \( \mathcal{V}_0 \) of 0, choose a strategy profile \( z \in [0, 1] \) and a strategy profile \( x' \in \mathcal{V}_0 \) such that \( \max\{1/2 + \epsilon, y^0\} < z < 1/2 + y^0 \) and \( x' < \epsilon \), where \( 0 < \epsilon < \min\{1/2, y^0\} \). Then, by \( z > y^0 \) and \( 1 - z > 1/2 - y^0 \), we have \( \psi(z, y^0) > \psi(y^0, y^0) \). However, since \( z > x' \) and \( 1/2 + \epsilon < 1/2 + \epsilon < z \), we have \( 1 - z < 1/2 - x' \), and consequently \( \psi(z, x') < \psi(x', x') \).
Case 2. $y^0 = 0$. Note that $\psi(z, y^0) > \psi(y^0, y^0)$ if and only if $0 < z < 1/2$. Then, for any neighborhood $V_0$ of 0, choosing a positive number $\epsilon$ such that $(\epsilon/2, \epsilon) \subset V_0$, $z = \epsilon/2$ and a strategy profile $x' \in V_0$ such that $x' \in (\epsilon/2, \epsilon)$, we have $\psi(z, y^0) > \psi(y^0, y^0)$ but $\psi(z, x') = -z < 1/2 - x' = \psi(x', x')$.

Thus, we cannot find any $y^0 \in [0, 1]$ and any neighborhood $V_0$ of 0 such that $\psi(z, V_0) > \psi(V_0, V_0)$ for every deviation profile $z$ that recursively upsets $y^0$. Therefore, $\psi$ is not recursively diagonal transfer continuous on $X_0$, and thus, by Theorem 4.1, there is no symmetric pure strategy Nash equilibrium on $X$.

**Example 4.2 (Bagh and Jofre)** The following two-person concession quasi-symmetric game on the unit square considered by Bagh and Jofre (2006) is a special case of a class of timing games on the unit square considered by Reny (1999). The payoffs are:

$$u_i(x_1, x_2) = \begin{cases} 
10, & \text{if } x_i < x_{-i} \\
1, & \text{if } x_i = x_{-i} < 0.5 \\
0, & \text{if } x_i = x_{-i} \geq 0.5 \\
-10, & \text{if } x_i > x_{-i}
\end{cases}$$

We now show that $\psi$ is recursively diagonal transfer continuous, and thus the game possesses a symmetric pure strategy equilibrium. Indeed, let $\psi(y, x) = u_i(y, x)$. Suppose $\psi(y, x) > \psi(x, x)$ for $x \in X_0$ and $y \in X_0$. Let $y^0 = 0$ and $V_x$ be a neighborhood of $x$. It is clear that $\psi(y, y^0) \leq \psi(y^0, y^0)$ for all $y \in X$, and thus it is impossible to find any securing strategy profile $y^1$ such that $\psi(y^1, y^0) > \psi(y^0, y^0)$. Hence, the recursive diagonal transfer continuity holds, and thus by Theorem 4.1, this game has a pure strategy Nash equilibrium.

Besides, since all the games in Examples 3.1-3.5 are quasisymmetric, it is even easier to show the existence/nonexistence of pure strategy (symmetric) Nash equilibrium by working on a single payoff function $\psi$, instead of the aggregate payoff function $U$ that is the sum of individual payoff functions.

Similar to Proposition 3.1, we have the following proposition.

**Proposition 4.1** Suppose a game $G = (X_i, u_i)_{i \in I}$ is quasi-symmetric, compact, and deviational transitive. Then it possesses a pure strategy symmetric Nash equilibrium if and only if $\psi(x, y)$ defined by (4) is 1-recursively diagonal transfer continuous on $X$.

We now provide some sufficient conditions for a game $G = (X_i, u_i)_{i \in I}$ to be deviational transitive and 1-recursively diagonal transfer continuous.

Similarly, we can show that diagonal better-reply security and its strength imply diagonal transfer continuity which in turn implies 1-recursively diagonal transfer continuity. The fol-
lowing shows that under diagonal monotonicity, upper semi-continuity also implies 1-recursively diagonal transfer continuity.

**Definition 4.3** A quasi-symmetric game $G = (X_i, u_i)_{i \in I}$ with $X_i \subset \mathbb{R}$ is diagonally monotonic if either (i) for each $\bar{x} \in X_0$, $u_i(\bar{x}, \ldots, x, \ldots, \bar{x})$ is decreasing in $x$ for all $x < \bar{x}$ and $x \geq \bar{x}$ or (ii) for each $\bar{x} \in X_0$, $u_i(\bar{x}, \ldots, x, \ldots, \bar{x})$ is increasing in $x$ for all $x \leq \bar{x}$ and $x > \bar{x}$.

**Lemma 4.1** Suppose $X_i$ is a subset of $\mathbb{R}$. If a quasi-symmetric game $G = (X_i, u_i)_{i \in I}$ is diagonally monotonic, $\psi$ is deviational transitive.

The proof is the same as that of Lemma 3.2, and omitted here.

**Lemma 4.2** Suppose $X_i$ is a subset of $\mathbb{R}$ and a game $G = (X_i, u_i)_{i \in I}$ is diagonally monotonic and upper semi-continuous on $X$. Then it is 1-recursively diagonal transfer continuous in $x$.

**Proof.** We only need to prove the case where $\psi$ is increasing in $x$. The proof of the case where $\psi$ is decreasing in $x$ is similar.

Suppose $\psi(y, x) > \psi(x, x)$ for $x, y \in X_0$. We need to show that there exists a point $y^0 \in X_0$ and a neighborhood $V_x$ of $x$ such that $\psi(z, V_x) > \psi(V_x, V_x)$ whenever $\psi(z, y^0) > \psi(y^0, y^0)$. Indeed, since $\psi(y, x) > \psi(x, x)$, we have $y > x$ by diagonal monotonicity of $\psi$. Let $y^0 = x + \delta < y$ for some positive $\delta > 0$. We have $\psi(y^0, x) > \psi(x, x)$ by diagonal monotonicity of $\psi$. Then, by upper semi-continuity, there is a neighborhood $V_x = \{x' \in X_0 : |x' - x| < \epsilon\}$ such that $\psi(y^0, x') > \psi(x', x)$ for all $x' \in V_x$. Since $u_i(\bar{x}, \ldots, x, \ldots, \bar{x})$ is increasing in $x$ on $X_0 \setminus \bar{x}$ for all $\bar{x} \in X_0$, we particularly have $\psi(y^0, x') > \psi(x', x')$ for all $x' = x' \in V_x$. Thus, whenever $\psi(z, y^0) > \psi(y^0, y^0)$, we have $z > y^0$ by diagonal monotonicity of $\psi$, and therefore, we have $\psi(z, x') > \psi(y^0, x') > \psi(x', x')$ for all $x' \in V_x$, which means $\psi$ is 1-recursively diagonal transfer continuous in $x$.  

Then, by Proposition 4.1 and Lemmas 4.1 - 4.2, we can have the following corollary.

**Corollary 4.1** Suppose that a game $G = (X_i, u_i)_{i \in I}$ is quasi-symmetric, compact, diagonally monotonic, and $X_i$ is a subset of $\mathbb{R}$. Then it possesses a pure strategy symmetric Nash equilibrium if any of the following conditions is satisfied:

(a) It is diagonally better-reply secure.

(b) It is diagonally transfer continuous.

(c) It is upper semi-continuous.
5 Conclusion

This paper fully characterizes the existence of pure strategy Nash equilibrium in games arbitrary strategy space and payoffs/preferences. Strategy space may be discrete, continuum, non-compact or non-convex; preferences may be nontotal, nontransitive, discontinuous, nonconvex, or non-monotonic. We establish a condition, called recursive diagonal transfer continuity for aggregate payoffs or recursive weak transfer quasi-continuity for individuals’ preferences, which is necessary, and further, under compactness of strategy space, sufficient for the existence of pure strategy Nash equilibrium. We also provide a stronger version of recursive diagonal transfer continuity (or recursive weak transfer quasi-continuity) that is necessary and sufficient for the existence of pure strategy Nash equilibrium in noncompact games.

The approach developed in the paper can similarly used to fully characterize the existence of mixed strategy Nash and Bayesian Nash equilibria in games with general strategy spaces and payoffs. It can also allow us to ascertain the existence of equilibria in important classes of economic games. Tian (2012a, 2012b) show how they can be employed to fully characterize the existence of competitive equilibrium for economies with excess demand functions and stable matchings.

We end the paper by remarking that the main purpose of this paper is not intent to provide conditions that is easy to check, but to characterize the essence of equilibrium in general games. Recursive transfer continuity provides a way of understanding the essence of equilibrium, more than necessarily providing a way to check its existence. It helps us to understand what kind of games can have or cannot have equilibria. In general, the weaker the conditions in an existence theorem, the harder it is to verify whether the conditions are satisfied in a particular game. Nevertheless, our characterization results may be used to develop new sufficient conditions for the existence of equilibrium. In the paper, we also provide some new classes of sufficient conditions for the existence of equilibrium in discontinuous and nonconvex games. A potential future work may be attempted to find more sufficient conditions for recursive diagonal transfer continuity.
References


