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Equilibria in first price auctions with participation costs ☆

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ABSTRACT

This paper characterizes the equilibria of first price auctions with participation costs in the independent private values environment. Bidders use cutoff strategies to decide whether they will participate in the auction. It is shown that, when bidders are homogeneous, there always exists a unique symmetric equilibrium, and further, there is no other equilibrium when valuation distribution functions are inelastic. When distribution functions are elastic at the symmetric equilibrium, there exists an asymmetric equilibrium. Inelasticity/elasticity includes concavity/convexity of distribution functions as a special case. We find similar results when bidders are heterogeneous.

1. Introduction

Auction is an efficient way to enhance the competition among buyers and, in turn, to increase the efficiency of allocating scarce resources in the presence of private information. However, they are generally not freely implemented. In many situations, a pre-bid cost is required for bidders to attend an auction. Sometimes the cost can be very high. As Mills (1993) points out, the bidding cost incurred by a typical bidder in a government procurement auction often runs into millions of dollars. This paper studies (Bayesian–Nash) equilibria of sealed-bid first price auctions with bidder participation costs in the independent private values environment.

1.1. Motivation

The fundamental structure of a first price auction with participation costs is one through which an indivisible object is allocated to one of many potential buyers who must incur some costs[1] to be able to participate in the auction. After the cost is incurred, a bidder can submit a bid. The bidder who submits the highest bid wins the object and pays his own bid.

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[1] Related terminology includes participation cost, participation fee, entry cost or opportunity costs of participating in the auction. See Green and Laffont (1984), Samuelson (1985), McAfee and McMillan (1987a, 1987b), etc.

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There are many sources for participation costs. For instance, sellers may require that those who submit bids have a certain minimum amount of bidding funds, which may compel some bidders to borrow; bidders themselves may have transportation costs to go to an auction place, incur some costs to learn the rules of the auction and how to submit bids, or bear opportunity costs to attend an auction.

In the presence of participation costs, bidders' behavior may change. If a bidder's expected revenue from participating in an auction is less than the participation cost, he will choose not to enter the auction. Even if he decides to participate, since the number of bidders who submit bids is endogenous, his bidding behavior may not be the same as it would be in the standard auction without participation costs. The number of bidders can affect the strategic behavior among the bidders greatly (cf. McAfee and McMillan, 1987a, 1987b; Harstad et al., 1990; and Levin and Smith, 1996). For example, in first price auctions, bidders shade more of their valuations as fewer bidders submit bids in the auctions.

Some studies have been conducted on the information acquisition in auctions. A bidder may want to learn how he and others value the item, and thus he may incur a cost in the information acquisition about their valuations.[2] A main difference between participation costs and information acquisition costs is that information acquisition costs are avoidable while participation costs are not. If a bidder does not acquire information about his own or others' valuations, he does not incur any cost, but can still submit bids. Some researchers, such as McAfee and McMillan (1987a, 1987b), Harstad (1990) and Levin and Smith (1994), combine the ideas of participation costs and information acquisition costs. Compte and Jehiel (2007) investigate the advantage of using dynamic auctions in the presence of information acquisition cost only. However, information acquisition costs and participation costs can both be regarded as sunk costs after the bidders submit bids.

1.2. Related literature

The studies of participation costs in auctions so far have mainly focused on the second price auction due to its simplicity of bidding behavior. In second price auctions (Vickrey, 1961), bidders cannot do better than bid their true values when they find participating optimal. Much of the existing literature investigates equilibria of second price auctions with participation costs. Green and Laffont (1984) study the second price auction with participation costs in a general framework where bidders' valuations and participation costs are both private information and establish the existence of symmetric equilibrium with uniform distribution. Gal et al. (2007) study equilibria in a two-dimensional framework with more general distributions, focusing on symmetric equilibrium only. Campbell (1998), Tan and Yilankaya (2006) and Miralles (2008) study equilibria and their properties of second price auctions in an economic environment with equal participation costs when bidders' values are private information. Cao and Tian (2008a) investigate equilibria in second price auctions where bidders may have differentiated participation costs. They introduce the notions of monotonic equilibrium and neg-monotonic equilibrium. Kaplan and Sela (2006) consider a private entry model in second price auctions in which they assume all bidders' valuations are common knowledge while participation costs are private information.

Studies of first price auctions in the presence of participation costs, however, have received little attention, although they are used more often in practice,[3] like the auctions for tendering, particularly for government contracts and auctions for mining leases. The difficulty partly lies in the fact that in first price auctions, bidding strategies are not so explicit as compared with the strategies in second price auctions. Bidders in first price auctions do not bid their true valuations. The degree of shading relies heavily on who else enters the auction and what information is inferred from the entrance behavior of those bidders. The effect of the information inferred on the bidding strategies of first price auctions is greater than that on second price auctions. Moreover, when bidders use different thresholds to enter an auction, the valuation distributions updated from their entrance behavior are different so that there may be no explicit bidding function and some bidders may use mixed strategies. As such, it is technically more difficult to find the cutoffs since they are determined by the expected revenue from participating in the auction at the thresholds, which in turn depends on the more complicated bidding functions of bidders who submit bids.

Some studies on equilibrium behavior in economic environments with different valuation distributions can be used to study the equilibria of first price auctions with participation costs. Kaplan and Zamir (2000, 2007) discuss the properties of bidding functions when valuations are uniformly distributed with different supports. Martinez-Pardina (2006) studies the first price auction in which bidders' valuations are common knowledge. They show that at equilibrium bidders whose valuations are common knowledge randomize their bids.

1.3. The results of the paper

In this paper, we investigate Bayesian–Nash equilibria of sealed-bid first price auctions in the independent private values environment with participation costs. We assume bidders know their valuations and participation costs before they make their decisions. Participation costs are assumed to be the same across all the bidders.

[2] Persico (2000) studies the incentives of information acquisition in auctions. He finds that bidders have more incentives for information acquisition in first price auctions than in second price auctions.

[3] Samuelson (1985) studies the entrance equilibrium of first price competitive procurement auctions and related welfare problem, focusing on the symmetric cutoff threshold. Menezes and Monteiro (2000) is another example.

When bidders are homogeneous, there is a unique symmetric equilibrium in terms of cutoff point. We show that there is no other equilibrium when valuation distribution function is inelastic, i.e., $vf(v) \leqslant F(v)$, which is satisfied when it is a concave function.[4] However, when the valuation distribution function $F(\cdot)$ is elastic at the symmetric equilibrium v^s, i.e., $v^s f(v^s) > F(v^s)$, which is satisfied when it is strictly convex, there always exists an asymmetric equilibrium. It may be remarked that, when a distribution function is strictly convex, it is elastic everywhere, especially at the symmetric equilibrium, and then there exists an asymmetric equilibrium. In this case, when bidders are divided into two different groups randomly, the cutoffs used by one group can always be different from those used by the other group.

The existence of asymmetric equilibria has important consequences for the strategic behavior of bidders and the efficiency of the auction mechanism. In the presence of participation cost, a bidder would expect fewer bidders to submit their bids. When there is only symmetric equilibrium, every bidder has to follow the symmetric cutoff strategy. However, when asymmetric equilibria exist, bidders may choose an equilibrium that is more desirable. In this case, some bidders may form a collusion to cooperate at the entrance stage by choosing a smaller cutoff point that may decrease the probability that other bidders enter the auction. Consequently, it reduces the competition in the bidding stage. An asymmetric equilibrium may become more desirable when an auction can run repeatedly. Also, an asymmetric equilibrium may be ex-post inefficient. The item being auctioned is not necessarily allocated to the bidder with the highest valuation.

We also consider the existence of equilibria in an economy with heterogeneous bidders in the sense that the distribution functions are different. Specifically, we consider the case where one distribution (called a weak bidder) is first order dominated by another (called a strong bidder). We concentrate on equilibria at which the bidders in the same group use the same threshold. We show that there is always an equilibrium at which the strong bidders use a smaller cutoff for valuations. When the distribution functions are concave, the equilibrium is unique. However, when the distribution functions for the weak bidders are strictly convex and the participation cost is sufficiently large, there exists an equilibrium at which weak bidders use a smaller cutoff.

The remainder of the paper is structured as follows. Section 2 presents a general setting of economic environment. Section 3 studies the existence and uniqueness of equilibria for homogeneous bidders. Section 4 studies equilibria for heterogeneous bidders. Concluding remarks are provided in Section 5. All the proofs are presented in Appendix A.

2. Economic environment

We consider an independent private values economic environment with one seller and $n \geqslant 2$ risk-neutral buyers (bidders). Let I denote the set of bidders. The seller is also risk-neutral and has an indivisible object to sell to one of the buyers. The seller values the object as 0. Each buyer i's valuation for the object is v_i $(i = 1, 2, \ldots, n)$, which is private information to the other bidders. It is assumed that v_i is independently distributed with a cumulative distribution function $F_i(\cdot)$ that has continuously differentiable density $f_i(\cdot) > 0$ everywhere with support $[0, 1]$.

The auction format is the sealed-bid first price auction. The bidder with the highest bid wins the auction and pays the price equal to his bid. His payoff is equal to the difference between his valuation and the price. The other bidders have zero payoff from submitting a bid. If the highest bid is submitted by more than one bidder, there is a tie which will be broken by a fair lottery.

There is a participation cost, common to all bidders, denoted by $c \in (0, 1)$. Bidders must incur c in order to submit bids. It is assumed that each bidder knows his own valuation and who will participate, but not the others' valuations so that we are in the interim information setting. Specifically, the timing of the game is as follows:

- Nature draws a valuation v_i for each bidder i and tells the bidder only what his own valuation is.
- Bidder i decides whether or not to submit a bid. If he chooses to submit a bid, he pays the participation cost c that is not refundable, otherwise the game ends for him.
- All the bidders who pay the participation costs observe who else also participates in the auction and submit a bid. The item is awarded to the bidder who submits the highest bid and pays his own bid. If more than one bidder submits the highest bid, the allocation is determined by a fair lottery.

The individual action set for any bidder can be characterized as $No \cup [0, 1]$, where "*No*" denotes not submitting a bid. Bidder i incurs the participation cost c if and only if his action is different from "*No*." While it is always a weakly dominant strategy to bid one's true valuation in second price auctions, this is not true for first price auctions. In first price auctions, a bidder may not bid his true valuation. Nevertheless, given the strategies of all other bidders, a bidder's expected revenue from participating in the auction is a nondecreasing function of his valuation.[5] Thus, a bidder submits a bid if and only if his valuation is greater than or equal to a cutoff point and does not enter otherwise.[6]

[4] The elasticity of the function $y = g(x)$ with respect to x refers to the percentage change in y induced by a small percentage change in x so that it is given by $\frac{dy/y}{dx/x}$. For the distribution function $F(v)$, its elasticity is then given by $\frac{vf(v)}{F(v)}$. We call F elastic if $\frac{vf(v)}{F(v)} > 1$ or inelastic otherwise.

[5] Lu and Sun (2007) show that for any auction mechanism with participation costs, the participating and nonparticipating types of any bidder are divided by a nondecreasing and equicontinuous shutdown curve. Thus in our framework, when participation cost is given, the participating and nonparticipating types of any bidder can be divided by a cutoff value and the threshold form is the only form of equilibria.

[6] In Milgrom and Weber (1982), the term of "screening level" is used instead of "cutoff point."

An equilibrium strategy of whether to participate is then given by a profile of the bidders' cutoff points, which are a vector of the minimum valuations for each bidder i to cover the cost. Let $v^* = (v_1^*, \ldots, v_n^*)$ denote the profile of bidders' cutoff points and $S_i(v^*)$ denote the set of bidders who also participate in the auction beside bidder i. The bidding decision function $b_i(\cdot)$ of each bidder is characterized by

$$b_i\big(v_i, v^*, S_i\big(v^*\big)\big) = \begin{cases} \lambda_i(v_i, v^*, S_i(v^*)) & \text{if } 1 \geqslant v_i \geqslant v_i^*, \\ No & \text{if } v_i < v_i^*, \end{cases}$$

where $\lambda_i(v_i, v^*, S_i(v^*))$ is a contingent bidding function when bidder i participates in the auction. Note that, if bidder i enters the auction while all the others do not enter, bidder i bids zero. If some other bidders also participate in the auction, the bid depends on the cutoff points and the valuation distributions of all the others. For notational simplicity, we use $b_i(v_i, v^*)$ and $\lambda_i(v_i, v^*)$ to denote $b_i(v_i, v^*, S_i(v^*))$ and $\lambda_i(v_i, v^*, S_i(v^*))$ respectively in the remainder of the paper.

For the game described above, each bidder's action is to choose a cutoff and decide how to bid when he participates. Thus, a (Bayesian–Nash) equilibrium of the sealed-bid first price mechanism with participation cost is composed of bidders' cutoff strategies and participants' bidding strategies.

Formally, we have the following definitions:

Definition 1. A strategy profile $(v^*, \mathbf{b}(v_i, v^*)) = ((v_1^*, b_1(v_1, v^*)), \ldots, (v_n^*, b_n(v_n, v^*))) \in \mathbb{R}_+^{2n}$ is a (Bayesian–Nash) equilibrium of the first price auction with participation cost if each bidder i's action $(v_i^*, b_1(v_i, v^*))$ is optimal, given others' strategies.

Note that, once the cutoff points are determined, the game is reduced to the standard first price auction and the optimal bidding functions for participating bidders are uniquely determined (see Maskin and Riley, 2003). As such, an equilibrium is fully characterized by the profile of cutoff points $v^* = (v_1^*, \ldots, v_n^*) \in \mathbb{R}_+^n$. Then all the results in the paper should be interpreted in terms of cutoffs.

As usual, when bidders' distribution functions are the same, i.e., $F_1(\cdot) = F_2(\cdot) = \cdots = F_n(\cdot) = F(\cdot)$, we may have symmetric and/or asymmetric equilibria.

Definition 2. For the economic environment with the same distribution functions, an equilibrium $v^* = (v_1^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ of the first price auction with participation cost is a symmetric (resp. asymmetric) equilibrium if the bidders have the same cutoff points, i.e., $v_1^* = v_2^* = \cdots = v_n^*$ (resp. different cutoff points). Denote by $v_s = (v^s, \ldots, v^s)$ the symmetric equilibrium.

Remark 1. It is worthwhile to mention the following facts on the cutoff points:

(1) $v_i^* > 1$ means that bidder i will never participate in the auction, no matter what his valuation is. This happens when the bidder's revenue from participating in the auction is less than c even when $v_i = 1$.
(2) When $v_i^* < v_i \leqslant 1$, bidder i will enter the auction and submit a bid $\lambda_i(v_i, v^*)$. When $v_i = v_i^*$, bidder i is indifferent between participating in the auction and holding out. For discussion convenience, we assume he enters the auction. When $v_i < v_i^*$, bidder i does not participate in the auction.
(3) $v_i^* \geqslant c$. A bidder needs at least a value c to participate in the auction.
(4) By the same reasoning as in Cao and Tian (2008a), $v_i^* \leqslant 1$ for at least one bidder i.

Note that, once a bidder enters the auction, he can observe who has also entered the auction and thus update his belief about others' valuation distributions. If we observe that bidder i participates in the auction, it can be inferred that bidder i's value is higher than or equal to v_i^*. Then, by Bayes' rule, bidder i's value is distributed on $[v_i^*, 1]$ with

$$Pr\big(\xi \leqslant v \big| v \geqslant v_i^*\big) = \frac{Pr(v_i^* \leqslant \xi < v)}{Pr(\xi \geqslant v_i^*)} = \frac{F_i(v) - F_i(v_i^*)}{1 - F_i(v_i^*)}.$$

The corresponding density function is given by $\frac{f_i(v)}{1-F_i(v_i^*)}$.

3. Homogeneous bidders

In this section we assume bidders' valuations are drawn from the same distribution function, i.e., $F_i(\cdot) = F(\cdot)$ for all i. We first consider the existence of symmetric equilibrium. For bidders using the same cutoff point v^s, the supports of their updated valuation distributions have the same lower bound when they participate in the auction. Then the minimal bids they submit should be the same. Thus when $v_i = v^s$, bidder i can win the item only when all others do not participate. At equilibrium we have

$$c = v^s F\big(v^s\big)^{n-1}.$$

Since $\rho(v) = vF(v)^{n-1} - c$ is an increasing function with $\rho(0) < 0$ and $\rho(1) > 0$, there exists a unique symmetric equilibrium. To illustrate how bidders submit bids when they face different numbers of other bidders who enter the auction, consider the following example:

Example 1. Suppose $F(v)$ is uniform on $[0, 1]$. Then by $v^s F(v^s)^{n-1} = c$, we have $v^s = \sqrt[n]{c}$. Then when $v_i \geqslant \sqrt[n]{c}$, the bidding function for i is $\lambda_i(v_i, v^*) = v_i - \frac{v_i - \sqrt[n]{c}}{1+|S_i(v^*)|}$ if $S_i(v^*) \subset I \setminus \{i\}$ is nonempty and zero if $S_i(v^*)$ is empty, where $|S_i(v^*)|$ denotes the number of elements of $S_i(v^*)$. Otherwise, bidder i will not participate in the auction. Hence, the unique symmetric equilibrium is $(\sqrt[n]{c}, \sqrt[n]{c}, \ldots, \sqrt[n]{c})$ and the bidding function is given by

$$b_i(v_i, v^*) = \begin{cases} \lambda_i(v_i, v^*) & 1 \geqslant v_i \geqslant \sqrt[n]{c}, \\ No & v_i < \sqrt[n]{c}, \end{cases}$$

where

$$\lambda_i(v_i, v^*) = \begin{cases} 0 & \text{if } S_i(v^*) \text{ is empty}, \\ v_i - \frac{v_i - \sqrt[n]{c}}{1+|S_i(v^*)|} & \text{if } S_i(v^*) \subset I \setminus \{i\} \text{ is nonempty}. \end{cases}$$

Now we consider the existence of asymmetric equilibria. Suppose there are only two different cutoff points used by bidders. Bidders $i = 1, \ldots, m$ use v_1^* and bidders $j = m+1, \ldots, n$ use v_2^* as the cutoff point. Without loss of generality, we assume $v_1^* < v_2^*$. By Remark 1, we must have $v_1^* \leqslant 1$. Thus we partition the bidders into two types or groups. Bidders in type 1 use v_1^* and bidders in type 2 use v_2^* as their cutoffs separately.

When bidder i in group 1 participates in the auction, his updated valuation is distributed on $[v_1^*, 1]$ with cumulative distribution function $G_i(v) = \frac{F(v)-F(v_1^*)}{1-F(v_1^*)}$, and when bidder j in group 2 participates in the auction, his updated valuation is distributed on $[v_2^*, 1]$ with cumulative distribution function $G_2(v) = \frac{F(v)-F(v_2^*)}{1-F(v_2^*)}$. The two distributions have the same upper bounds but different lower bounds. Thus if both types of bidders participate in the auction, we have an asymmetric first price auction in the sense that bidders have the same valuation distributions but different supports. To get the expected revenue at the cutoffs, we need to know how the bidders bid when both types of bidders participate in the auction.

Without loss of generality, we assume that a bidder with zero probability of winning bids his true value when he participates.[7] Then, by Maskin and Riley (2003), there is a unique optimal bidding strategy, which is characterized in the following lemma:

Lemma 1. *Suppose that both k_1 bidders in type 1 whose values are distributed on the interval $[v_1^*, 1]$ with cumulative distribution function $G_1(v) = \frac{F(v)-F(v_1^*)}{1-F(v_1^*)}$ and k_2 bidders in type 2 whose values are distributed on the interval $[v_2^*, 1]$ with cumulative distribution function $G_2(v) = \frac{F(v)-F(v_2^*)}{1-F(v_2^*)}$ participate in the auction, where $v_1^* < v_2^*$. Let $\underline{b} = \max \arg\max_b (F(b) - F(v_1^*))^{k_1} (F(b) - F(v_2^*))^{k_2-1} (v_2^* - b)$. The optimal inverse bidding functions $v_1(b)$ and $v_2(b)$ are uniquely determined by*

(1) *$v_1(b) = b$ for $v_1^* \leqslant b \leqslant \underline{b}$;*
(2) *for $\underline{b} < b \leqslant \bar{b}$, the inverse bidding functions are determined by the following differential equation system:*

$$\begin{cases} \frac{k_1 f(v_1(b)) v_1'(b)}{F(v_1(b)) - F(v_1^*)} + \frac{(k_2-1) f(v_2(b)) v_2'(b)}{F(v_2(b)) - F(v_2^*)} = \frac{1}{v_2(b) - b}, \\ \frac{(k_1-1) f(v_1(b)) v_1'(b)}{F(v_1(b)) - F(v_1^*)} + \frac{k_2 f(v_2(b)) v_2'(b)}{F(v_2(b)) - F(v_2^*)} = \frac{1}{v_1(b) - b}, \end{cases}$$

with boundary conditions $v_2(\underline{b}) = v_2^*$, $v_1(\underline{b}) = \underline{b}$ and $v_1(\bar{b}) = v_2(\bar{b}) = 1$.

Bidders in type 2 who have an advantage in distribution can benefit from the auction. Indeed, for a bidder in type 2 with any value on his support, he has a positive probability to win the auction. However, bidders in type 1, when $v_1 \in [v_1^*, \underline{b})$, have no chance to win the auction when any bidder in type 2 also submits a bid. By Lemma 1, when there are two bidders using the same cutoff participating in the auction, the bidder with the value equal to the cutoff has zero expected revenue from the auction.

Remark 2. When there are k bidders in type 1 and one bidder in type 2 participating in the auction, the lower bound of the bid submitted by bidders in type 2 is $\max \arg\max_b (F(b) - F(v_1^*))^k (v_2^* - b)$.

[7] A bidder who has zero probability of winning can bid more than his value. However, this bidding strategy can be eliminated by a trembling-hand argument. Once a bidder bids above his value, he may have a positive probability to win the object which gives him a negative revenue. Bidding below his value when a bidder has zero probability of winning can also be supported in an optimal bidding strategy. However, the allocation is the same as the optimal bidding strategy where he bids his value.

Bidder $i \in \{1, \ldots, m\}$ with $v_i = v_1^*$ can win the object only when none of the others enters the auction. He bids zero when he is the only participant. Indeed, if another bidder $i' \in \{1, 2, \ldots, i'-1, i'+1, \ldots, m\}$ participates, we have $v_{i'} \geqslant v_i = v_1^*$, and thus $\lambda_{i'}(v_{i'}, v^*) \geqslant \lambda_i(v_i, v^*)$. Then bidder i gains zero revenue from the participation. When bidder $j = m+1, \ldots, n$ also enters the auction, we have $v_j \geqslant v_2^* > v_1^* = v_1$, thus bidder i will surely lose the auction.

At equilibrium, we then have

$$c = v_1^* F\left(v_1^*\right)^{m-1} F\left(v_2^*\right)^{n-m}.$$

For bidder $j \in \{m+1, \ldots, n\}$ with $v_j = v_2^*$, he bids zero and has revenue v_2^* when none of the others enters the auction. If any other bidder in type 2 enters the auction, he will lose the bid. If only $k \leqslant m$ bidders in type 1 enter the auction, the optimal bid $\underline{b}_k$ for bidder j is determined by

$$\underline{b}_k = \max \arg\max_b \left(F(b) - F\left(v_1^*\right)\right)^k \left(v_2^* - b\right).$$

The first order condition for $\underline{b}_k$ gives

$$\underline{b}_k + \frac{F(\underline{b}_k) - F(v_1^*)}{kf(\underline{b}_k)} = v_2^*.$$

$\underline{b}_k$ is chosen with probability $C_m^k F(v_1^*)^{m-k}(1 - F(v_1^*))^k$. $C_m^k = \frac{m!}{k!(m-k)!}$ is the combination number for choosing k candidates from the n items that are available. Thus, at equilibrium, we have

$$c \geqslant v_2^* F\left(v_1^*\right)^m F\left(v_2^*\right)^{n-m-1} + F\left(v_2^*\right)^{n-m-1} \sum_{k=1}^{m} C_m^k F\left(v_1^*\right)^{m-k} \left(F(\underline{b}_k) - F\left(v_1^*\right)\right)^k \left(v_2^* - \underline{b}_k\right),$$

where the first part is the expected revenue when none of the others enters the auction, which happens with probability $F(v_1^*)^m F(v_2^*)^{n-m-1}$; the second part is the expected revenue when no other bidders in type 2 participate in the auction and there are exactly $k \leqslant m$ bidders in type 1 in the auction, which happens with probability $F(v_2^*)^{n-m-1} F(v_1^*)^{m-k}$. The inequality holds whenever bidders in type 2 do not participate in the auction, i.e., $v_2^* > 1$.

Summarizing the discussion, we have the following proposition.

Proposition 1. *In an economic environment with homogeneous bidders,*

(1) *there is a unique symmetric equilibrium at which all bidders use the same cutoff point v^s that is determined by $v^s F(v^s)^{n-1} = c$;*
(2) *if $F(\cdot)$ is elastic at v^s, i.e., $F(v^s) < v^s f(v^s)$, then there exists an asymmetric equilibrium at which $m \leqslant n-1$ bidders use the cutoff point v_1^* and the others use the cutoff point v_2^* that satisfy*

$$c = v_1^* F\left(v_1^*\right)^{m-1} F\left(v_2^*\right)^{n-m},$$

$$c \geqslant v_2^* F\left(v_1^*\right)^m F\left(v_2^*\right)^{n-m-1} + F\left(v_2^*\right)^{n-m-1} \sum_{k=1}^{m} C_m^k F\left(v_1^*\right)^{m-k} \left(F(\underline{b}_k) - F\left(v_1^*\right)\right)^k \left(v_2^* - \underline{b}_k\right),$$

with equality whenever $v_2^* \leqslant 1$ and $v_1^* < v^s < v_2^*$, where

$$\underline{b}_k = \max \arg\max_b \left(F(b) - F\left(v_1^*\right)\right)^k \left(v_2^* - b\right),$$

and $C_m^k = \frac{m!}{k!(m-k)!}$;
(3) *if $F(\cdot)$ is inelastic, i.e., $F(v) \geqslant vf(v)$ for all $v \in [0, 1]$, there exists no asymmetric equilibrium.*

In words, when $F(\cdot)$ is elastic at the symmetric equilibrium, if the bidders are randomly divided into two groups, there is an equilibrium where all bidders within one group use the same cutoff that is different from the cutoff used by bidders in the other group. One implication of this result is that some bidders can coordinate by choosing a smaller cutoff threshold so that they can reduce the probability of the others' entering the auction which, in turn, can decrease the competition among the bidders who participate in the auction. However, when $F(\cdot)$ is inelastic, there is no such equilibrium.

Remark 3. When $F(\cdot)$ is strictly convex, it is elastic everywhere on its support, and therefore there exists an asymmetric equilibrium. For instance, when $c = 0.1$, $F(v) = v^2$ and $n = 2$, there is a symmetric equilibrium $(0.466, 0.466)$ and an asymmetric equilibrium $(0.141, 0.842)$.

To get some intuition for the existence of asymmetric equilibria when $F(\cdot)$ is strictly convex, consider the case when $n=2$ and suppose initially both bidders use the symmetric equilibrium v^s. Now bidder 1 decreases his cutoff slightly to v_1^1. For bidder 2 with $v_2 = v^s$, his expected payoff is $v^s F(v_1^1) + (v^s - \underline{b}_1^1)(F(\underline{b}_1^1) - F(v_1^1))$, where $\underline{b}_1^1 = \max\arg\max_b (F(b) - F(v_1^1))(v^s - b)$ with $v_1^1 < \underline{b}_1^1 < v^s$. Rewrite bidder 2's expected revenue as

$$v_s F\big(\underline{b}_1^1\big) - \underline{b}_1^1 F\big(\underline{b}_1^1\big) + \underline{b}_1^1 F\big(v_1^1\big),$$

which is less than $v^s F(v^s) + (F(v^s) - v_1^1)F(v_1^1) < c$ by noting that $v_1^1 < \underline{b}_1^1 < v^s$ and $F(\cdot)$ is strictly convex. Thus bidder 2's expected revenue is less than c when $v_2 = v^s$ and then he will adjust his cutoff to $v_2^1 > v^s$. Given that bidder 2 adjusts his cutoff to v_2^1, bidder 1 will adjust his cutoff to $v_1^2 < v^s$. Repeating this best-reply adjustment process, it will converge to an asymmetric equilibrium (v_1^*, v_2^*).

Remark 4. Further remarks can be given as follows:

(1) When the lower bound of support is positive, $F(\cdot)$ may be elastic at v^s even if $F(\cdot)$ is concave on its support. Thus we may have an asymmetric equilibrium.
(2) As $c \to 0$, one can check that both symmetric equilibrium and asymmetric equilibrium (if it exists) go to zero. Thus when $c = 0$, all bidders participate in the auction.
(3) Similar to Cao and Tian (2008a, 2008b), one can study equilibrium properties when bidders may have differentiated participation costs or both values and participation costs are private information.

It may be remarked that asymmetric equilibria inevitably lead to inefficient allocation in first price auctions with participation costs. Indeed, like second price auctions with participation costs, they are not efficient because the object may not be allocated to the bidder with the highest valuation when bidders use different cutoff points. However, unlike second price auctions with participation costs that are weakly efficient (Miralles, 2008), first price auctions are not weakly efficient in the sense that the bidder who wins the object may not have the highest valuation among those who participate. To see this, suppose $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ are the equilibrium bidding functions for any two bidders who use different cutoff points. We assume that $\lambda_1(v) < \lambda_2(v)$. Then, by the continuity of the functions, $\lambda_1(v+\epsilon) < \lambda_2(v-\epsilon)$ when ϵ is sufficiently small. Thus bidder 2 will win the object even though he has a lower valuation. In conclusion, we have the following proposition:

Proposition 2. *First price auctions with participation costs are not efficient and not even weakly efficient at asymmetric equilibrium.*

In reality, we can expect that if the auction runs repeatedly, bidders may use asymmetric equilibria at earlier periods while using the symmetric equilibrium at later periods. We now investigate the welfare effect of participation costs on sellers in the case of a one-shot game, focusing on the symmetric equilibrium.[8] When bidders use the same threshold and participate in the auction, the optimal bidding function is unique, which is symmetric and monotonically increasing, given by

$$\lambda(v_i, v_s) = v_i - \frac{\int_{v^s}^{v_i} (F(y) - F(v^s))^{k-1}\, dy}{(F(v_i) - F(v^s))^{k-1}}$$

when there are $k \geqslant 2$ participants. Thus the seller's expected revenue from the auction is

$$R = \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} \int_{v^s}^{1} \left(v_i - \frac{\int_{v^s}^{v_i} (F(y) - F(v^s))^{k-1}\, dy}{(F(v_i) - F(v^s))^{k-1}} \right) k\big(F(v_i) - F\big(v^s\big)\big)^{k-1} f(v_i)\, dv_i,$$

and consequently, we have

$$R = n(n-1) \int_{v^s}^{1} \big(1 - F(x)\big) x f(x) F(x)^{n-2}\, dx \tag{1}$$

with integration by parts and changing the order of the integration in the double integrals.[9]

There are several effects of increasing the magnitude of participation costs. First, as c increases, the probability to have k participants decreases. Secondly, participants bid more aggressively in that to win the auction, a player has to bid the expected value of the highest among his opponents with values between v^s and 1. Lower participation reduces the expected revenue while more aggressive bidding increases it. From Eq. (1), we have the following result:

[8] The welfare analysis for the case of asymmetric equilibrium is much more complicated. Letting the bidders know the number of other bidders who submit bids may have different welfare implications for the sellers. We leave the welfare analysis at the asymmetric equilibrium for future research.

[9] See details in Appendix A.

Proposition 3. *At the symmetric equilibrium, the seller's expected revenue decreases as the participation cost c increases.*

One implication of the above proposition is that in reality, the seller may give the potential bidders some subsidies to encourage them to participate in the auction to increase the expected revenue.

Remark 5. The result in Proposition 3 is opposite to the result with a reserve price r. It is always optimal to set $r > 0$. Reserve price makes the seller run a risk that the object remains unsold. However, unlike participation cost, it ensures the seller a minimal revenue r so long as there is at least one bidder submitting a bid.

Remark 6. When participation costs are part of the seller's revenue, like the entry fee, the conclusion in Proposition 3 no longer holds. In this case, the seller's expected revenue is

$$n(n-1)\int_{v^s}^{1}\big(1-F(x)\big)xf(x)F(x)^{n-2}\,dx + nc\big(1-F\big(v^s\big)\big),$$

which is equivalent to

$$n(n-1)\int_{v^s}^{1}\big(1-F(x)\big)xf(x)F(x)^{n-2}\,dx + nv^sF\big(v^s\big)^{n-1}\big(1-F\big(v^s\big)\big).$$

First order condition $v^s f(v^s) = 1 - F(v^s)$ determines the optimal entry fee from the perspective of the seller.

Remark 7. Menezes and Monteiro (2000) also consider first price auctions with participation costs, but they adopt a different specification on information structure. A bidder does not know who else is in the auction when he is to submit a bid. Besides, they only focus on the symmetric equilibrium at which all bidders use the same cutoff point (which is equal to v^s) and submit bids via the same bidding function. They mainly focus on comparing the revenues from first price auctions and second price auctions and investigate the effect of the number of potential bidders on the seller's revenue. Within their framework, when a bidder decides to participate in the auction, he will bid as if all others are in the auction since he cannot observe any other's entrance behavior and the bidding function is given by

$$\lambda^*(v_i, v_s) = \frac{\int_{v^s}^{v_i}(n-1)yF(y)^{n-2}f(y)\,dy}{F(v)^{n-1}}$$

with $v_i \geqslant v^s$, and consequently the expected revenue is given by

$$\widetilde{R} = \int_{v^s}^{1}\lambda^*(v_i, v_s)nF^{n-1}(x)f(x)\,dx,$$

which can be shown as equivalent to (1). Thus at symmetric equilibrium, letting the bidders observe or not who else participates gives the seller the same expected revenue.

4. Heterogeneous bidders

Now consider the case where we have n_1 strong bidders with value distribution $F_1(\cdot)$ and n_2 weak bidders with value distribution $F_2(\cdot)$ so that $F_1(v) < F_2(v)$ for all $v \in (0,1)$. The total number of bidders is $n = n_1 + n_2$. We concentrate on type-symmetric equilibrium at which all strong (resp. weak) bidders use the same cutoff point.

We first assume, provisionally, that the cutoff points v_1^* and v_2^* satisfy $v_1^* < v_2^*$. Then for a strong bidder i with $v_i = v_1^*$, he can win the object only when all the other strong and weak bidders do not participate in the auction. (If any strong bidder i' enters the auction, he must have a value greater than v_1^* and thus bids higher than bidder i; or if any weak bidder j enters, then it must be the case that $v_j \geqslant v_2^* > v_1^*$. As shown in the previous section, bidder i will lose the item for sure.) Thus, at equilibrium we have

$$c = v_1^*F_1\big(v_1^*\big)^{n_1-1}F_2\big(v_2^*\big)^{n_2}.$$

For a weak bidder j with $v_j = v_2^*$, we have the following three cases:

Case 1: All the other bidders do not enter the auction. Then bidder j bids zero and gains a surplus of v_2^*. The probability of this event is $F_1(v_1^*)^{n_1}F_2(v_2^*)^{n_2-1}$. In this case the expected revenue for bidder j is $v_2^*F_1(v_1^*)^{n_1}F_2(v_2^*)^{n_2-1}$.

Case 2: At least another weak bidder enters. Then bidder j will lose the auction, deriving zero revenue from participating.

Case 3: None of the other weak bidders enters and there are $k \in \{1, 2, \ldots, n_1\}$ strong bidders participating in the auction. In this case bidder j with value v_2^* will submit a bid

$$\underline{b}_k = \max \arg\max_b \left[\left(F_1(b) - F_1\left(v_1^*\right)\right)^k \left(v_2^* - b\right) \right].$$

The first order condition for $\underline{b}_k$ gives

$$\underline{b}_k + \frac{F_1(\underline{b}_k) - F_1(v_1^*)}{k f_1(\underline{b}_k)} = v_2^*.$$

The probability of this event is $C_{n_1}^k F_1(v_1^*)^{n_1-k}(1 - F_1(v_1^*))^k$. The expected revenue in this case is $C_{n_1}^k F_1(v_1^*)^{n_1-k} \times F_2(v_2^*)^{n_2-1}(F_1(\underline{b}_k) - F_1(v_1^*))^k(v_2^* - \underline{b}_k)$.

Then at equilibrium we have

$$c \geqslant v_2^* F_1\left(v_1^*\right)^{n_1} F_2\left(v_2^*\right)^{n_2-1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1\left(v_1^*\right)^{n_1-k} F_2\left(v_2^*\right)^{n_2-1} \left(F_1(\underline{b}_k) - F_1\left(v_1^*\right)\right)^k \left(v_2^* - \underline{b}_k\right).$$

Proposition 4. *When $F_1(v) < F_2(v)$ for all $v \in (0, 1)$, there always exists a type-symmetric equilibrium at which $v_1^* < v_2^*$. Further, the type-symmetric equilibrium $v_1^* < v_2^*$ is unique when both distributions are inelastic.*

Similarly for the case where $v_1^* \geqslant v_2^*$, at equilibrium we have

$$c = v_2^* F_2\left(v_2^*\right)^{n_2-1} F_1\left(v_1^*\right)^{n_1},$$

and

$$c \geqslant v_1^* F_2\left(v_2^*\right)^{n_2} F_1\left(v_1^*\right)^{n_1-1} + \sum_{k=1}^{n_2} C_{n_2}^k F_2\left(v_2^*\right)^{n_2-k} F_1\left(v_1^*\right)^{n_1-1} \left(F_2(\underline{b}_k) - F_2\left(v_2^*\right)\right)^k \left(v_1^* - \underline{b}_k\right),$$

where the first part on the right side of the inequality is the expected revenue when none of the others (no matter they are strong or weak bidders) participates in the auction. The second part is the expected revenue when at least one weak bidder participates and no other strong bidders participate.

Proposition 5. *Suppose $F_1(v) < F_2(v)$ for all $v \in (0, 1)$. In the heterogeneous economy involving any number of bidders,*

(1) *if $F_2(\cdot)$ is concave, there is no type-symmetric equilibrium with $v_2^* \leqslant v_1^*$;*
(2) *if $F_2(\cdot)$ is strictly convex, there exists $c^* < 1$ such that there exists a type-symmetric equilibrium with $v_2^* \leqslant v_1^*$ for all $c > c^*$.*

This result indicates that, when participation cost is sufficiently large, strong bidders may choose a higher cutoff point. The intuition behind this is that, when c is sufficiently large and a weak bidder is more likely to have a higher valuation, the expected revenue of a strong bidder from entering the auction is low. Strong bidders' advantage in valuations is attenuated by weak bidders' value distributions and the high participation cost.

5. Conclusion

This paper investigates the nature of Bayesian–Nash equilibria of sealed-bid first price auctions with participation costs. Bidders use cutoff strategies in which each bidder participates in the auction if and only if his value is greater than or equal to his cutoff point. Once a bidder participates in the auction, the bidding strategy depends on the valuation distributions and cutoff points of other bidders.

When bidders are ex-ante homogeneous with the same valuation distribution, there exists a unique symmetric equilibrium at which all bidders use the same cutoff to enter the auction and there may also exist an asymmetric equilibrium. In particular, there is no asymmetric equilibrium when $F(\cdot)$ is inelastic, while there exists an asymmetric equilibrium when $F(\cdot)$ is elastic at the symmetric equilibrium. When bidders can be ranked by their valuation distributions, we find that there always exists an equilibrium at which the strong bidders use a smaller cutoff. However, the opposite can be obtained when the participation cost is sufficiently large and weak bidders' valuation distributions are strictly convex.

In the presence of participation costs, not all bidders participate in the auction and the seller's expected revenue decreases as the participation costs increase. Then, it may be profitable for the seller to subsidize the buyers to encourage their participation in the auction. How to implement this should be a potentially interesting question which will be left for future research.

Appendix A. Proofs

Proof of Lemma 1. Denote the inverses of the bidding function as $v_1(b)$ with support $[\underline{b}_1, \bar{b}_1]$ and $v_2(b)$ with support $[\underline{b}_2, \bar{b}_2]$. Let $(\underline{b}, \bar{b}]$ be the range in which a bidder has a positive probability to win the object if he participates in the auction. First from Maskin and Riley (2003), the upper endpoint of the support of the valuation distributions is the same for all bidders and thus the upper endpoints of the supports of all buyers' equilibrium bid distributions are the same. Thus $\bar{b}_1 = \bar{b}_2 = \bar{b}$ and $v_1(\bar{b}) = v_2(\bar{b}) = 1$.

Also from Maskin and Riley (2003), we have $\underline{b}_1 < \underline{b}_2 = \underline{b}$, which indicates that the minimum bid of a bidder in type 1 is always less than that of bidders in type 2 since bidders in type 2 have an advantage in valuation distribution.

Below $\underline{b}$, type 1 bidder has no chance to win the auction and bids his true value, so $v_1(\underline{b}) = \underline{b}$. For bidders in type 2, when $v_2 = v_2(\underline{b}) = v_2^*$, bidding $\underline{b}$ is their best strategy. Again, from Maskin and Riley (2003), $\underline{b} = \max\arg\max_b (F(b) - F(v_1^*))^{k_1}(F(b) - F(v_2^*))^{k_2-1}(v_2^* - b)$.

In the interval $[\underline{b}, \bar{b}]$, a bidder in type 1 bids b which is determined by the following maximization problem:

$$\max_b \left(\frac{F(v_2(b)) - F(v_2^*)}{1 - F(v_2^*)}\right)^{k_2} \left(\frac{F(v_1(b)) - F(v_1^*)}{1 - F(v_1^*)}\right)^{k_1-1} (v_1 - b).$$

Similarly, a bidder in type 2 solves the following problem:

$$\max_b \left(\frac{F(v_1(b)) - F(v_1^*)}{1 - F(v_1^*)}\right)^{k_1} \left(\frac{F(v_2(b)) - F(v_2^*)}{1 - F(v_2^*)}\right)^{k_2-1} (v_2 - b).$$

First order conditions give us

$$\begin{cases} \frac{k_1 f(v_1(b))v_1'(b)}{F(v_1(b)) - F(v_1^*)} + \frac{(k_2-1) f(v_2(b))v_2'(b)}{F(v_2(b)) - F(v_2^*)} = \frac{1}{v_2(b) - b}, \\ \frac{(k_1-1) f(v_1(b))v_1'(b)}{F(v_1(b)) - F(v_1^*)} + \frac{k_2 f(v_2(b))v_2'(b)}{F(v_2(b)) - F(v_2^*)} = \frac{1}{v_1(b) - b}. \end{cases}$$

The boundary conditions for the differential equation system are $v_2(\underline{b}) = v_2^*$, $v_1(\underline{b}) = \underline{b}$ and $v_1(\bar{b}) = v_2(\bar{b}) = 1$. □

Proof of Proposition 1. (1) The existence and uniqueness of symmetric equilibrium is obvious, thus the proof is omitted here.

(2) Suppose $F(\cdot)$ is elastic at v^s so that $F(v^s) < v^s f(v^s)$. Consider the following two equations:

$$c = xF(x)^{m-1}F(y)^{n-m},$$
$$c \geqslant yF(x)^m F(y)^{n-m-1} + F(y)^{n-m-1} \sum_{k=1}^{m} C_m^k F(x)^{m-k} \big(F(\underline{b}_k) - F(x)\big)^k (y - \underline{b}_k),$$

where $\underline{b}_k$ satisfies $\underline{b}_k + \frac{F(\underline{b}_k) - F(x)}{kf(\underline{b}_k)} = y$, x corresponds to the cutoff point used by bidders in the first group, and y corresponds to the cutoff point used by bidders in the second group. Let v^s satisfy $c = v^s F(v^s)^{m-1} F(v^s)^{n-m}$ and define $x = \phi(y)$ implicitly from $c = xF(x)^{m-1}F(y)^{n-m}$. Notice that $\phi(y)$ is continuously differentiable and $\phi(v^s) = v^s$. Since $x \leqslant y$, $x = \phi(y)$ with $y \geqslant v^s$. Then we have

$$\phi'(y) = -\frac{(n-m) f(y) xF(x)}{(F(x) + (m-1)xf(x))F(y)},$$

and thus

$$\phi'(v^s) = -\frac{(n-m) v^s f(v^s)}{F(v^s) + (m-1)v^s f(v^s)}.$$

Define

$$h(y) = F(y)^{n-m-1}\left[yF\big(\phi(y)\big)^m + \sum_{k=1}^{s} C_m^k F\big(\phi(y)\big)^{m-k}\big(F\big(\underline{b}_k(y)\big) - F\big(\phi(y)\big)\big)^k \big(y - \underline{b}_k(y)\big)\right] - c$$

with $y \geqslant v^s$. Notice that $h(y)$ is continuously differentiable and $\underline{b}_k(y) = v^s$ when $y = v^s$. So $h(v^s) = 0$. In order to obtain an asymmetric equilibrium, we only need to show that *either* there exists a $y^* \in (v^s, 1]$ such that $h(y^*) = 0$ (in which case we have $v_2^* = y^*$ and $v_1^* = h(v_2^*) < v^s$ as our asymmetric cutoff equilibrium) *or* $h(1) < 0$ (in which case $v_2^* > 1$ and $v_1^* = c$). Thus if $h(1) < 0$, it is proved.

Suppose $h(1) > 0$. Since $h(\cdot)$ is continuous with $h(v^s) = 0$ and $h(1) > 0$, when $h(y)$ is decreasing at v^s, then there exists a $y^* \in (v^s, 1]$ such that $h(y^*) = 0$. This is true when $F(\cdot)$ is elastic at v^s. Indeed,

$$h'(y) = I(y) + F(y)^{n-m-1}\left[II(y) + \sum_{k=1}^{m} C_s^k\big(III(y) + IV(y)\big)\right],$$

where

$$I(y) = (n-m-1)F(y)^{n-m-2}f(y)\left[yF\big(\phi(y)\big)^m + \sum_{k=1}^{m} F\big(\phi(y)\big)^{m-k}\big(F\big(\underline{b}_k(y)\big) - F\big(\phi(y)\big)\big)^k\big(y - \underline{b}_k(y)\big)\right],$$

$$II(y) = F\big(\phi(y)\big)^m + y.mF\big(\phi(y)\big)^{m-1}f\big(\phi(y)\big)\phi'(y),$$

$$III(y) = (m-k)F\big(\phi(y)\big)^{m-k-1}f\big(\phi(y)\big)\phi'(y)\big(F\big(\underline{b}_k(y)\big) - F\big(\phi(y)\big)\big)^k\big(y - \underline{b}_k(y)\big),$$

$$IV(y) = F\big(\phi(y)\big)^{m-k}\big[k\big[F\big(\underline{b}_k(y)\big) - F\big(\phi(y)\big)\big]^{k-1}\big(f\big(\underline{b}_k(y)\big)\underline{b}_k'(y) - f\big(\phi(y)\big)\phi'(x)\big)\big(y - \underline{b}_k(y)\big)$$
$$+ \big(F\big(\underline{b}_k(y)\big) - F\big(\phi(y)\big)\big)^k\big(1 - \underline{b}_k'(y)\big)\big].$$

When $x = y = v^s$, we have $\underline{b}_k(v^s) = v^s$. Then,

$$I\big(v^s\big) = (n-m-1)F\big(v^s\big)^{n-m-2}f\big(v^s\big)v^sF\big(v^s\big)^m = (n-m-1)F\big(v^s\big)^{n-2}v^sf\big(v^s\big),$$

$$II\big(v^s\big) = F\big(v^s\big)^m + v^s.mF\big(v^s\big)^{m-1}f\big(v^s\big)\phi'\big(v^s\big),$$

$$III\big(v^s\big) = IV\big(v^s\big) = 0$$

and thus

$$h'\big(v^s\big) = F\big(v^s\big)^{n-2}\big[(n-m-1)v^sf\big(v^s\big) + mv^sf\big(v^s\big)\phi'\big(v^s\big) + F\big(v^s\big)\big].$$

Thus, $h'(v^s) < 0$ if and only if

$$\big|\phi'\big(v^s\big)\big| = \frac{(n-m)v^sf(v^s)}{F(v^s) + (m-1)v^sf(v^s)} > \frac{(n-m-1)v^sf(v^s) + F(v^s)}{mv^sf(v^s)},$$

which is true when $F(\cdot)$ is elastic at v^s. Indeed, when $F(\cdot)$ is elastic at v^s, we have $v^sf(v^s) > F(v^s)$. So $F(v^s) + (m-1)v^sf(v^s) < mv^sf(v^s)$ and at the same time $(n-m)v^sf(v^s) > (n-m-1)v^sf(v^s) + F(v^s)$. Then if $h(1) > 0$, we have an asymmetric equilibrium at which $v_1^* < v^s < v_2^* \leqslant 1$, otherwise there is an asymmetric equilibrium at which bidders in group 2 never participate in the auction.

(3) When $F(\cdot)$ is inelastic, we prove the nonexistence of asymmetric equilibrium by way of contradiction.

First we prove that when we divide the bidders into two groups, there exists no asymmetric equilibrium at which bidders in each group use the same cutoff. Suppose there is an asymmetric equilibrium with $v_1^* < v_2^*$. Then

$$c = v_1^*F\big(v_1^*\big)^{m-1}F\big(v_2^*\big)^{n-m},$$

$$c \geqslant v_2^*F\big(v_1^*\big)^mF\big(v_2^*\big)^{n-m-1} + F\big(v_2^*\big)^{n-m-1}\sum_{k=1}^{m} C_s^kF\big(v_1^*\big)^{m-k}\big(F(\underline{b}_k) - F\big(v_1^*\big)\big)^k\big(v_2^* - \underline{b}_k\big).$$

One necessary condition for the system of these equations above to be true is

$$v_1^*F\big(v_1^*\big)^{m-1}F\big(v_2^*\big)^{n-m} > v_2^*F\big(v_1^*\big)^mF\big(v_2^*\big)^{n-m-1},$$

i.e., $\frac{F(v_2^*)}{v_2^*} > \frac{F(v_1^*)}{v_1^*}$, which cannot be true when $F(\cdot)$ is inelastic and $v_2^* > v_1^*$.[10] Following the same procedures above, we can prove there is no asymmetric equilibrium at which $v_1^* > v_2^*$.

More generally, suppose there exists an asymmetric equilibrium at which the cutoffs used by all bidders satisfy $v_1^* \leqslant v_2^* \leqslant \cdots \leqslant v_i^* < v_j^* \leqslant v_{j+1}^* \leqslant \cdots \leqslant v_n^*$. Note that, when $v_i = v_i^*$, bidder i will lose the bid when a bidder with a higher cutoff enters the auction. Thus for bidder i, at equilibrium we have

$$c = v_i^*\prod_{k\neq i} F\big(v_k^*\big) + \pi_i^1 + \pi_i^2 + \cdots + \pi_i^{i-1},$$

where the first part on the right is the expected revenue when none of the other bidders enters the auction, and π_i^k $(k = 1, 2, \ldots, i-1)$ is bidder i's expected revenue when there are only k bidders (the possible cases for this are C_{i-1}^k) with cutoffs less than v_i^* entering the auction.

[10] This is true by noting that $\frac{F(\cdot)}{v}$ is nonincreasing when it is inelastic.

For bidder j with $v_j = v_j^*$, at equilibrium

$$c \geqslant v_j^* \prod_{k \neq j} F\left(v_k^*\right) + \pi_j^1 + \pi_j^2 + \cdots + \pi_j^{i-1} + \pi_{ji},$$

with equality whenever $v_j \leqslant 1$, where the first part on the right is the expected revenue when none of the other bidders enters the auction. π_j^k $(k = 1, 2, \ldots, i-1)$ is bidder j's expected revenue when there are only k bidders with cutoffs less than v_i^* entering the auction. π_{ji} is j's expected profit when bidder i also participates in the auction and no bidder with a cutoff higher than v_j^* enters.

Note that $\pi_i^k < \pi_j^k$ since a bidder with a higher value has more expected revenue given the same rivals. Also note that $\pi_{ij} > 0$. Thus, for the above two equations to be true simultaneously, we must have

$$v_i^* \prod_{k \neq i} F\left(v_k^*\right) > v_j^* \prod_{k \neq j} F\left(v_k^*\right)$$

or equivalently $\frac{F(v_j^*)}{v_j^*} > \frac{F(v_i^*)}{v_i^*}$ which cannot be true when $F(\cdot)$ is inelastic and $v_j^* > v_i^*$. □

Proof of Eq. (1). Rewrite

$$R = \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} \int_{v^s}^{1} \left(v_i - \frac{\int_{v^s}^{v_i} (F(y) - F(v^s))^{k-1}\, dy}{(F(v_i) - F(v^s))^{k-1}} \right) k\left(F(v_i) - F\left(v^s\right)\right)^{k-1} f(v_i)\, dv_i$$

as

$$R = \int_{v^s}^{1} \left\{ v_i \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} k\left(F(v_i) - F\left(v^s\right)\right)^{k-1} - \int_{v^s}^{v_i} \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} k\left(F(y) - F\left(v^s\right)\right)^{k-1} dy \right\} dF(v_i).$$

Integrating by parts for $\int_{v^s}^{v_i} \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} k(F(y) - F(v^s))^{k-1}\, dy$ and making simplifications, we have

$$\begin{aligned}
R &= \int_{v^s}^{1} \int_{v^s}^{v_i} \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} k(k-1)\left(F(y) - F\left(v^s\right)\right)^{k-2} y\, dy\, f(v_i)\, dv_i \\
&= \int_{v^s}^{1} \int_{v^s}^{v_i} \sum_{k=2}^{n} C_{n-2}^{k-2} n(n-1) F(v_s)^{n-k} \left(F(y) - F\left(v^s\right)\right)^{k-2} y\, dy\, f(v_i)\, dv_i \\
&= n(n-1) \int_{v^s}^{1} \int_{v^s}^{v_i} F(y)^{n-2} y\, dy\, f(v_i)\, dv_i \\
&= n(n-1) \int_{v^s}^{1} \left(1 - F(x)\right) x f(x) F(x)^{n-2}\, dx,
\end{aligned}$$

where the second line comes from the fact that $C_n^k k(k-1) = n(n-1) C_{n-2}^{k-2}$ and the last line comes from changing the order of integration in the double integral. □

Proof of Proposition 3. From Eq. (1), the seller's expected revenue is a decreasing function of v^s. Thus as c increases, v^s increases and R decreases accordingly. □

Proof of Proposition 4. Now consider the following two equations:

$$\begin{aligned}
&c = x F_1(x)^{n_1 - 1} F_2(y)^{n_2}, \\
&c \geqslant y F_1(x)^{n_1} F_2(y)^{n_2 - 1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1 - k} F_2(y)^{n_2 - 1} \left(F_1(\underline{b}_k) - F_1(x)\right)^k (y - \underline{b}_k),
\end{aligned}$$

with $c \leqslant x < y \leqslant 1$, where x corresponds to the cutoff point used by the strong bidders and y corresponds to the cutoff point used by the weak bidders. Let v_1^s satisfy $v_1^s F_1(v_1^s)^{n_1 - 1} F_2(v_1^s)^{n_2} = c$. Note that $\theta(v_1^s) = v_1^s F_1(v_1^s)^{n_1 - 1} F_2(v_1^s)^{n_2}$ is an

increasing function with $\theta(1) = 1 > c$, so we have $v_1^s < 1$. For $y \geqslant v_1^s$, define $x = \phi(y)$ from $c = xF_1(x)^{n_1-1}F_2(y)^{n_2}$. Then x is a decreasing function of y and $\phi(v_1^s) = v_1^s$. Now let

$$h(y) = yF_1\big(\phi(y)\big)^{n_1}F_2(y)^{n_2-1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1\big(\phi(y)\big)^{n_1-k}F_2(y)^{n_2-1}\big(F_1\big(\underline{b}_k(y)\big) - F_1\big(\phi(y)\big)\big)^k\big(y - \underline{b}_k(y)\big) - c.$$

Then $h(y)$ is a continuous function of $y \geqslant v_1^s$. The remainder of the proof is based on the following two lemmas:

Lemma 2. *There always exists a type-symmetric equilibrium with $v_1^* < v_2^*$.*

Proof. Note that $x \leqslant \underline{b}_k \leqslant y$. When $y = v_1^s$, we have $\underline{b}_k = v_1^s$. Then

$$h\big(v_1^s\big) = v_1^s F_1\big(v_1^s\big)^{n_1}F_2\big(v_2^s\big)^{n_2-1} - c < v_1^s F_1\big(v_1^s\big)^{n_1-1}F_2\big(v_1^s\big)^{n_2} - c = 0$$

since $F_1(v_1^s) < F_2(v_2^s)$ by assumption. We also have

$$h(1) = F_1\big(\phi(1)\big)^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1\big(\phi(1)\big)^{n_1-k}\big(F_1\big(\underline{b}_k(1)\big) - F_1\big(\phi(1)\big)\big)^k\big(1 - \underline{b}_k(1)\big) - c.$$

Now if $h(1) \geqslant 0$, by the mean value theorem, there exists a $y = v_2^* \in (v_1^s, 1]$ such that $h(v_2^*) = 0$ so that there is an equilibrium at which $v_1^* = \phi(v_2^*) < v_1^s < v_2^* \leqslant 1$. Otherwise if $h(1) < 0$, then there is an equilibrium at which $v_1^* = \phi(1) < 1$ and $v_2^* > 1$, i.e., weak bidders never participate in the auction. □

Lemma 3. *When $F_1(\cdot)$ and $F_2(\cdot)$ are both inelastic and $F_1(v) < F_2(v)$ for all $v \in (0, 1)$, there exists a unique type-symmetric equilibrium with $v_1^* < v_2^*$.*

Proof. Suppose $y \leqslant 1$. Substituting $c = xF_1(x)^{n_1-1}F_2(y)^{n_2}$ into

$$c = yF_1(x)^{n_1}F_2(y)^{n_2-1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}F_2(y)^{n_2-1}\big(F_1(\underline{b}_k) - F_1(x)\big)^k(y - \underline{b}_k)$$

and making simplifications, we have

$$yF_1(x)^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}\big(F_1(\underline{b}_k) - F_1(x)\big)^k(y - \underline{b}_k) - xF_1(x)^{n_1-1}F_2(y) = 0. \tag{2}$$

We claim that the above equation implicitly defines x as a strictly increasing function of y. Consequently, it *either* has a unique intersection with $x = \phi(y)$ (which is strictly decreasing), *or* does not intersect with $x = \phi(y)$, in which case the unique equilibrium is given by $x = \phi(1)$ and $y > 1$ (weak bidders never participate).

To see this, taking derivatives with respect to y (notice that $\underline{b}_k$ is also a function of y) on both sides of the above equation, we have

$$\begin{aligned}
0 &= F_1(x)^{n_1} + n_1 yF_1(x)^{n_1-1}f_1(x)\frac{dx}{dy} - xF_1(x)^{n_1-1}f_2(y) - F_2(y)\big(F_1(x)^{n_1-1} + (n_1-1)xf_1(x)F_1(x)^{n_1-2}\big)\frac{dx}{dy} \\
&\quad + \sum_{k=1}^{n_1} C_{n_1}^k \bigg\{(n_1-k)F_1(x)^{n_1-k-1}f_1(x)\big(F_1(\underline{b}_k) - F_1(x)\big)^k(y - \underline{b}_k)\frac{dx}{dy} \\
&\quad + F_1(x)^{n_1-k}\big(F_1(\underline{b}_k) - F_1(x)\big)^{k-1}\bigg[\big(F_1(\underline{b}_k) - F_1(x)\big)\big(1 - \underline{b}_k'\big) + k(y - \underline{b}_k)\bigg(f(\underline{b}_k)\underline{b}_k' - f_1(x)\frac{dx}{dy}\bigg)\bigg]\bigg\},
\end{aligned}$$

where

$$\big(F_1(\underline{b}_k) - F_1(x)\big)\big(1 - \underline{b}_k'\big) + k(y - \underline{b}_k)\bigg(f(\underline{b}_k)\underline{b}_k' - f_1(x)\frac{dx}{dy}\bigg) = F_1(\underline{b}_k) - F_1(x) - k(y - \underline{b}_k)f_1(x)\frac{dx}{dy}$$

by noting that $F_1(\underline{b}_k(y)) - F_1(x) = kf_1(\underline{b}_k(y))(y - \underline{b}_k(y))$. Thus we have

$$\begin{aligned}
0 &= F_1(x)^{n_1} + n_1 yF_1(x)^{n_1-1}f_1(x)\frac{dx}{dy} - xF_1(x)^{n_1-1}f_2(y) \\
&\quad - F_2(y)\big(F_1(x)^{n_1-1} + (n_1-1)xf_1(x)F_1(x)^{n_1-2}\big)\frac{dx}{dy}
\end{aligned}$$

$$+\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)\big(F_1(\underline{b}_k)-F_1(x)\big)^{k-1}(y-\underline{b}_k)\frac{dx}{dy}\big\{n_1\big(F_1(\underline{b}_k)-F_1(x)\big)-kF_1(\underline{b}_k)\big\}$$
$$+\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}\big(F_1(\underline{b}_k)-F_1(x)\big)^k.$$

Then

$$\frac{dx}{dy}=\frac{F_1(x)^{n_1}+\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k)-F_1(x))^k - xF_1(x)^{n_1-1}f_2(y)}{-n_1yF_1(x)^{n_1-1}f_1(x)-II+F_2(y)(F_1(x)^{n_1-1}+(n_1-1)xf_1(x)F_1(x)^{n_1-2})}$$

with

$$II=\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)\big(F_1(\underline{b}_k)-F_1(x)\big)^{k-1}(y-\underline{b}_k)\big\{n_1\big(F_1(\underline{b}_k)-F_1(x)\big)-kF_1(\underline{b}_k)\big\}$$
$$=I-\alpha,$$

where

$$I=n_1\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)\big(F_1(\underline{b}_k)-F_1(x)\big)^{k}(y-\underline{b}_k),$$
$$\alpha=\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)\big(F_1(\underline{b}_k)-F_1(x)\big)^{k-1}(y-\underline{b}_k)kF_1(\underline{b}_k)>0.$$

Now we prove the denominator and numerator are strictly positive separately. First we prove the numerator is positive. From Eq. (2), we have

$$yF_1(x)^{n_1}-xF_1(x)^{n_1-1}F_2(y)=-\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}\big(F_1(\underline{b}_k)-F_1(x)\big)^k(y-\underline{b}_k).$$

When $F_2(\cdot)$ is inelastic, we have

$$F_1(x)^{n_1}+\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}\big(F_1(\underline{b}_k)-F_1(x)\big)^k - xF_1(x)^{n_1-1}f_2(y)$$
$$\geqslant F_1(x)^{n_1}+\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}\big(F_1(\underline{b}_k)-F_1(x)\big)^k - xF_1(x)^{n_1-1}\frac{F_2(y)}{y}.$$

Then,

$$yF_1(x)^{n_1}+y\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}\big(F_1(\underline{b}_k)-F_1(x)\big)^k - xF_1(x)^{n_1-1}F_2(y)$$
$$=-\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}\big(F_1(\underline{b}_k)-F_1(x)\big)^k(y-\underline{b}_k)+y\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}\big(F_1(\underline{b}_k)-F_1(x)\big)^k$$
$$=\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}\big(F_1(\underline{b}_k)-F_1(x)\big)^k\underline{b}_k>0.$$

So the numerator is positive.

We now prove the denominator is also positive. Again from (2) we have

$$\begin{aligned}-I-n_1yF_1(x)^{n_1-1}f_1(x)&=-n_1f_1(x)/F_1(x)\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}\big(F_1(\underline{b}_k)-F_1(x)\big)^k(y-\underline{b}_k)-n_1yF_1(x)^{n_1-1}f_1(x)\\&=-n_1f_1(x)/F_1(x)\big(xF_1(x)^{n_1-1}F_2(y)-yF_1(x)^{n_1}\big)-n_1yF_1(x)^{n_1-1}f_1(x)\\&=n_1f_1(x)\big(-xF_1(x)^{n_1-2}F_2(y)+yF_1(x)^{n_1-1}-yF_1(x)^{n_1-1}\big)\\&=-n_1f_1(x)xF_1(x)^{n_1-2}F_2(y).\end{aligned}$$

The denominator then becomes

$$\begin{aligned}&\alpha - n_1 f_1(x)xF_1(x)^{n_1-2}F_2(y) + F_2(y)\big(F_1(x)^{n_1-1} + (n_1-1)xf_1(x)F_1(x)^{n_1-2}\big)\\&\quad = \alpha + F_2(y)\big(F_1(x)^{n_1-1} - xf_1(x)F_1(x)^{n_1-2}\big) > 0\end{aligned}$$

since $\alpha > 0$ and $F_1(x) \geqslant xf_1(x)$ by the inelasticity of $F_1(\cdot)$. Thus we have $\frac{dx}{dy} > 0$. The uniqueness of the equilibrium is established. □

Proof of Proposition 5. We first prove that when $F_2(\cdot)$ is inelastic, there is no type-symmetric equilibrium with $v_1^* \geqslant v_2^*$. Suppose not. Then a necessary condition is

$$v_2^* F_2\big(v_2^*\big)^{n_2-1} F_1\big(v_1^*\big)^{n_1} \geqslant v_1^* F_2\big(v_2^*\big)^{n_2} F_1\big(v_1^*\big)^{n_1-1},$$

or

$$\frac{F_1(v_1^*)}{v_1^*} \geqslant \frac{F_2(v_2^*)}{v_2^*}.$$

Note that when $F_2(\cdot)$ is inelastic and $v_1^* \geqslant v_2^*$, we have $\frac{F_2(v_2^*)}{v_2^*} \geqslant \frac{F_2(v_1^*)}{v_1^*}$, and thus $\frac{F_1(v_1^*)}{v_1^*} \geqslant \frac{F_2(v_1^*)}{v_1^*}$ which cannot be true since $F_2(v_1^*) > F_1(v_1^*)$ by assumption.

We now show that when $F_2(\cdot)$ is strictly convex, there exists an equilibrium at which $v_1^* \geqslant v_2^*$ when c is sufficiently large.

Let v_2^s satisfy

$$c = v_2^s F_2\big(v_2^s\big)^{n_2-1} F_1\big(v_2^s\big)^{n_1}$$

and v_1^s satisfy

$$c = v_1^s F_2\big(v_1^s\big)^{n_2-1}.$$

For $y \in [v_1^s, v_2^s]$, define $x = \phi(y)$ from $c = yF_2(y)^{n_2-1}F_1(x)^{n_1}$. Then x is a decreasing function of y satisfying $\phi(v_2^s) = v_2^s$ and $\phi(v_1^s) = 1$. Now define

$$h(y) = \phi(y)F_2(y)^{n_2}F_1\big(\phi(y)\big)^{n_1-1} + \sum_{k=1}^{n_2} C_{n_2}^k F_2(y)^{n_2-k} F_1\big(\phi(y)\big)^{n_1-1}\big(F_2\big(\underline{b}_k(y)\big) - F_2(y)\big)^k\big(\phi(y) - \underline{b}_k(y)\big) - c.$$

There is the required equilibrium if $\exists y \in [v_1^s, v_2^s]$ with $h(y) = 0$. Note that

$$h\big(v_2^s\big) = v_2^s F_2\big(v_2^s\big)^{n_2} F_1\big(v_2^s\big)^{n_1-1} - c > v_2^s F_2\big(v_2^s\big)^{n_2-1} F_1\big(v_2^s\big)^{n_1} - c = 0$$

since $F_2(v_2^s) > F_1(v_1^s)$ by assumption. Since $h(y)$ is continuous, we only need

$$h\big(v_1^s\big) = F_2\big(v_1^s\big)^{n_2} + \sum_{k=1}^{n_2} C_{n_2}^k F_2\big(v_1^s\big)^{n_2-k}\big(F_2\big(\underline{b}_k\big(v_1^s\big)\big) - F_2\big(v_1^s\big)\big)^k\big(1 - \underline{b}_k\big(v_1^s\big)\big) - c < 0.$$

From the definition we know v_1^s is a monotonically increasing function of c, denoted by $v_1^s(c)$. It is obvious that $v_1^s(1) = 1$ and $v_1^{s\prime}(c) = \frac{F_2(v_1^s)v_1^s}{c(F_2(v_1^s)+(n_2-1)f_2(v_1^s))}$, so we have $v_1^{s\prime}(1) = \frac{1}{1+(n_2-1)f_2(1)}$. It then suffices to show

$$\hat{h}(c) = F_2\big(v_1^s(c)\big)^{n_2} + \sum_{k=1}^{n_2} C_{n_2}^k F_2\big(v_1^s(c)\big)^{n_2-k}\big(F_2\big(\underline{b}_k\big(v_1^s(c)\big)\big) - F_2\big(v_1^s(c)\big)\big)^k\big(1 - \underline{b}_k\big(v_1^s(c)\big)\big) - c < 0$$

for some c. Note that we have $\hat{h}(1) = 0$ and

$$\begin{aligned}\hat{h}'(c) &= n_2 F_2\big(v_1^s(c)\big)^{n_1-1} f_2\big(v_1^s(c)\big)v_1^{s\prime}(c)\\&\quad + \sum_{k=1}^{n_2} C_{n_2}^k\big[(n_2-k)F_2\big(v_1^s(c)\big)^{n_2-k-1} f_2\big(v_1^s(c)\big)v_1^{s\prime}(c)\big(F_2\big(\underline{b}_k\big(v_1^s(c)\big)\big) - F_2\big(v_1^s(c)\big)\big)^k\big(1 - \underline{b}_k\big(v_1^s(c)\big)\big)\\&\quad + F_2\big(v_1^s(c)\big)^{n_2-k}\big(k\big(F_2\big(\underline{b}_k\big(v_1^s(c)\big)\big) - F_2\big(v_1^s(c)\big)\big)^{k-1}\big(1 - \underline{b}_k\big(v_1^s(c)\big)\big)\big(f_2\big(\underline{b}_k\big(v_1^s(c)\big)\big)\underline{b}_k'\big(v_1^s(c)\big)\\&\quad - f_2\big(v_1^s(c)\big)\big)\big)v_1^{s\prime}(c) - \big(F_2\big(\underline{b}_k\big(v_1^s\big)\big) - F_2\big(v_1^s\big)\big)^k\underline{b}_k'\big(v_1^s(c)\big)\big] - 1.\end{aligned}$$

As $c \to 1$, we have $\underline{b}_k(v_1^s(1)) \to 1$, and thus

$$\hat{h}'(1) = n_2 f_2(1) v_1^{s\prime}(1) - 1 = \frac{f_2(1) - 1}{1 + (n_2 - 1) f_2(1)} > 0$$

when $F_2(\cdot)$ is strictly convex. Hence, $\exists c^* < 1$ s.t. $\hat{h}(c) < 0$ whenever $c > c^*$. $\square$

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