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NASH-IMPLEMENTATION OF THE LINDAHL CORRESPONDENCE WITH DECREASING RETURNS TO SCALE TECHNOLOGIES*

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This paper considers the problem of designing "well-behaved" mechanisms whose Nash allocations coincide with Lindahl allocations in the presence of decreasing returns to scale (DRS) technologies. The mechanism presented here is individually feasible, balanced, continuous, and differentiable around Nash equilibria. Further, the mechanism has a message space of minimal dimension. Moreover we show that, in contrast to mechanisms dealing with constant returns to scale, an important characteristic of mechanisms implementing the Lindahl correspondence with DRS technologies is that at least one individual's personalized prices depend on his own messages, provided the mechanisms are balanced and smooth.

1. INTRODUCTION

Since Groves and Ledyard (1977) first proposed a mechanism to solve the "free rider" problem for public goods economies, there have been many mechanisms which implement the (constrained) Lindahl or Walrasian correspondence such as those in Hurwicz (1979a), Schmeidler (1980), Walker (1981), Hurwicz, Maskin, and Postlewaite (1984), Hurwicz (1986a), Groves and Ledyard (1987), Nakamura (1989), Tian (1989, 1990, 1991, 1992, 1993), and Tian and Li (1991) among others. However most mechanisms for public goods economies assume that the technology for producing public goods displays constant returns to scale (CRS). Although it is natural for economists to design "well-behaved" mechanisms for the simple case of CRS first, the "well-behaved" mechanisms for the more general case of decreasing returns to scale (DRS) are also quite desirable.

This paper gives a "well-behaved" mechanism whose Nash allocations coincide with Lindahl allocations for public goods economies with DRS technologies. The mechanism presented is completely feasible (individually feasible and balanced),[2] continuous, differentiable around Nash equilibria, and almost everywhere differentiable. This allows the differential approach to be used in finding Nash equilibrium points. Moreover, the mechanism uses a message space of minimal dimension. The results are shown to be true for any economy with at least three agents whose preferences satisfy the usual regularity assumptions (strict monotonicity, convexity, and differentiability) and in addition satisfies the condition that any

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[2] That is, the resulting allocations are in the consumption space and are balanced for all messages.

interior allocation is strictly preferred to a boundary allocation.[3] Briefly, the mechanism proceeds as follows: The reported messages are used to determine the personal price paid by each consumer of a public good, and these prices are summed to determine the amount received by the producer for each unit of that public good supplied. Next, find the profit maximizing level of output for the public goods subject to global material constraint. Although the profit maximizing vector may not be consistent with individual budget balance, an adjustment is then made so that the individual budget constraint is satisfied for every consumer, even out of equilibrium; by selecting the public goods vector that is closest to the profit maximizing one. We will have profit maximization at equilibrium.[4] The outcome for the private good is then determined such that the budget constraint of each agent holds with equality. This mechanism has all the desirable properties mentioned above. Further, under conditions imposed in the paper, we show that the set of Nash allocations coincides with the set of Lindahl allocations.

It is important to have a mechanism which implements the Lindahl correspondence for the following reasons. First, Lindahl allocations result in Pareto efficient allocations. Much of the literature on the design and evaluation of allocation mechanisms has adopted the Pareto-efficiency correspondence as an ideal for performance comparisons. Economists desire Pareto efficiency as a basic social goal partly because of the known and satisfactory efficiency properties of competitive markets and partly because of the acceptability of the concept of Pareto-efficiency as a minimal welfare criteria. Second, it results in individually rational allocations in the usual sense that they are not worse than the initial endowment. Third, the concept of Lindahl equilibrium is very similar to the conventional concept of Walrasian equilibrium with attention to the well-known duality which reverses the role of prices and quantities between private and public goods, and between Walrasian and Lindahl allocations. In the Walrasian case, prices must be equalized while quantities are individualized; in the Lindahl case the quantities of the public good must be the same for everyone, while prices charged for the public good are individualized. In addition, the concepts of Walrasian and Lindahl equilibria are both relevant to private-ownership economies.

Including DRS technology in a Lindahl correspondence is interesting for two reasons. First, DRS technology is more common than CRS for producing a public good. The causes of DRS in the production of public goods are largely the same as for private goods. For example, consider the production of uranium for nuclear weapons that are used for national defense, it may be difficult or impossible to double all inputs, in that increasing the scale of operation would necessarily force the miners to operate on less rich veins of rock. DRS could also arise if some inputs

[3] This assumption cannot be dropped for implementation of the Lindahl correspondence by a completely feasible mechanism, otherwise the Lindahl correspondence violates Maskin's (1977) monotonicity condition which is a necessary condition for Nash implementation. For details, see Tian (1988).

[4] We may not have profit maximization out of equilibrium. This may be a serious defect of the mechanism. However, it is true that we do not have profit maximization out of equilibrium in a real market economy, either. Many things occur to frustrate the aspirations of producers. But our mechanism, like a firm in a real market economy, does select a production *plan* that is closest to the profit maximizing one at every point along the path to equilibrium.

were inadvertently omitted in estimating a production function. Decreasing returns to scale are also likely to be the case when the scale of operation is very large. Upward sloping average and marginal cost curves may be present at high output levels because coordination and control become increasingly difficult. Information may be lost or distorted as it is transmitted from workers to lower-level management and then to top level management, and the reverse may be equally likely. Channels of communication become more complex and more difficult to monitor, additional monitoring services cannot generally be obtained as readily as can additional inputs. Decisions require more time to make and implement.

Second, our mechanism for DRS technologies is drastically different from mechanisms for CRS technologies. In the CRS case, the personalized prices of all agents in the existing mechanisms are independent of their own messages. In our mechanism however, the personalized prices of agents with nonzero profit shares depend on their own messages while only the personalized prices of agents without profit shares are independent of their own messages. We prove in Theorems 3 and 4 that this is a generic property for any balanced and smooth mechanism which implements the Lindahl correspondence. It seems to us that this is an important distinction between CRS and DRS technologies when designing a balanced mechanism which implements the Lindahl correspondence. As an application of Theorems 3 and 4, one can conclude that the mechanisms proposed by Walker (1981) and Tian (1990, 1991) which implement the Lindahl correspondence for CRS technologies cannot be modified easily to implement the Lindahl correspondence for public goods economies with DRS technologies, because in those mechanisms the personalized prices of all agents are independent of their own messages.

Our mechanism also improves the results of Walker (1981) and Nakamura (1989) in several aspects. Walker (1981) considered a general production technology, but his mechanism has several undesirable properties for economies with more than one private good.[5] Specifically, his mechanism is a combination of a game form and the market mechanism, so it is not really a "pure" game form but a quasi-game form in the sense that it requires that producers are nonplayer participants[6]; it is not individually feasible, balanced, or single valued. Nakamura (1989) constructed two "pure" mechanisms which implement the Lindahl correspondence. However, one of them is neither (weakly) balanced nor individually feasible. The other is balanced but still not individually feasible and uses the strong assumption that all agents share the profits. For the importance of designing individually feasible, balanced, single-valued, and continuous mechanisms, see Groves and Ledyard (1987, pp. 72–75) and Tian (1989, 1991).

The plan of this paper is as follows. Section 2 sets forth a public goods economy model and presents a "pure" mechanism which has the desirable properties mentioned above. Section 3 shows that this mechanism fully implements the

[5] For one private good and one public good economies, Walker (1981) considered only the case of constant returns to scale and thus his mechanism has only one undesirable property, namely, individual infeasibility.

[6] This is because producers obey certain behavior rules but do not have preferences, and hence are not players in games ("soulless" production managers). Hurwicz (1979b) called this kind of game structure a quasi-game.

Lindahl correspondence. Section 4 proves that, for any balanced and smooth "pure" mechanism implementing the Lindahl correspondence, the personalized prices of consumers with nonzero profit shares must depend on their own messages and the personalized prices of those consumers without profit shares must be independent of their own messages. Finally, in Section 5, we give some concluding remarks.

2. PUBLIC GOODS MODEL AND MECHANISM

2.1. *Economic Environments.* We consider public goods economies with n agents who consume one private good x (the numeraire) and K public goods y. Throughout this paper subscripts are used to index agents and superscripts are used to index goods unless otherwise stated. Denote by $N = \{1, 2, \dots, n\}$ the set of agents. Each agent's characteristic is denoted by $e_i = (w_i, R_i)$, where w_i is the initial endowment of the private good and R_i the preference ordering (P_i denotes the asymmetric part of the preference R_i) defined on $\mathbb{R}_+^{1+K}$. We assume that there are no initial endowments of public goods, but that public goods can be produced from the private good. We further assume that there is only one producer[7] whose production technology is given by a cost function C: $\mathbb{R}_+^K \to \mathbb{R}_+$, and each consumer i has a nonnegative profit share θ_i with $\sum_{i \in N} \theta_i = 1$. An economy is the full vector $e = (e_1, \dots, e_n, C)$ and the set of all such economies is denoted by E. The following additional assumptions are made on E.

ASSUMPTION 1. $n \geqq 3$.[8]

ASSUMPTION 2. $w_i > 0$ *for all* $i \in N$.

ASSUMPTION 3. *Each consumer's preference* R_i *can be represented by a twice continuously differentiable and strictly quasi-concave utility function* u_i *which satisfies strictly differentiable monotonicity (i.e.,* $\partial u_i / \partial x_i > 0$ *and* $\partial u_i / \partial y > 0$) *on* $\mathbb{R}_{++}^{1+K}$.

ASSUMPTION 4. *For all* $i \in N$, $u_i(x_i, y) > u_i(x_i', y')$ *for all* $(x_i, y) \in \mathbb{R}_{++}^{1+K}$ *and* $(x_i', y') \in \partial \mathbb{R}_+^{1+K}$, *where* $\partial \mathbb{R}_+^{1+K}$ *is the boundary of* $\mathbb{R}_+^{1+K}$.

ASSUMPTION 5. *The cost function* C: $\mathbb{R}_+^K \to \mathbb{R}_+$ *is twice continuously differentiable, strictly increasing, strictly convex in* y, *and* $C(0) = 0$.[9] *Thus,* $DC \geqq 0$ *for all* $y \geqq 0$ *and* D^2C *is positive definite. Here* DC *and* D^2C *represent the first and second derivatives, respectively.*

[7] Extension to the presence of any number of producers is straightforward.

[8] As usual, vector inequalities are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \dots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \dots, m$.

[9] This is actually an input requirement function. In this model, there is only one private good which is a numeraire, so that we can interpret $C(y)$ as a cost function.

REMARK 1. The familiar Cobb-Douglas utility function satisfies Assumptions 3 and 4.

2.2. *Lindahl Allocations.* An allocation $(x, y) = (x_1, \dots, x_n, y)$ is *feasible* for an economy e if $(x, y) \in \mathbb{R}_+^{n+K}$ and $\sum_{i=1}^n x_i + C(y) \leqq \sum_{i=1}^n w_i$. An allocation (x, y) is *Pareto efficient* for an economy e if it is feasible and there does not exist another feasible allocation (x', y') such that $u_i(x_i', y') \geqq u_i(x_i, y)$ for all $i \in N$ and $u_j(x_j', y') > u_j(x_j, y)$ for some $j \in N$.[10] An allocation (x, y) is *individually rational* for an economy e if $u_i(x_i, y) \geqq u_i(w_i, 0)$ for all $i \in N$.

An allocation (x^*, y^*) is a *Lindahl allocation* for an economy e if it is feasible and there are personalized price vectors $q_i^* \in \mathbb{R}_+^K$, one for each i, such that

1) $x_i^* + q_i^* \cdot y^* \leqq w_i + \theta_i(q^* \cdot y^* - C(y^*))$ for all $i \in N$;
2) $u_i(x_i, y) > u_i(x_i^*, y^*)$ implies $x_i + q_i^* \cdot y > w_i + \theta_i(q^* \cdot y^* - C(y^*))$ for all $i \in N$;
3) $q^* \cdot y^* - C(y^*) = \max_{y \in \mathbb{R}_+^K}(q^* \cdot y - C(y))$,

where $\sum_{i=1}^n q_i^* = q^*$. Denote by $L(e)$ the set of all such allocations.

REMARK 2. Condition (3) is the profit maximization condition. Note that the maximum profit is greater than zero when the maximizing level of output y^* is semi-positive (i.e., $y^* \geq 0$). Also, under Assumptions 2 to 4, the Lindahl allocation exists (compare Milleron 1972, p. 443).

REMARK 3. Every Lindahl allocation is clearly Pareto efficient and individually rational. However, even though a Lindahl allocation is individually rational, it may not be autarkically individually rational.[11]

2.3. *Mechanism.* Let F be a *social choice rule*, i.e., a correspondence from E to the set Z of resource allocations. In the rest of this paper, we will use the Lindahl correspondence as the social choice rule.

Let M_i denote the ith message (strategy) domain. Its elements are written as m_i and called messages. Let $M = \Pi_{i=1}^n M_i$ denote the message (strategy) space. Let $h: M \to Z$ denote the outcome function, or more explicitly, $h_i(m) = (X_i(m), Y(m))$, where $X_i(m)$ is the ith agent's outcome function for the private good, and $Y(m)$ the outcome function for public goods. A mechanism consists of $\langle M, h \rangle$ which is defined on E.

A message $m^* = (m_1^*, \dots, m_n^*) \in M$ is a *Nash equilibrium* (NE) of the mechanism $\langle M, h \rangle$ for an economy e if for any $i \in N$ and for all $m_i \in M_i$,

$$u_i(X_i(m^*), Y(m^*)) \geqq u_i(X_i(m^*/m_i, i), Y(m^*/m_i, i)), \tag{1}$$

[10] Note that, in contrast to the efficiency for private goods, for public goods economies, a weak Pareto efficient allocation may not be Pareto efficient even if preferences satisfy strict monotonicity and continuity (compare Tian 1988).

[11] An allocation (x^*, y^*) is *autarkically (free access) individually rational* if for each agent i, $u_i(x_i^*, y^*) \geqq u_i(\bar{x}_i, \bar{y}_i)$. Here $(\bar{x}_i, \bar{y}_i)$ is a utility maximizer subject to $x_i + C(y) \leqq w_i$. This concept is due to Gevers (1986), Moulin (1989), and Saijo (1991).

where $(m^*/m_i, i) = (m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. The $h(m^*)$ is then called a *Nash (equilibrium) allocation*. Denote by $\mathcal{V}_{M,h}(e)$ the set of all such Nash equilibria and by $N_{M,h}(e)$ the set of all such Nash (equilibrium) allocations. A mechanism $\langle M, h\rangle$ fully *Nash-implements* the Lindahl correspondence L on E, if, for all $e \in E$, $N_{M,h}(e) \neq \emptyset$ and $N_{M,h}(e) = L(e)$. A mechanism $\langle M, h\rangle$ is *individually feasible* if $(X(m), Y(m)) \in \mathbb{R}_+^{n+K}$ for all $m \in M$. A mechanism $\langle M, h\rangle$ is *balanced* if for all $m \in M$

$$\sum_{j=1}^{N} X_j(m) + C(y(m)) = \sum_{j=1}^{n} w_j. \tag{2}$$

A mechanism $\langle M, h\rangle$ is *completely feasible* if it is individually feasible and balanced.

Now we construct a completely feasible and continuous mechanism with a message space of minimal dimension which fully Nash-implements the Lindahl correspondence under the assumptions that the cost function and endowments are known to the designer.

For each $i \in N$, the agent's message domain is of the form

$$M_i = \mathbb{R}^K. \tag{3}$$

To define the personalized prices, we need to distinguish two cases: (i) $\theta_i \in [0, 1)$ for all $i \in N$ and (ii) there is one consumer (say, consumer 1) who owns all the profit shares (i.e., $\theta_1 = 1$, $\theta_i = 0$ for $i = 2, \ldots, n$).

Case (i).

$$q_i(m) = \theta_i m_i + (1 - \theta_{i+1})m_{i+1}, \tag{4}$$

where $n + 1$ is to be read as 1.

Case (ii).

$$q_1(m) = m_1; \tag{5}$$

$$q_i(m) = m_{i+1} \qquad \text{for } i = 2, \ldots, n-1; \tag{6}$$

$$q_n(m) = m_2. \tag{7}$$

REMARK 4. Observe that, by construction, consumers with zero profit shares cannot change their personalized prices by changing their own messages, although the personalized prices of consumers with nonzero profit shares depend on their own messages.

Define the producer's price vector $q(m)$ by summing all the consumers' personalized prices. Then for both cases we have

$$q(m) = \sum_{i=1}^{n} q_i(m) = \sum_{i=1}^{n} m_i \equiv \tilde{m}. \tag{8}$$

Let $A = \{y \in \mathbb{R}_+^K: 0 \leqq C(y) \leqq w\}$ be the feasible production set under given resources $w = \sum_{i=1}^n w_i$. Then A is nonempty, compact, and convex (by noting that $C(y)$ is convex).

The profit maximizing level of output for public goods is given by

$$\tilde{y}(m) = \arg\min \{\|q(m) - DC(y)\|: y \in A\}, \tag{9}$$

choosing $\tilde{y}(m)$ such that its marginal cost vector is the closest to the producer's price vector. Note that $\tilde{y}(m)$ is single valued, continuous,[12] and by construction,

$$\tilde{y}(m) = (DC)^{-1}(q(m)) \tag{10}$$

if $q(m) \in DC[A]$.

Define the feasible correspondence $B: M \rightarrow\rightarrow \mathbb{R}_+^K$ by

$$B(m) = \{y \in \mathbb{R}_+^K: w_i + \theta_i[q(m) \cdot y - C(y)] - q_i(m) \cdot y \geqq 0 \ \forall \ i \in N\}, \tag{11}$$

which is clearly nonempty, compact, convex, and continuous on M.

Define the outcome function for public goods $Y: M \rightarrow B$ by

$$Y(m) = \arg\min \{\|y - \tilde{y}(m)\|: y \in B(m)\}, \tag{12}$$

choosing $Y(m)$ closest to the profit maximizing level of output $\tilde{y}(m)$. Then $Y(m)$ is single valued and continuous on M. Note that $Y(m)$ also satisfies the global material balance condition because $B(m) \subset A$ for all $m \in M$.

Define the taxing outcome function $T_i(m): M \rightarrow \mathbb{R}$ by

$$T_i(m) = q_i(m) \cdot Y(m). \tag{13}$$

Then

$$\sum_{i=1}^n T_i(m) = q(m) \cdot Y(m). \tag{14}$$

Define the ith agent's outcome function for private good $X_i: M \rightarrow R_+$ by

$$X_i(m) = w_i + \theta_i[q(m) \cdot Y(m) - C(Y(m))] - q_i(m) \cdot Y(m), \tag{15}$$

which corresponds to the budget constraint. Because $\sum_{i \in N} \theta_i = 1$ and $\sum_{i \in N} q_i(m) = q(m)$, we have

$$\sum_{i=1}^n X_i(m) + C(Y(m)) = \sum_{i=1}^n w_i, \tag{16}$$

which means that the balanced condition holds for all $m \in M$.

[12] This is because $\tilde{y}(m)$ is an upper semicontinuous correspondence by Berge's Maximum Theorem (see Berge 1963, p. 116) and single valued (see Mas-Colell 1985, p. 28). It might be remarked that the convexity of the cost function is playing a big role in the paper. For example, in the single valuedness of $\tilde{y}(\cdot)$ and $Y(\cdot)$ defined in (12).

Thus the mechanism specified above is individually feasible, balanced, continuous, and almost everywhere differentiable on M.

3. EQUIVALENCE OF NASH AND LINDAHL ALLOCATIONS

This section is devoted to the proof of equivalence between Nash allocations and Lindahl allocations, as stated in Theorem 1 and Theorem 2 below. Lemma 1 and Lemma 2 are preliminary results used to prove Theorem 1 and Theorem 2.

LEMMA 1. *If* $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, *then* $(X(m^*), Y(m^*)) \in \mathbb{R}^{n+K}_{++}$.

PROOF. We argue by contradiction. Suppose $(X(m^*), Y(m^*)) \in \partial\mathbb{R}^{n+K}_{+}$. Then $Y(m^*) \in \partial\mathbb{R}^K_+$ or $X_i(m^*) = 0$ for some $i \in N$.

Let $\iota \in \mathbb{R}^K_+$ be a vector with ones. Let

$$\delta = \frac{\bar{w}}{2(\iota \cdot DC(\iota) + 2\hat{m}^*)},$$

where $\bar{w} = \min_{i\in N} |w|$ and $\hat{m}^* = \sum_{k=1}^K \sum_{i=1}^n |m_i^{*k}|$. Let $b = \min\{1, \delta\}$ and $\tilde{y} = (1/K)(b, \dots, b)'$.

Now suppose that consumer i chooses his/her message $m_i = DC(\tilde{y}) - \sum_{l\neq i}^n m_l^*$. Then $q(m^*/m_i, i) = m_i + \sum_{l\neq i}^n m_l^* = DC(\tilde{y})$ and

$$\begin{aligned}
&w_j + \theta_j[q(m^*/m_i, i)\cdot\tilde{y} - C(\tilde{y})] - q_j(m^*/m_i, i)\cdot\tilde{y}\\
&\quad \geqq w_j - q_j(m^*/m_i, i)\cdot\tilde{y}\\
&\quad \geqq w_j - \frac{1}{K}\sum_{k=1}^K [|m_i^k| + \hat{m}^*]b \qquad \text{(by noting } |\iota\cdot q_j(m)| \leqq \textstyle\sum_{k=1}^K \sum_{i=1}^n |m_i^k|)\\
&\quad \geqq w_j - [\iota\cdot DC(\tilde{y}) + 2\hat{m}^*]b \qquad \text{(by noting } m_i = DC(\tilde{y}) - \textstyle\sum_{l\neq i}^n m_l^*)\\
&\quad \geqq w_j - [\iota\cdot DC(\iota) + 2\hat{m}^*]\delta \qquad \text{(by noting } b \leqq 1 \text{ and } b \leqq \delta)\\
&\quad = w_j - \frac{\bar{w}}{2} \geqq \frac{\bar{w}}{2} > 0
\end{aligned}$$

for all $j \in N$. Therefore $\tilde{y} \in B(m^*/m_i, i)$ and thus

$$Y(m^*/m_i, i) = \tilde{y} > 0,$$

$$X_j(m^*/m_i, i) = w_j + \theta_j[q(m^*/m_i, i)\cdot\tilde{y} - C(\tilde{y})] - q_j(m^*/m_i, i)\cdot\tilde{y} > 0.$$

Hence, by Assumption 4, we have

$$u_i(X_i(m^*/m_i, i), Y(m^*/m_i, i)) > u_i(X_i(m^*), Y(m^*)),$$

which contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

LEMMA 2. *If* $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, *then* $Y(m^*) \in$ *int* $B(m^*)$ *and* $q(m^*) \in DC[A]$. *Therefore* $Y(m^*) = \tilde{y}(m^*) = (DC)^{-1}(q(m^*))$.

PROOF. Suppose, by way of contradiction, that $Y(m^*) \in \partial B(m^*)$.[13] Then either $Y^k(m^*) = 0$ for some k or $0 = w_i + \theta_i[q \cdot Y(m^*) - C(Y(m^*))] - q_i(m^*) \cdot Y(m^*) = X_i(m^*)$ for some $i \in N$. But both cases are impossible by Lemma 1. So $Y(m^*) \in$ int $B(m^*)$ and thus $Y(m^*) = \tilde{y}(m^*)$. Similarly, we must have $q(m^*) \in DC[A]$ for otherwise, $\tilde{y}(m^*) = 0$ or $DC(\tilde{y}(m^*)) = \sum_{i=1}^{n} w_i$ and thus $X_i(m^*) = 0$ for all $i \in N$. Hence $Y(m^*) = \tilde{y}(m^*) = (DC)^{-1}(q(m^*))$. Q.E.D.

REMARK 5. From Lemma 2 and the twice differentiability of the cost function, we know that the outcome functions are differentiable on some neighborhood of $m^* \in \mathcal{V}_{M,h}(e)$ and thus we can use the differential approach to find Nash equilibrium points. (For the examples of using the differential approach to locating Nash equilibria, see Chapter 16 of Varian 1992). Lemma 2 also shows that the outcome for public goods is equal to the profit maximizing level of output at Nash equilibrium.

We now prove the main results of this paper in the following theorems.

THEOREM 1. *Under Assumptions* 1 *to* 5, *if the mechanism defined above has a Nash equilibrium* m^*, *then* $(X(m^*), Y(m^*))$ *is a Lindahl allocation with the Lindahl price system* $(q_1(m^*), \dots, q_n(m^*))$, *i.e.*, $N_{M,h}(e) \subseteq L(e)$.

PROOF. Let m^* be a Nash equilibrium. We prove that $(X(m^*), Y(m^*))$ is a Lindahl allocation with the price vector $(q_1(m^*), \dots, q_n(m^*)) \in \mathbb{R}_+^{nK}$. By the construction of the mechanism and Lemma 2, we know that $(X(m^*), Y(m^*))$ is feasible, $Y(m^*)$ is the profit maximizing level of output, and $(X_i(m^*), Y(m^*))$ satisfies the budget constraint of agent i. So we need only show that each individual is maximizing his/her preferences.

By Lemma 1, we know that $(X_i(m^*), Y(m^*)) \in \mathbb{R}_{++}^{1+K}$. By Lemma 2, $Y(m^*) \in$ int $B(m^*)$ and $Y(m^*) = (DC)^{-1}(q(m^*))$. Therefore there exists a neighborhood $\mathcal{O}(m^*)$ of m^* such that $Y(m) \in$ int $B(m^*)$ for all $m \in \mathcal{O}(m^*)$. Hence all outcome functions $q_i(m)$, $q(m)$, $X_i(m)$, and $Y(m)$ are differentiable on $\mathcal{O}(m^*)$. Thus, by using the first order condition, we have

$$D_{m_i} u_i(X_i(m^*), Y(m^*)) = 0. \tag{17}$$

Therefore,

$$(D_x u_i)[\theta_i(Y \cdot D_{m_i} q + q \cdot D_{m_i} Y - DC(Y) \cdot D_{m_i} Y) - q_i \cdot D_{m_i} Y - Y \cdot D_{m_i} q_i] + D_y u_i(D_{m_i} Y) = 0. \tag{18}$$

By the constructions of $q(m)$ and $q_i(m)$ and Lemma 2, we know that for all $m \in \mathcal{O}(m^*)$, $q(m) = DC(Y(m))$, $D_{m_i} q = I_K$, $D_{m_i} q_i = \theta_i I_K$,[14] and thus $\theta_i Y(m) \cdot D_{m_i} q$

[13] $\partial B(m)$ denotes the boundary of $B(m)$.

[14] This is true for both cases by the definition of $q_i(m)$ and by noting that θ_i can be zero for some $i \in N$.

$- Y(m) \cdot D_{m_i} q_i(m) = \theta_i Y(m) - \theta_i Y(m) = 0$. Here I_K is a $K \times K$ identity matrix. Hence (18) becomes

$$[(D_x u_i)(-q_i(m^*)) + (D_y u_i)] D_{m_i} Y = 0. \tag{19}$$

Because $D_{m_i} Y = (D^2 C)^{-1}$ is a nonsingular matrix,

$$D_x u_i(-q_i(m^*)) + D_y u_i = 0 \tag{20}$$

and thus

$$\frac{1}{D_x u_i} D_y u_i = q_i(m^*), \tag{21}$$

which is the first order necessary and sufficient condition for utility maximization since u_i is strictly quasi-concave. Q.E.D.

THEOREM 2. *Under Assumptions* 1 *to* 5, *if* (x^*, y^*) *is a Lindahl allocation with the Lindahl price system* $(q_1^*, \ldots, q_n^*)$, *then there is a Nash equilibrium* m^* *of the mechanism such that* $Y(m^*) = y^*$, $X_i(m^*) = x_i^*$, $q_i(m^*) = q_i^*$, *for all* $i \in N$, *i.e.*, $L(e) \subseteq N_{M,h}(e)$.

PROOF. We want to find a message m^* such that (x^*, y^*) is a Nash allocation. We first assume that at least two agents have nonzero profit shares (i.e., $\theta_i \in [0, 1)$ for all $i \in N$, which belongs to Case (i)). Consider the following linear equations system:

$$\Theta m = q^*, \tag{22}$$

where

$$\Theta = \begin{pmatrix} \theta_1 I_K & (1 - \theta_2 I_K & 0 & \cdots & 0 & 0 \\ 0 & \theta_2 I_K & (1-\theta_3) I_K & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \theta_{n-1} I_K & (1-\theta_n) I_K \\ (1-\theta_1) I_K & 0 & 0 & \cdots & 0 & \theta_n I_K \end{pmatrix} \tag{23}$$

and $q^* = (q_1^*, \ldots, q_n^*)'$. Expanding the determinant $|\Theta|$ by the first K columns, we have $|\Theta| = \Pi_{i=1}^n \theta_i^K + (-1)^{n+1} \Pi_{i=1}^n (1 - \theta_i)^K \neq 0$ because $\theta_i \in [0, 1)$ for all $i \in N$.[15] Hence the linear equations system (22) has a unique solution m^*. Now if only one agent receives all the profit, the linear equations system is specified by (5) to (7) and thus clearly has a unique solution. Then, for both cases, we can find a unique m^* such that $q_i(m^*) = q_i^*$, $q(m^*) = \sum_{i=1}^n q_i(m^*) = \sum_{i=1}^n q_i^*$, $q(m^*) = DC(Y(m^*))$, $X_i(m^*) = x_i^*$, $Y(m^*) = y^*$.

From (18) and (21), we know that the first order condition for Lindahl allocations is the same as the first order condition for Nash allocations. So we need only show

[15] This is true by the following reasoning. If $\theta_i = 0$ for some $i \in N$, then $|\Theta| = (-1)^{n+1} \Pi_{i=1}^n (1 - \theta_i)^K \neq 0$. If $\theta_i \neq 0$ for all $i \in N$, then $\theta_i < 1 - \theta_{i+1}$ since $n \geqq 3$ and $\sum_{i=1}^n \theta_i = 1$. Thus $\Pi_{i=1}^n \theta_i^K < \Pi_{i=1}^n (1 - \theta_i)^K$. So $|\Theta| = \Pi_{i=1}^n \theta_i^K + (-1)^{n+1} \Pi_{i=1}^n (1 - \theta_i)^K \neq 0$.

that the first order necessary condition for Nash equilibrium is actually sufficient for Nash equilibrium by showing that the following function (which gives consumer i's budget constraint)

$$(24)\quad X_i(m^*/m_i, i) = w_i + \theta_i[q(m^*/m_i, i)\cdot Y(m^*/m_i, i) - C(Y(m^*/m_i, i))] - q_i(m^*/m_i, i)\cdot Y(m^*/m_i, i)$$

is concave in $Y(m^*/m_i, i)$ for all $m_i \in M_i$.[16]

We first consider those m_i such that $(X_i(m^*/m_i, i), Y(m^*/m_i, i)) \in \mathbb{R}^{1+K}_{++}$. Then $Y(m^*/m_i, i) \in \operatorname{int} B(m^*/m_i, i)$ and thus all the outcome functions are differentiable with respect to m_i on some neighborhood of m_i. Differentiating (24) with respect to m_i, we have

$$(25)\quad D_Y X_i \cdot D_{m_i} Y = \theta_i[q \cdot D_{m_i} Y + Y \cdot D_{m_i} q - DC(Y)\cdot D_{m_i} Y] - q_i \cdot D_{m_i} Y - Y \cdot D_{m_i} q_i.$$

By repeating the arguments that lead to equation (20), we get

$$(26)\quad D_Y X_i = -q_i(m).$$

The second order derivative matrix is

$$(27)\quad D^2_Y X_i D_{m_i} Y = -D_{m_i} q_i(m) = -\theta_i I_K.$$

Then

$$(28)\quad D^2_Y X_i = -\theta_i (D_{m_i} Y)^{-1} = -\theta_i D^2 C,$$

which is negative semi-definite. Thus $X_i(m^*/m_i, i)$ is concave in $Y(m^*/m_i, i)$ when $(X_i(m^*/m_i, i), Y(m^*/m_i, i)) \in \mathbb{R}^{1+K}_{++}$. Now consider those m_i such that outcomes $(X_i(m^*/m_i, i), Y(m^*/m_i, i))$ are boundary points of $\mathbb{R}^{1+K}_{+}$. Since these outcomes are also boundary points of the curve specified by (24) which are continuous in m_i, and concave in Y in $\mathbb{R}^{1+K}_{++}$, $X_i(m^*/m_i, i)$ is concave in $Y(m^*/m_i, i)$ for all $m_i \in M_i$. Thus consumer i maximizes a strictly quasi-concave utility function subject to a convex constraint when others' messages are given. Hence the first order condition is sufficient (compare Arrow and Enthoren 1961). Q.E.D.

REMARK 6. From Theorem 2.4 of Reichelstein and Reiter (1988) we know that Nash implementation is always at least as costly, in message space size, as (decentralized) realization. Because the minimal dimension required for decentralized realization of the Lindahl correspondence under prescribed behavior is nK (compare Sato 1981 or Hurwicz 1986b), the mechanism presented in this paper has a message space of minimal dimension and thus is informationally efficient. Also, because every Lindahl allocation is Pareto efficient and individually rational, the mechanism yields Pareto efficient and individually rational allocations.

[16] Note that choosing m_i is equivalent to choosing $Y(m^*/m_i, i)$ for consumer i when others' messages are given.

Summarizing the above discussions, we conclude that for public goods economies E satisfying Assumptions 1 to 5, there exists a completely feasible, continuous, and almost everywhere differentiable mechanism with a message space of minimal dimension which fully Nash-implements the Lindahl correspondence.

4. CHARACTERISTICS OF MECHANISMS FOR DRS

Notice that the above mechanism is drastically different from the usual CRS mechanisms. Groves and Ledyard (1987, p. 77) argued that, for a given mechanism in public goods economies with CRS, if the joint message m^* is a Nash equilibrium that yields a Lindahl allocation, the personalized prices $q_i(m^*)$ may depend on the messages of other agents, but not on the agent's own message. In our mechanism dealing with DRS, the personalized prices of consumers with nonzero profit shares depend on their own messages while only the personalized prices of consumers with zero profit shares are independent of their own messages. One may wonder if this distinction is valid for any balanced and smooth mechanism which fully implements the Lindahl correspondence. The following results answer this question.

THEOREM 3. *Let $\langle M, h\rangle$ be a mechanism which fully Nash-implements the Lindahl correspondence on the set of sufficiently large economic environments with DRS technologies. Suppose that the outcome function h is differentiable on some neighborhood of every Nash equilibrium and satisfies the budget constraints with equality. Then, if all consumers' profit share are nonzero, at least one consumer's personalized prices depend on his own messages.*

PROOF. Consider the following class of public goods economies: one private and one public good, decreasing returns in producing y from x by the cost function $C(y) = y^2$, endowments $w_i > 0$, and preferences R_i which are represented by the differentiable utility functions u_i of the form

$$u_i(x_i, y) = x_i^{a_i} y^{1-a_i}, \qquad \forall\, a_i \in (0, 1)$$

for all $i \in N$. Denote by E_c the set of all such economies. Then one can show that $(x^*, y^*) \in L(e)$ for $e \in E_c$ if and only if

$$x_i^* = w_i + \theta_i[q^{*2}/4 - q_i^*(q^*/2)], \qquad \forall\, i \in N \tag{29}$$

$$y^* = q^*/2, \tag{30}$$

where

$$q_i^* = \frac{2(1 - a_i)}{q^*}[w_i + \theta_i(q^{*2}/4)], \qquad \forall\, i \in N \tag{31}$$

and

$$q^* = 2\sqrt{\frac{\sum_{i=1}^{n} (1 - a_i)w_i}{[2 - \sum_{i=1}^{n} (1 - a_i)\theta_i]}}. \tag{32}$$

From (29) to (32), we know that for all Lindahl allocations for the economies E_c, $x_i^* \in (0, w_i)$ and $y^* \in (0, \sqrt{w})$ since $q_i^* \in (0, (w_i + \theta_i w)/\sqrt{w})$ and $q^* \in (0, 2\sqrt{w})$.

Now for any mechanism $\langle M, h\rangle$, $X_i(m)$, by assumption, is defined by

$$X_i(m) = w_i + \theta_i[q(m) \cdot Y(m) - C(Y(m))] - q_i(m) \cdot Y(m) \tag{33}$$

and thus the mechanism is balanced. Because the mechanism, by assumption, is differentiable on some neighborhood of every Nash equilibrium $m \in V_{M,h}(E_c)$ and fully implements the Lindahl correspondence on E_c, we have the first order condition for Nash equilibrium

$$(D_x u_i)[\theta_i(Y \cdot D_{m_i} q + q \cdot D_{m_i} Y - DC(Y) \cdot D_{m_i} Y) - q_i \cdot D_{m_i} Y - Y \cdot D_{m_i} q_i] + D_y u_i(D_{m_i} Y) = 0. \tag{34}$$

Because $(X(m), Y(m))$ is also a Lindahl allocation by assumption, from the first order condition for Lindahl allocations, we have

$$[(D_x u_i)(-q_i(m)) + (D_y u_i)]D_{m_i} Y = 0. \tag{35}$$

Combining (34) and (35), we obtain

$$(D_x u_i)[\theta_i(Y \cdot D_{m_i} q + q \cdot D_{m_i} Y - DC(Y) \cdot D_{m_i} Y) - Y \cdot D_{m_i} q_i] = 0. \tag{36}$$

Because $(D_x u_i) > 0$, $q(m) - DC(Y(m)) = 0$ by the profit maximization condition, and $Y(m) \in (0, \sqrt{w})$, we must have

$$\theta_i \partial q(m)/\partial m_i = \partial q_i(m)/\partial m_i \tag{37}$$

or

$$\theta_i \partial DC(Y(m))/\partial m_i = \partial q_i(m)/\partial_{m_i} m_i \tag{38}$$

for all $m \in V_{M,h}(E_c)$.

Now suppose by way of contradiction, that for all $i \in N$, $\partial q_i(m)/\partial m_i = 0$ for all $m \in \mathcal{V}_{M,h}(E_c)$. Because $\theta_i > 0$ for all $i \in N$, then $\partial DC(Y(m))/\partial m_i = 0$ for all $m \in \mathcal{V}_{M,h}(E_c)$ and all $i \in N$. Hence, by Sard's theorem, $DC(Y(\mathcal{V}_{M,h}(E_c))) = DC(y^*(E_c))$ is of measure zero, where $y^*(e)$ is the Lindahl allocation for the pubic goods at $e \in E_c$. But by Nash implementation, $DC(Y(\mathcal{V}_{M,h}(E_c))) = DC(y^*(E_c))$ has a positive measure, which is a contradiction. Q.E.D.

From the proof of the above theorem, we have the following result:

THEOREM 4. *Let $\langle M, h\rangle$ be a mechanism which fully Nash-implements the Lindahl correspondence on the set of sufficiently large economic environments with*

DRS technologies. Suppose that the outcome function h is differentiable on some neighborhood of every Nash equilibrium and satisfies the budget constraints with equality. Then, at least on $\mathcal{V}_{M,h}(E)$, *the personalized prices of consumers without profit shares must be almost everywhere independent of their own messages, and the personalized prices of consumers with nonzero profit shares must depend on their own messages if their outcome functions* $(X_i(m), Y(m))$ *are not almost everywhere independent of their own messages.*

PROOF. Under E_c specified in the proof of Theorem 3, we know that

$$\theta_i \partial q(m)/\partial m_i = \partial q_i(m)/\partial m_i \tag{39}$$

for all $m \in \mathcal{V}_{M,h}(E_c)$. Thus, if the profit share of agent i is zero (i.e., $\theta_i = 0$), then $\partial q_i(m)/\partial m_i = 0$ for all $m \in \mathcal{V}_{M,h}(E_c)$. By Sard's theorem, $q_i(m)$ is of measure zero for all Nash equilibrium strategies of consumer i. Thus q_i is almost everywhere independent of the agent's own messages on $\mathcal{V}_{M,h}(E_c)$ which proves the first part of the theorem.

Now we prove the second part of the theorem. Suppose by way of contradiction, that for some i with $\theta_i > 0$, $\partial q_i(m)/\partial m_i = 0$ for all $m \in \mathcal{V}_{M,h}(E_c)$. We then have $\partial q(m)/\partial m_i = 0$ and $\partial DC(Y(m))/\partial m_i = 0$ for all $m \in \mathcal{V}_{M,h}(E_c)$. Consequently, $\partial X_i(m)/\partial m_i = 0$ and $\partial Y(m)/\partial m_i = 0$ for all $m \in \mathcal{V}_{M,h}(E_c)$, which means $(X_i(m), Y(m))$ is almost everywhere independent of the agent's own messages on $\mathcal{V}_{M,h}(E_c)$, a contradiction. Q.E.D.

REMARK 7. Since in the case of CRS, every agent's profit share can be considered as zero, the above theorem actually supports the arguments for the CRS technology made by Groves and Ledyard (1987).

REMARK 8. As an application of Theorems 3 and 4, consider the mechanisms proposed by Walker (1981) and Tian (1990) for one private and one public good economies. These mechanisms are differentiable around Nash equilibria, satisfy the budget constraints with equality, and yield personalized prices of agents independent of their own messages. Although they implement the Lindahl correspondence for CRS technologies, we know by Theorems 3 and 4, that simple modifications of these mechanisms cannot implement the Lindahl correspondence under DRS.

5. CONCLUDING REMARKS

In the above sections we have given a simple mechanism which is well-behaved in the sense that the mechanism is individually feasible, balanced, continuous, and has a message space of minimal dimension. Moreover, the mechanism is almost everywhere differentiable on the message space and differentiable on some neighborhood of every Nash equilibrium so that we can use the differential approach. We have also shown that mechanisms for DRS are drastically different from the mechanisms for CRS in a way that some agents' personalized prices must depend on their own messages if the mechanisms are balanced and smooth. In this section, we would like to mention some possible extensions.

First, the mechanism presented here deals only with public goods economies with one private good. However, by using techniques similar to those given in Tian and Li (1991), the mechanism presented above can be generalized to include economies with an arbitrary number of private goods and a DRS technology by combining the above mechanism and a mechanism given in Tian (1992). The resulting mechanism would be completely feasible, continuous, and implement the Lindahl correspondence.

Second, the mechanism obtained in this paper works only for public goods economies that have strictly decreasing returns to scale. However, this restriction can be relaxed easily to include economies with general convex production possibility sets (including CRS and DRS technologies) by combining the above mechanism and the mechanism given in Tian (1991). Such techniques have been used in Tian and Li (1995). The resulting mechanism will have all the desirable properties of the above mechanism.

Third, similar to those mechanisms proposed by Hurwicz, Maskin, and Postlewaite (1984), and Tian (1993), the mechanism presented above can be extended to allow for endowments unknown to the designer. This case of course certainly increases the size of the message space but reduces the information requirements on the designer.

Finally, like many mechanisms in the literature (such as those in Hurwicz 1979, Walker 1981, Tian 1990), we have assumed that the technology is known to the designer in the above mechanism. This is clearly not a satisfactory assumption. This assumption however, can also be relaxed if we do not insist on the requirement of the minimal sized message space by using techniques similar to those in Hurwicz, Maskin, and Postlewaite (1984), Nakamura (1989), Tian (1989).

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