Studies on The Identification Problem of The Simultaneous Economic Models From The Viewpoint of Unique Determination of Parameters (II)

Tian Guoqiang (田国强)

(Huazhong (Central of China) University of Science and Technology)

Abstract

This paper is a continuation of [14]. Using our viewpoint and methods, we study the identification of the dynamic models and obtain a number of results. We remove the usual assumption that the roots of the polynomial equation \( |B_0 + B_1L + \ldots + B_pL^p| = 0 \) lie outside the unit circle, so that we can study the identification of the unstable dynamic models. We also remove the general assumption that the exogenous variables are not connected by any linear identities. In the following paper, we shall discuss the identification problem of the nonlinear models, the error-shock models (including the unstable ones) by using the same viewpoint and methods.

§ 1. Introduction

In [14], proceeding from the viewpoint that the parameters to be estimated should be uniquely determined, we defined the concepts of the distinction and identification of vectors, and discussed the identification of the very general contemporaneous simultaneous models by the one-step identification method and two-step identification method. We also gave the concepts of almost identification and completely under identification. By using our viewpoint and methods, the statement of many of the principal results in identification theory was concised. In this paper we discuss the identification of the dynamic models. The concepts, terminologies, symbols relative to the identification can be seen in [14].

§ 2. Dynamic Models

A lot of work on identification for dynamic models has been done\(^{(2)}\)(\(^{(7)}\)(\(^{(8)}\)). But most study was made on how to restrict the admissible matrix \(F(L)\) of polynomials in the lag operator \(L\) to an identity matrix. Thus there are more difficulties for identifying the models, and identification is confined to a part of the models.

In this section, we remove the general assumptions that the roots of \(|B_0 + B_1 L + \cdots + B_p L^p| = 0\) lie outside the unit circle\(^{(8)}\) and that \(X_i\) are not connected by any linear identity as well as the conditions which are difficult to verify and satisfy\(^{(8)}\). We discuss the identification of more general dynamic models with the points and methods in the present paper and obtain the theorems for identification which are similar to contemporaneous models.

The assumption that the roots of \(|B_0 + B_1 L + \cdots + B_p L^p| = 0\) lie outside the unit circle serves to ensure that \(\{y_t\}\) is weakly stationary and the variance sequence \(\text{var}(y_t)\) is not divergent \(^{(1)}\). The autoregressive process \(\sum_{i=0}^{p} B_i y_{t-i} + \sum_{i=0}^{p} \alpha_i x_{t-i} = u_t\) can then be expressed as the infinite moving average process \(y_t = \sum_{i=0}^{p} \pi_i x_{t-i} + \sum_{i=0}^{p} \rho_i u_{t-i}\) \(^{(1)}\) However consistent estimators of parameters may exist without the assumption as long as the parameters of the model are uniquely determined*, and from the view-point of economic theory, some people do consider that economic systems are almost unstable, and disturbance at every initial point will keep on dispersing with the shift of time, one relies on artificial restrictions to hold back such dispersion \(^{(1)}\). Thus,

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*To show this, we consider the simple model

\[ y_t = \beta y_{t-1} + \epsilon_t \]

where \(\beta > 1\), \(E \epsilon_t = 0\), and \(\epsilon_t\) is weakly stationary. We have

\[
\hat{\beta} = \frac{\sum_{t=1}^{T} y_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2} - \beta \frac{\sum_{t=1}^{T} \epsilon_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2}
\]

We only need to prove

\[
\frac{\sum_{t=1}^{T} \epsilon_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2} \rightarrow 0 \quad (T \rightarrow \infty).
\]

Since

\[
\frac{\sum_{t=1}^{T} \epsilon_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2} \leq \frac{\sum_{t=1}^{T} |\epsilon_t| |y_{t-1}|}{\sum_{t=1}^{T} y_{t-1}^2} \leq M \frac{\sum_{t=1}^{T} |y_{t-1}|}{\sum_{t=1}^{T} y_{t-1}^2}
\]

where \(M\) is a positive integer, we only need to prove
it seems necessary to study the identifiability of unstable dynamic models.

Consider the following model

$$\sum_{i=0}^{p} B_i y_{t-i} + \sum_{i=0}^{q} \Gamma_i x_{t-i} = u_t, \quad (2.1)$$

where $y_t, x_t$, and $u_t$ are respectively $G \times 1, K \times 1$ and $G \times 1$ vectors of observed endogenous variables, exogenous variables, and unobserved disturbances terms at time $t$; $B_i$ and $\Gamma_i$ are respectively $G \times G$ and $G \times K$ parameter matrices. Rewriting (2.1) in terms of the lag operator $L$, $LY_t = Y_{t-1}$, we have

$$B(L)Y_t + \Gamma(L)x_t = u_t, \quad (2.2)$$

where

$$B(L) = B_0 + B_1 L + \cdots + B_p L^p,$$

$$\Gamma(L) = \Gamma_0 + \Gamma_1 L + \cdots + \Gamma_q L^q. \quad (2.3)$$

**Assumption 2.1** $B_0$ is nonsingular.

**Assumption 2.2** $x_t$ is stationary in a wide sense, ergodic and has finite second moments, and is independent of the $u_t$ sequences**.

**Assumption 2.3** $EU_t = 0$, and $\{u_t\}$ is weakly stationary.

We denote the sequence of the second order moments by

$$C_U(\tau) = EU_t u_{t-\tau} \quad (\tau = 0, 1, \cdots, +\infty). \quad (2.5)$$

As $B > 1$, we know $\frac{|y_{t-1}|}{\sum_{i=1}^{T} y_{i-1}} \not\to r (T \to +\infty)$.

Let

$$V_T = \sum_{i=1}^{T} |y_{i-1}|, \quad Z_T = \sum_{i=1}^{T} y_{i-1}^2,$$

then $V_T \not\to +\infty, Z_T \not\to +\infty (T \to +\infty)$, and there exists a positive integer $N$,

$$Z_T - Z_{T-1} = y_{T-1}^2 > 0 (a, e).$$

Thus, by the O.stolz theorem in mathematical analysis, we have

$$\frac{\sum_{i=1}^{T} |y_{i-1}|}{\sum_{i=1}^{T} y_{i-1}^2} \to 0 \quad (T \to +\infty). \quad \frac{V_T - V_{T-1}}{Z_T - Z_{T-1}} = \frac{|y_{T-1}|}{z_{T-1}} - \frac{1}{z_{T-1}^2} \to 0 \quad (T \to +\infty),$$

$$\sum_{i=1}^{T} |y_{i-1}| \not\to r \quad (T \to +\infty).$$

Which proves $\frac{\sum_{i=1}^{T} |y_{i-1}|}{\sum_{i=1}^{T} y_{i-1}^2} \not\to r \quad (T \to +\infty)$, hence $B$ is the consistent estimator of $B$.

In fact when $U_t$ is serially uncorrelated, we do approach the identifiability of the dynamic models as if they are contemporaneous. For the lagged endogenous variables in this case can be regarded as predetermined.

** The assumption that $x_t$ is weakly stationary can be removed.
Denote

\[ B = (B_0, B_1, \ldots, B_p), \]

\[ \Gamma = (\Gamma_0, \Gamma_1, \ldots, \Gamma_\tau), \]

\[ \vec{Y}_t = \begin{bmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p} \end{bmatrix}, \quad \vec{X}_t = \begin{bmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-q} \end{bmatrix}. \]

Then (2.1) can be rewritten as

\[ B\vec{Y}_t + \Gamma\vec{X}_t = U_t. \]

By Assumption 2.2, we have

\[ BC_{\vec{Y}X}(\tau, t-\tau) + \Gamma C_{\vec{X}}(\tau) = 0 \quad (\tau = 0, 1, \ldots, +\infty), \]

where

\[ C_{\vec{X}} = E\vec{X}_t\vec{X}_{t-i} \]

\[ = \begin{bmatrix} C_X(\tau) & C_X(\tau+1) & \cdots & C_X(\tau+q) \\ C_X(\tau+1) & C_X(\tau) & \cdots & C_X(\tau+q-1) \\ \vdots & \vdots & \ddots & \vdots \\ C_X(\tau+q) & C_X(\tau+q-1) & \cdots & C_X(\tau) \end{bmatrix} \quad (\tau = 0, \ldots, +\infty) \]

is a \((k+1)\times(k+1)\) symmetric matrix, in which \(C_X(j) = EX_{t-i}X'_{t-i} (j = i - k)\)

is a \(k\times k\) symmetric matrix,

\[ C_{\vec{Y}X}(t, t-\tau) = E\vec{Y}_t\vec{X}_{t-\tau} \]

\[ = \begin{bmatrix} C_{YX}(t, t-\tau) & C_{YX}(t, t-\tau-1) & \cdots & C_{YX}(t, t-\tau-q) \\ C_{YX}(t-1, t-\tau) & C_{YX}(t-1, t-\tau-1) & \cdots & C_{YX}(t-1, t-\tau-q) \\ \vdots & \vdots & \ddots & \vdots \\ C_{YX}(t-p, t-\tau) & C_{YX}(t-p, t-\tau-1) & \cdots & C_{YX}(t-p, t-\tau-q) \end{bmatrix} \quad (t = 0, \ldots, +\infty) \]

is a \(G(p+1)\times k(q+1)\) matrix, in which \(C_{YX}(t-i, t-\tau-j)\)

\[ = EY_{t-i}X'_{t-i-1} \quad (i = 0, \ldots, p, j = 0, \ldots, q) \]

is a \((p+1)\times(q+1)\) matrix

\[ C_{\vec{X}}(\tau) = EU_{\vec{Y}X}' \]

\[ = E(B\vec{Y} + \Gamma\vec{X})(B\vec{Y} + \Gamma\vec{X})', \]

\[ = BC_{\vec{Y}}(t, t-\tau)B' + \Gamma C_{\vec{Y}X}(t, t-\tau)B' + BC_{\vec{Y}X}(t, t-\tau)\Gamma' + \Gamma C_{\vec{X}}(\tau)\Gamma' \quad (\tau = 0, 1, \ldots, +\infty), \]

where

\[ C_{\vec{Y}}(\tau) = E\vec{Y}_t\vec{Y}_{t-\tau} \]

\[ = \begin{bmatrix} C_Y(t, t-\tau) & C_Y(t, t-\tau-1) & \cdots & C_Y(t, t-\tau-p) \\ C_Y(t-1, t-\tau) & C_Y(t-1, t-\tau-1) & \cdots & C_Y(t-1, t-\tau-p) \\ \vdots & \vdots & \ddots & \vdots \\ C_Y(t-p, t-\tau) & C_Y(t-p, t-\tau-1) & \cdots & C_Y(t-p, t-\tau-p) \end{bmatrix} \quad (\tau = 0, 1, \ldots, +\infty) \]
is a \( G(p+1) \times G(p+1) \) matrix, in which \( C_{Y}(t-i, t-j-\tau) = EY_{t-i}Y'_{t-j-\tau} \) \((i, j = 0, \ldots, p)\) is a \((p+1) \times (p+1)\) matrix.

When \( \tau = 0 \), by (2.10), (2.13) becomes
\[
C_{\lambda}(0) + \Gamma C_{T}(0) = BC_{T}(t, t)B'.
\] (2.15)

If \( B \) and \( \Gamma \) are identified, from (2.13), \( C_{\lambda}(\tau) \) can also be uniquely determined \((\tau = 0, 1, \ldots, +\infty)\). Rewrite (2.10) as
\[
(B, \Gamma) \begin{bmatrix}
C_{Y}(t, t) & C_{Y}(t, t - 1) & \cdots \\
C_{T}(0) & C_{T}(1) & \cdots
\end{bmatrix} = 0.
\] (2.16)

Denote by \( W \) the matrix obtained by detecting the identical columns of the observed matrix in (2.16) and write
\[
A = (B, \Gamma),
\] (2.17)

(2.16) can be rewritten as
\[
AW = 0.
\] (2.18)

Clearly, for the \( g \)-th row \( A_{g} \) of \( A \), we have
\[
AW_{g} = 0,
\] (2.19)
where
\[
W = \begin{bmatrix}
C_{Y}(t, t) & C_{Y}(t, t - 1) & C_{Y}(t, t - 2) & \cdots \\
C_{Y}(t - 1, t) & C_{Y}(t - 1, t - 1) & C_{Y}(t - 1, t - 2) & \cdots \\
& \vdots & \ddots & \ddots \\
C_{Y}(t - p, t) & C_{Y}(t - p, t - 1) & C_{Y}(t - p, t - 2) & \cdots \\
C_{X}(0) & C_{X}(1) & C_{X}(2) & \cdots \\
& \vdots & \vdots & \ddots \\
C_{X}(q) & C_{X}(q - 1) & C_{X}(q - 2) & \cdots
\end{bmatrix}. \quad (2.20)
\]

Denote \( \text{rank}(W) = k^{*} \). Since \( \text{rank}(A) = \text{rank}(B_{a}) = G, \text{rank}(W) = k^{*} \leq Gp + k(q + 1) \) from Lemma 2.1 in Reference[14]. When \( k^{*} < Gp + k(q + 1) \), \( A \) is not a basis for the row kernel of \( W \). Then from Lemma 2.2 in Reference[14], there exist \( Gp + k(q + 1) - k^{*} \) solution vectors \( C_{1}, C_{2}, \ldots, C_{Gp+k(q+1)-k^{*}} \) linearly independent of the rows of \( A \) such that
\[
\hat{A} = \begin{bmatrix}
A \\
C_{1} \\
\vdots \\
C_{Gp+k(q+1)-k^{*}}
\end{bmatrix}
\] (2.21)
is a basis for the row kernel of \( W \), then we have
\[
\hat{A}W = 0.
\] (2.22)
By Theorem 2.1 in [14], $A_p$ is completely under-identifiable if observational information alone is available. However, if we also know the restrictions on $A_p$

$$A_p\phi_p = d_p,$$  \hspace{1cm} (2.23)

such that where $\phi_p$ is a $[G(p+1) + k(q+1)] \times R_p$ matrix with known elements, $d_p$ is a $1 \times R_p$ vector with known elements, then by Theorem 2.4 in [14] and corollary 2.1, in [14] we have

**Theorem 2.1** Under Assumptions 2.1—2.3 and the restrictions (2.19) and (2.23), $A_p$ is identified if and only if

$$\text{rank}(\bar{A}_p \phi_p) = \begin{cases} G(p+1) + k(q+1) - k^*, & d_p \neq 0, \\ G(p+1) + k(q+1) - k^* - 1, & d_p = 0. \end{cases}$$

**Corollary 2.1** Under Assumptions 2.1—2.3, and the restrictions (2.19) and (2.23), a necessary condition for the identifiability of $A_p$ is

$$R_p \geq \begin{cases} G(p+1) + k(q+1) - k^*, & d_p \neq 0, \\ G(p+1) + k(q+1) - k^* - 1, & d_p = 0. \end{cases}$$

**Corollary 2.2** Under Assumptions 2.1—2.3 and the restrictions (2.19) and (2.23), when $k^* = Gp + k(q+1)$, $A_p$ is identified if and only if

$$\text{rank}(A_p \phi_p) = \begin{cases} G, & d_p \neq 0, \\ G - 1, & d_p = 0. \end{cases}$$

Next, we discuss the identifiability of $A_p$ or $A$ by applying the one-step identification method.

(2.19) and (2.23) can be rewritten as

$$A_p(\phi_p : W) = (d_p : 0),$$  \hspace{1cm} (2.24)

**Theorem 2.2** Under Assumptions 2.1—2.3 and the restriction (2.24), $A_p$ is identified if and only if

$$\text{rank}(\phi_p : W) = \begin{cases} G(p+1) + k(q+1), & d_p \neq 0, \\ G(p+1) + k(q+1) - 1, & d_p = 0. \end{cases}$$

Stacking $A$ in one row, i.e. $\bar{A} = (A_1, A_2, \ldots, A_\phi)^T$, then (2.18) can be rewritten as

$$(I_G \otimes W) \bar{A} = 0.$$  \hspace{1cm} (2.25)

$\bar{A}$ has the restriction

$$\phi \bar{A} = d,$$  \hspace{1cm} (2.26)

where $\phi$ is an $R \times [G(p+1) + k(q+1)]$G matrix with known elements, $d_e$ is an $R \times 1$ column vector with known elements and $d \neq 0$. Denote

$$Q = [I_G \otimes W], \quad \bar{d} = \begin{bmatrix} d \end{bmatrix}.$$  \hspace{1cm} (2.27)

Then we have

$$Q \bar{A} = \bar{d}.$$  \hspace{1cm} (2.28)
Theorem 2.3 Under Assumptions 2.1—2.3 and the restriction (2.28), \( \mathcal{A} \) is identified if and only if rank (\( Q \)) is equal to \( \lceil G(p+1) + k(q+1) \rceil \).

Corollary 2.3 Under Assumptions 2.1—2.3 and the restriction (2.28), a necessary condition for the identifiability of \( \mathcal{A} \) is

\[
R > G(G(p+1) + k(q+1) - k^*)
\]

Example. To illustrate the application of the conditions for identifiability by our identification method, we consider a two-equation simple dynamic system

\[
B_0 Y_t + B_1 Y_{t-1} + TX_t = U_t,
\]

i.e.,

\[
\begin{bmatrix}
B_{1.0} & B_{1.2} & \cdots & B_{1, t-1} & r_{11} & r_{12} \\
B_{2.0} & B_{2.2} & \cdots & B_{2, t-1} & r_{21} & r_{22}
\end{bmatrix}
\begin{bmatrix}
Y_{t1} \\
Y_{t2}
\end{bmatrix}
= \begin{bmatrix}
U_{t1} \\
U_{t2}
\end{bmatrix},
\] (2.29)

In this case,

\[
A = \begin{bmatrix}
B_{1.0} & B_{1.2} & B_{1.3} & \cdots & B_{1, t-1} & r_{11} & r_{12} \\
B_{2.0} & B_{2.2} & B_{2.3} & \cdots & B_{2, t-1} & r_{21} & r_{22}
\end{bmatrix},
\]

\[
W = \begin{bmatrix}
C_{YX}(t, t) & C_{YX}(t, t-1) & \cdots \\
C_{YX}(t-1, t) & C_{YX}(t-1, t-1) & \cdots \\
C_{X}(0) & C_{X}(1) & \cdots
\end{bmatrix}
\]

\[
= \begin{bmatrix}
C_{YX}(t, t) & C_{YX}(t, t-1) & \cdots \\
C_{YX}(t, t) & C_{YX}(t, t-1) & \cdots \\
C_{YX}(t-1, t) & C_{YX}(t-1, t-1) & \cdots \\
C_{X}(0) & C_{X}(1) & \cdots \\
C_{X}(0) & C_{X}(1) & \cdots
\end{bmatrix}
\]

We normalize the equations by letting \( B_{1.0} = 1 = B_{2.0} \). Suppose the constraint for parameters of the first equation is

\[
r_{11} = 0,
\]

and the constraints for parameters of the second equation are

\[
B_{2.1} = 0 \quad r_{21} + r_{22} = 1.
\]

Then

\[
\Phi_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}, \quad d_1 = (1, 0),
\]

\[
\Phi_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad d_2 = (1, 0, 1),
\]

\[
(2.32)
\]

\[
(2.33)
\]
1° When \( \text{rank}(W) = 4 \), then \( k^o = k \). Thus
\[
\text{rank}(A\phi_1) = \text{rank} \begin{bmatrix} 1 & 0 \\ B_{z1.0} & r_{z1.1} \end{bmatrix} = 2,
\]
\[
\text{rank}(A\phi_2) = \text{rank} \begin{bmatrix} B_{1z.0} & B_{1z.1} & r_{1z.1} \\ 1 & 0 & 1 \end{bmatrix} = 2.
\]

By Theorem 2.1, they are both identified.

2° \( \text{rank}(W) = k^o = 3 \).

Since \( A \) is not a basis for the row kernel of \( W \), there exists a solution vector \( a = (a_1, a_2, \ldots, a_8) \) of Eq. \( xW = 0 \) such that
\[
\tilde{A} = \begin{bmatrix} B_{11.0} & B_{12.0} & B_{11.1} & B_{12.1} & r_{11} & r_{12} \\ B_{21.0} & B_{22.0} & B_{21.1} & B_{22.1} & r_{21} & r_{22} \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{bmatrix}
\]
constitutes a basis for the row kernel of \( W \); we thus have
\[
\text{rank}(\tilde{A}\phi_1) = \text{rank} \begin{bmatrix} 1 & 0 \\ B_{z1.0} & r_{z1.1} \\ a_1 & a_6 \end{bmatrix} = 2 < G(p + 1) + k(q + 1) - k^o = 3,
\]
\[
\text{rank}(\tilde{A}\phi_2) = \text{rank} \begin{bmatrix} B_{12.0} & B_{12.1} & r_{12} \\ 1 & 0 & 1 \\ a_2 & a_4 & a_5 + a_6 \end{bmatrix} = \text{rank} \begin{bmatrix} B_{12.0} & B_{12.1} & r_{12} - B_{12.0} \\ 1 & 0 & 0 \\ a_2 & a_4 & a_5 + a_6 - a_2 \end{bmatrix}
\]

By Theorem 2.1, the first equation is completely underidentifiable, when \( B_{z1.1}(a_5 + a_6 - a_2) \neq a_4(r_{12} - B_{1z.0}) \), the second equation is still identified by Theorem 2.1. Since the Lebesgue measure of the point set on which \( \text{rank}(\tilde{A}\phi_2) \neq 3 \) is equal to zero, the second equation is almost identifiable.

3° \( \text{rank}(W) = k^o = 2 \).

Since \( R_1 = 2 < G(p + 1) + k(q + 1) - k^o = 4 \),
\[
R_2 = 3 < 4,
\]
hence the two equations are both completely underidentifiable.
References


从参数唯一确定的观点论联立经济模型的识别问题（II）

田国强
（华中工学院）

提 要

本文继续利用文献（14）的观点和方法研究了动态模型的识别问题，并且得到了若干有意义的结果。文中移去了“要求矩阵多项式方程 $| B_n + B_{n-1}L + \cdots + B_1L^{n-1} | = 0$ 的根全部位于单位圆内” 的假定，这样使得我们能够讨论非稳定的动态模型。文中还去掉“外生变量不只任何线性恒等式连接”的通常假定。由于这两个假定的去掉，使得我们利用一步识别法和二步识别法大大地扩大了讨论的范围。同前文，我们将利用同样的观点和方法来探讨非线性模型、误差冲击模型（包括非稳定的动态误差模型），也将得到类似的结果。