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Ratio-Lindahl equilibria and an informationally efficient and implementable mixed-ownership system ☆

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Abstract

In this paper, we formalize an informationally efficient and implementable mixed-ownership economic institutional framework by using the Ratio-Lindahl equilibrium that yields Pareto-efficient and individually rational allocations for public goods economies with general variable returns. We consider the incentive aspects of the system by giving an "incentive-compatible," informationally efficient, and "privacy preserving" mechanism whose Nash allocations coincide with Ratio-Lindahl allocations. Moreover, the mechanism is "well-behaved" in the sense that it uses an individually feasible, balanced, and continuous outcome function and further, has a message space of minimal dimension.

Keywords: Ratio-Lindahl equilibrium; Mixed-ownership system; Incentive; Information; Mechanism design

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1. Introduction

The notion of self-interested behaviour is a basic and key concept in microeconomics and the theory of individuals' behaviour. However, most investigations have been concerned with analysing self-interested behaviour only under conditions of private-ownership – in particular, market – institutions. Initially in this literature, the central question asked was whether a certain mechanism (especially the competitive mechanism) generated Pareto-efficient allocations, and if so – for what categories of economic environments. Subsequently, the question was reversed in the mechanism design literature: instead of regarding mechanisms as given and seeking the class of environments for which they work one seeks mechanisms which will implement some socially desirable goals (especially those which result in Pareto-efficient and individually rational allocations) for a given class of environments without destroying participants' incentives, and which have a low cost of operation and other desirable properties.

The reverse question was stimulated by two major lines in the history of economics. Within the capitalist/private-ownership economy literature, a stimulus arose from studies focusing upon the failure of the competitive market to function as a mechanism for implementing efficient allocations in many nonclassical economic environments. A second stimulus arose from the socialist/state-ownership economy literature, as evidenced in the "socialist controversy" – the debate between Mises-Hayek and Lange-Lerner. The controversy was provoked by von Mises's scepticism as to even a theoretical feasibility of rational allocation under socialism. The incentives structure and information structure are two basic features of any economic system. The study of these two features is attributed to these two major lines, culminating in the theory of mechanism design. The theory of economic mechanism design which was originated by Hurwicz (1960, Hurwicz, 1972a, Hurwicz, 1972b, Hurwicz, 1979a) is very general. All economic mechanisms and systems (including those known and unknown, private-ownership, state-ownership, and mixed-ownership systems) can be studied with this theory.

For public goods economies, the commonly used social choice rule is the Lindahl correspondence which is a general equilibrium concept for private-ownership institutions in which the profits (or losses) of firms are shared by individuals (stock-holders) and there are no state-owned firms. Since the Lindahl mechanism is not "incentive-compatible" (in the sense that it has the "free rider" problem), there are many mechanisms which implement the Lindahl correspondence at Nash equilibrium points such as those in Hurwicz (1979b), Walker (1981), Hurwicz et al. (1984), Hurwicz (1986a), Groves and Ledyard (1987), Tian (1989, Tian, 1990, Tian, 1991, Tian, 1993), Nakamura (1989), Tian and Li (1991), Li et al., (1990) among others.

However, when some firms in a society are owned by the government and the technologies of these firms do not exhibit constant returns to scale (CRS), but rather exhibit decreasing (DRS) or increasing returns to scale (IRS), either the conventional Lindahl mechanism fails since it is not clear how the profits should be distributed, or a Lindahl equilibrium does not exist as in the case of IRS. This is true if the system is a state-ownership or mixed-ownership system [1]. Recently, Tian and Li (1994a) showed that the (Generalized) Ratio equilibrium, which was first introduced by Kaneko (1977) and generalized by Mas-Colell and Silvestre (1989), can be viewed as an equilibrium concept for state-ownership and is an appropriate social choice rule because it results in Pareto-efficient and individually rational allocations. The idea of the Generalized Ratio equilibrium is to share the cost but not the profit of production and thus avoid the profit-distribution problem. Tian and Li further considered the incentive aspects of the system and gave a completely feasible and continuous mechanism which fully implements the Generalized Ratio correspondence while allowing for general variable returns at different scales of production: IRS, CRS, and DRS.

As we can see, all countries in the real-world have mixed-ownership systems to some degree. For some purposes, the government owns firms which produce goods that privately-owned firms cannot produce, while other goods are produced by privately-owned firms. However, all of these systems have incentive and/or information problems. Some natural questions to ask are: how to formalize these kinds of mixed-ownership systems (institutions), what social choice rules should be used in these types of economic systems so that they result in desirable outcomes (in particular Pareto-efficient and individually rational allocations) and, can the social choice rule chosen for the mixed-ownership system be implemented by an "incentive-compatible", "informationally decentralized" and "well-behaved" mechanism? These questions are the focus of this paper, which finds that each question can be answered affirmatively.

In this paper, we formalize an informationally efficient and implementable mixed-ownership economic institutional framework by using a general equilibrium concept which we call the Ratio-Lindahl equilibrium that yields Pareto-efficient and individually rational allocations for public goods economies with general variable returns. This equilibrium concept is in fact a combination of the Generalized Ratio equilibrium which is viewed as a state-ownership and the Lindahl equilibrium which is viewed as a privately-ownership. It allows for the coexistence of privately-owned firms and state-owned firms in which the technologies of state-owned firms can have general variable returns at different scales of production. Like Kaneko (1977) and Mas-Colell and Silvestre (1989), we assume that the government gives each state-owned firm an exclusive franchise to produce certain public goods. This may be a reasonable and realistic assumption. For instance, in certain situations in the real world, competition may be impractical. Realizing this,

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[1] Here, state-ownership means the government owns the firms and individuals do not share the profits (losses) but may need to share the cost of production. Mixed-ownership implies the coexistence of privately-owned firms and state-owned firms in the society.

governments often allow state-owned firms to take advantage of large economies of scale. The government also asks individuals to share the cost of these public goods. The other public goods are produced by privately-owned firms according to the profit maximizing rule and sold in competitive markets. Production possibility sets for the privately-owned firms are assumed to be convex while production possibility sets for state-owned firms are not necessarily restricted to convex sets so that state-owned firms can take advantage of their large economies of scale. Agents are self-interested and maximize their utilities under their budget constraints so that the sum of the expenditure on private goods and public goods produced by privately-owned firms and cost shares for public goods produced by state-owned firms does not exceed their wealth. A remarkable property of Ratio-Lindahl equilibrium is that it applies to a broader range of economic environments than those used in the literature by allowing the presence of by-products. In the existing literature, production technologies in public goods economies preclude the by-products technologies. But in our paper, we consider a more general case than those in the literature since our model includes by-products as a special case. In other words, our model applies not only to the standard public goods economies but also to public goods economies with by-products productions. As Tian (1994) showed, while this may make little difference for the Walrasian process, it makes a big difference for the Lindahl process in terms of informational efficiency. Furthermore, the Ratio-Lindahl equilibrium is more general than those in the literature which includes the private-ownership, state-ownership, Walrasian equilibrium, Lindahl equilibrium, Ratio equilibrium, and Generalized Ratio equilibrium as special cases.

We then consider the incentive aspects of the Ratio-Lindahl allocation process giving an "incentive-compatible," informationally efficient, and "privacy preserving" mechanism whose Nash allocations coincide with Ratio-Lindahl allocations for general variable returns. Further, the mechanism is "well-behaved" in the sense that it uses an individually feasible, balanced, and continuous outcome function and further, has a message space of minimal dimension.

The plan of this paper is as follows. In section 2 we give formal definitions of a mixed-ownership institutional model and Ratio-Lindahl equilibrium for public goods economies. Section 3 gives a parametric mechanism which has the desirable properties mentioned above to deal with incentive aspects of Ratio-Lindahl allocations. We also prove that this mechanism fully implements the Ratio-Lindahl correspondence. Concluding remarks are presented in section 4.

2. Model and Ratio-Lindahl allocations

2.1. A mixed-ownership model for public goods economies

In an economy with public goods, there are n agents who consume one private good x (the numeraire) and K public goods y. The single private good x can, and probably should, be thought of as a Hicksian composite commodity or money, and public goods y can be thought of as K projects. Denote by $N = \{1, 2, ..., n\}$ the set of agents. We assume there are no initial endowments of public goods, but that public goods can be produced from the private good by J private-owned firms and T state-owned firms. We will use the subscripts s and p to denote state-owned firms and privately-owned firms, respectively. We further assume that the government gives each state-owned firm t an exclusive franchise to produce K_{st} public goods, denoted by $y_{st} \in R_{+}^{K_{st}}$. Production technology for each state-owned firm t is represented by a production function $f_{st}: R_{+} \to R_{+}^{K_{st}}$ and thus $y_{st} = f_{st}(v_{st})$, where v_{st} is the input used by state-owned firm t. Note that this notation allows the presence of by-products (joint-production) when $K_{st} > 1$, that is, the output level of each public good produced by state-owned firm t is fixed for each given level of input. In other words, there is no substitution of production between public goods. Note that this assumption contains the usual assumption that each firm produces only one public good as a special case. This may be more a realistic and reasonable assumption since in general a firm has some by-products when it produces a good. An example of a by-product is a dam which can be used to hold water, but also be used to generate electricity. Let $f_s(v_s) = (f_{s1}(v_{s1}), ..., f_{sT}(v_{sT}))$. Let $\Sigma_{t=1}^{T} K_{st} = K^s$, which is the total number of public goods produced by the state-owned firms that may have CRS, DRS, or IRS technologies. The remaining $K_p \equiv K - K_s$ public goods are produced competitively by J privately-owned firms and thus $K_p + K_s = K$. Public goods produced by the privately-owned firm j are denoted by $y_{pj} \in \mathbb{R}_{+}^{K_p}$. The cost functions (the input demand function) of state-owned firm t and privately-owned firm j are denoted by $C_{st}(y_{st})$ and $C_{pj}(y_{pj})$, respectively [2]. We assume throughout that, for the state-owned firm t, $C_{st}(0) = 0$ and C_{st} is increasing and continuous; for the privately-owned firm j, $C_{pj}(0) = 0$, C_{pj} is strictly increasing, continuous, and convex. Thus we allow the technologies of state-owned firms to display IRS (so that the state-owned firms can take advantage of large economies of scale) but not privately-owned firms. Let $H = J + T$ the total number of firms, and let $C_s(y_s) = (C_{s1}(y_{s1}), ..., C_{sT}(y_{sT}))$, $C_p(y_p) = (C_{p1}(y_{p1}), ..., C_{pJ}(y_{pJ}))$, and $C(y) = (C_s(y_s), C_p(y_p))$, where $y_s = (y_{s1}, ..., y_{sT})$ and $y_p = (y_{p1}, ..., y_{pJ})$. Denote by $\hat{y}_p = \Sigma_{j=1}^{J} y_{pj}$ aggregate public goods produced by privately-owned firms.

Each agent's characteristic is denoted by $e_i = (w_i, R_i)$, where $w_i > 0$ which is the initial endowment of the private good and R_i is the preference ordering (P_i denotes the asymmetric part of the preference R_i) defined on $\mathbb{R}_{+}^{1+K}$ which is strictly monotonically increasing. Each agent i has J nonnegative profit shares $\theta_{ij} \in [0, 1]$ from the J private firms, satisfying $\Sigma_{i \in N} \theta_{ij} = 1$, $j = 1, ..., J$.

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[2] The cost function coincides with the input demand function since the private good is a numeraire good.

A mixed-ownership economy is the full vector $e=(e_1, ..., e_n, C_s(\cdot), C_p(\cdot), \{\theta_{ij}\})$, and the set of all such economies is denoted by E.

An allocation $(x, y)=(x_1, ..., x_n, y)$ is feasible for an economy e if $(x,y) \in \mathbb{R}_+^{n+K_s+JK_p}$ and

$$\sum_{i=1}^{n} x_i + \iota_H \cdot C(y) \leq \sum_{i=1}^{n} w_i \tag{1}$$

where L_H is a vector of ones of dimension H. An allocation (x, y) is *Pareto efficient* for an economy e if it is feasible and there is no other feasible allocation (x', y') such that $(x_i', y_s', \hat{y}_p')R_i(x_i, y_s, \hat{y}_p)$ for all $i \in N$ and $(x_k', y_s', \hat{y}_p')R_k(x_k, y_s, y_p)$ for some $k \in N$. An allocation (x, y) is individually rational for an economy e if $(x_i, y_s, \hat{y}_p)R_i(w_i, 0)$ for all $i \in N$.

2.2. Ratio-Lindahl allocations

An allocation (x^*, y^*) is a *Ratio-Lindahl allocation* for an economy e in a mixed-ownership system if it is feasible and there are ratio vectors $r_i^* \in R_+^T$ and personalized price vectors $q_i^* \in \mathbb{R}_+^{K_p}$, one for each i, such that

$$(1)\, x_i^* + r_i^* \cdot C_s(y_s^*) + q_i^* \cdot \hat{y}_p^* \leq w_i + \sum_{j=1}^{J} \theta_{ij}\left[q^* \cdot y_{pj}^* - C_{pj}(y_{pj}^*)\right] \forall i \in N;$$

$$(2) \forall i \in N, (x_i, y_s, y_p) P_i(x_i^*, y_s^*, \hat{y}_p^*) \text{ implies } x_i + r_i^* \cdot C_s(y_s) + q_i^* \cdot \hat{y}_p > w_i + \sum_{j=1}^{J} \theta_{ij}\left[q^* \cdot y_{pj}^* - C_{pj}(y_{pj}^*)\right];$$

$$(3)\, q^* \cdot y_{pj}^* - C_{pj}(y_{pj}^*) = \max_{y_{pj} \in \mathbb{R}_+^{K_p}}\left(q^* \cdot y_{pj} - C_{pj}(y_{pj})\right) \forall j \in \{1,...,J\},$$

where $\sum_{i=1}^{n} r_i^* = \iota_T$, and $\sum_{i=1}^{n} q_i^* = q^*$ is the market price vector for public goods produced by privately-owned firms. Denote by $RL(e)$ the set of all allocations.

Remark 1. Note that the Ratio-Lindahl equilibrium is a combination of the Generalized Ratio equilibrium introduced in Tian and Li (1994a) and Tian (1994) and the Lindahl equilibrium. The system reduces to state-ownership when $J=0$ and to private-ownership when $T=0$. Also, the Ratio-Lindahl equilibrium reduces to the Ratio equilibrium (RE) of Kaneko (1977) when $J=0$ and $K_t=1$ for all $t \in \{1, ..., T\}$, reduces to the Generalized Ratio equilibrium when $J=0$, reduces to the Walrasian equilibrium when $T=0$, and reduces to the Lindahl equilibrium when $T_S=0$. Thus the Ratio-Lindahl equilibrium includes the Ratio equilibrium, the Generalized Ratio equilibrium, the Walrasian equilibrium, and the Lindahl equilibrium as special cases and thus unifies the Ratio, Generalized Ratio, and Lindahl equilibria. Further, a Ratio-Lindahl equilibrium is a Balanced Linear Cost Share equilibrium of Mas-Colell and Silvestre (1989) when $J=0$ and $T=1$ (by taking the a vectors to zero).

It is clear that every Ratio-Lindahl allocation is individually rational. We now show that every Ratio-Lindahl allocation is Pareto-efficient.

Proposition 1. If (x, y) is a Ratio-Lindahl allocation, then it is a Pareto-efficient allocation.

Proof. Suppose, by way of contradiction, that there exists a feasible allocation (x', y') such that at least one consumer (say, consumer i) is better off and others are not worse off under (x', y'). Then we must have $x'_i + r_i^* C_s(y'_s) + q_i^* \hat{y}'_p > w_i + \sum_{j=1}^J \theta_{ij}[q^* \cdot y_{pj}^* - C_{pj}(y_{pj}^*)]$ and $x'_k + r_k^* \cdot C_s(y'_s) + q_k^* y'_p \geq w_k + \sum_{j=1}^J \theta_{kj}[q^* \cdot y_{pj}^* - C_{pj}(y_{pj}^*)]$ by monotonicity of preferences. Therefore, adding up these inequalities, we have

$$\sum_{k=1}^n w_k + q^* \cdot y_p^* - \iota \cdot C_p(y_p^*) < \sum_{k=1}^n x'_k + \iota \cdot C_s(y'_s) + q^* \cdot y'_p$$

$$\leq \sum_{k=1}^n w_k + q^* \cdot y'_p - \iota \cdot C_p(y'_p) \tag{2}$$

and thus

$$q^* \cdot y_p^* - \iota \cdot C_p(y_p^*) < q^* \cdot y'_p - \iota \cdot C_p(y'_p), \tag{3}$$

which contradicts the fact that y_p^* is the profit maximizing level of output. Q.E.D.

The remainder of this section considers the existence of Ratio-Lindahl equilibria. It may be remarked that we cannot follow the approaches used by Kaneko (1977) or Mas-Colell and Silvestre (1989) to prove the existence of a Generalized Ratio equilibrium since these approaches require that preference orderings be convex, and production technologies of state-owned firms not display IRS (i.e., production functions of state-owned firms need to be concave). Here we follow an approach given in Tian (1994) which shows that proving the existence of a Ratio-Lindahl equilibrium for the original mixed-ownership economy is equivalent to proving the existence of a Lindahl equilibrium for the transformed private-ownership economy. We call this approach the "Transformation Equivalence Principle" (TE-Principle) which plays an important role not only in proving the existence but also in considering the incentive and information aspects of Ratio-Lindahl allocations.

For any economy e with $e_i = (w_i, R_i)$ and production functions f_{st}, we can define a private-ownership economy e' with $e'_i = (R'_i, w_i)$, x being private good, $(v_s, \hat{y}_p) \in R_+^{T+K_p}$ being public goods, and R'_i being preferences which are defined by $(x_i, v_s, \hat{y}_p) R'_i (x_i, v'_s, \hat{y}'_p)$ if and only if $(x_i, f(v_s), \hat{y}_p) R_i (x_i, f(v'_s), \hat{y}'_p)$ [3].

Proposition 2. (TE-Principle): For any economy $e \in E$, if (x^*, y^*) is a Ratio-Lindahl allocation for e, then $(x^*, z^*) \equiv (x^*, C_s(y_s^*), y_p^*)$ is a Lindahl

[3] When preferences R_i can be represented by utility functions $u_i(x_i, y)$, R'_i are defined by utility functions $u'_i x_i, v_s, \hat{y}_p) \equiv u(x, f_s(v'_s), \hat{y}'_p)$.

allocation for the transformed economy e'; and if (x^*, z^*) is a Lindahl allocation for the transformed economy e', then $(x^*, y^*) \equiv (x^*, f_s(v_s^*), y_p^*)$ is a Ratio-Lindahl allocation for the original economy e.

Proof. We first note that if (x^*, y^*) is a Ratio-Lindahl allocation for e, then, by monotonicity of preferences, $f_s(C(y_s^*)) = y_s^*$ (i.e., y_s^* is produced in an efficient way). We then show that if (x^*, y^*) is a Ratio-Lindahl allocation for e then $(x^*, z^*) = (x^*, C_s(y_s^*), y_p^*)$ is a Lindahl allocation for e'. Since (x^*, z^*) is feasible and y_p^* is the profit maximizing level of output, we only need to show that each agent is maximizing his utility. Suppose, by way of contradiction, that there is some $i \in N$ and (x_i, z) such that

$$x_i + r_i^* \cdot v_s + q_i^* \cdot \hat{y}_p \leq w_i + \sum_{j=1}^{J} \theta_{ij}\left[q^* \cdot y_{pj}^* - C_{pj}(y_{pj}^*)\right]$$

and $(x_i, f_s(v_s), \hat{y}_p) P_i (x_i^*, f_s(v_s^*), \hat{y}_p^*)$. Let $y_s = f_s(v_s)$. Then $(x_i, y_s, \hat{y}_p) P_i (x_i^*, f_s(v^*), \hat{y}_p^*)$and yet

$$x_i + r_i^* \cdot C_s(y_s) + q_i^* \cdot \hat{y}_p \leq w_i + \sum_{j=1}^{J} \theta_{ij}\left[q^* \cdot y_{pj}^* - C_{pj}(y_{pj}^*)\right].$$

But this contradicts the hypothesis that (x^*, y^*) is a Ratio-Lindahl allocation.

Now we show that $(x^*, y^*) = (x^*, f_s(v_s^*), y_p^*)$ is a Ratio-Lindahl allocation if (x^*, z^*) is a Lindahl allocation for e'. Similarly, we only need to show that each agent is maximizing his utility. Suppose, by way of contradiction, that there is some $i \in N$ and (x_i, y) such that

$$x_i + r_i^* \cdot C_s(y_s) + q_i^* \cdot \hat{y}_p \leq w_i + \sum_{j=1}^{J} \theta_{ij}\left[q^* \cdot y_{pj}^* - C_{pj}(y_{pj}^*)\right]$$

and $(x_i, y_s, \hat{y}_p) P_i (x_i^*, y_s^*, \hat{y}_p^*)$. Since $f_s(C(y_s)) \geq y_s$, then $(x_i, f_s(C(y_s)), \hat{y}_p) P_i (x_i^*, f_s(v_s^*), \hat{y}_p^*)$. Let $v_s = C_s(y_s)$. Then we have $(x_i, v_s, \hat{y}_p) P_i' (x_i^*, v_s^*, \hat{y}_p^*)$ and

$$x_i + r_i^* v_s + q_i^* \cdot \hat{y}_p \leq w_i + \sum_{j=1}^{J} \theta_{ij}\left[q^* \cdot y_{pj}^* - C_{pj}(y_{pj}^*)\right].$$

But this contradicts the hypothesis that (x^*, v_s^*, y_p^*) is a Lindahl allocation. Q.E.D.

Theorem 1. For an economy e, if the transformed preference orderings R_i' are continuous and convex [4]*, and the technologies of privately-owned firms satisfy standard assumptions listed in Milleron (1972, p.443), then there exists a Ratio-Lindahl equilibrium for the economy e.*

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[4] A preference R_i' is convex if, for bundles a, b, c and $0 < \lambda \leq 1$, $c = \lambda a + (1-\lambda) b$, the relation a P_i' b implies c P_i' b.

Proof. For the transformed economy e' which is a private-ownership economy, technologies of firms either display CRS (if the firms are state-owned firms in the original economy) or satisfy standard assumptions listed in Milleron (1972) (if the firms are privately-owned firms in the original economy). Then we know the existence of Lindahl equilibrium by Theorem 3.1 of Milleron (1972, p. 443) for the transformed economy e'. Thus there is a Ratio-Lindahl equilibrium by the TE-Principle. Q.E.D.

Remark 2. Note that, when preferences R_i can be represented by a utility function $u_i(x_i, y)$, a sufficient condition for R'_i to be continuous and convex is that the composite function $u(x, f(v))$ is continuous and quasi-concave. Also, it can be verified easily that a sufficient condition for convexity of transformed preference orderings R'_i is that preferences R_i are convex and f_s is concave.

Remark 3. The following example shows that even if the production functions of state-owned firms exhibit IRS, the utility functions of agents after transformation can be quasi-concave. Thus by the TE-Principle we know the existence of a Ratio-Lindahl equilibrium provided the transformed utility functions are continuous and the technologies of privately-owned firms satisfy standard assumptions.

Example 1. Consider the case of $T = 1$, $J = 1$, and $K = 2$. Suppose that the utility function of agent i is given by $u_i(x_i, y) = x_i^{1/3} y_s^{1/6} y_p^{1/3}$ and the production function of the state-owned firm is given by $y_s = f_s(v_s) = v_s^2$ which exhibits IRS. Then the utility function after transformation is $u'_i(x_i, v_s, y_p) = x_i^{1/3} v^{1/3} y_p^{1/3}$ which is quasi-concave. Thus the existence of Ratio-Lindahl equilibrium is a consequence of the existence of Lindahl equilibrium.

In fact, as long as the transformed utility function u'_i is a Cobb-Douglas utility function, it is quasi-concave (cf. Berge (1963, p. 209)). Thus we have the following proposition which gives a sufficient condition for the transformed utility functions to be quasi-concave.

Fact 1. If both utility functions u_i and production functions f_s are Cobb-Douglas type functions (f_s can be any degree of IRS as long as it is a Cobb-Douglas type function), the transformed utility functions u'_i are quasi-concave.

3. Implementation of Ratio-Lindahl allocations

In this section we consider the implementation of Ratio-Lindahl allocations. The Ratio-Lindahl allocation process, like the Lindahl allocation process, has a "free rider" problem. So we need to design mechanisms which, in some sense, implement the Ratio-Lindahl correspondence. To do so, we first give some notation and definitions used in the mechanism design literature. We then give a completely feasible and continuous mechanism which fully implements the Ratio-Lindahl correspondence through Nash equilibria. Further, the mechanism has a message space of minimal dimension.

Let F be a social choice rule, i.e., a correspondence from E to the set Z of resource allocations. Let M_i denote the i-th message (strategy) domain. Its elements are written as m_i and called messages. Let $M = \prod_{i=1}^{n} M_i$ denote the message (strategy) space. Let $h: \mathscr{M} \to Z$ denote the outcome function, or more explicitly, $h_i(m) = (X_i(m), Y(m), V(m))$, where $X_i(m)$ is the i-th agent's outcome function for the private good, $Y(m)$ is the outcome function for the public goods, and $V(m)$ is the input demand outcome function needed to produce public goods $Y(m)$ – in particular, $Y_s(m) = f_s(V_s(m))$.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is a *Nash equilibrium* (NE) of the game form [5] $\langle M, h \rangle$ for an economy e if for any $i \in N$ and for all $m_i \in M_i$,

$$\left(X_i(m^*), \hat{Y}_i(m^*)\right) R_i \left(X_i(m^*/m_i, i), \hat{Y}_i(m^*/m_i, i)\right), \tag{4}$$

where $(m^*/m_i, i) = (m_{1*}, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. The outcome $h(m^*)$ is then called a *Nash (equilibrium) allocation*. Denote by $V_{M,h}(e)$ the set of all such Nash equilibria and by $N_{M,h}(e)$ the set of all such Nash (equilibrium) allocations. A mechanism $\langle M, h \rangle$ fully *Nash-implements* the social choice rule F on E, if, for all $e \in E$, $N_{M,h}(e) \neq \varnothing$ and $N_{M,h}(e) = F(e)$.A mechanism $\langle M, h \rangle$ is *individually feasible* if $(X_i(m), Y(m)) \in \mathbb{R}_+^{1+K}$ for all $i \in N$ and all $m \in M$. A mechanism $\langle M, h \rangle$ is *balanced* if for all $m \in M$

$$\sum_{j=1}^{N} X_j(m) + \iota_H \cdot C(Y(m)) = \sum_{j=1}^{n} w_j. \tag{5}$$

A mechanism $\langle M, h \rangle$ is *completely feasible* if it is individually feasible and balanced.

In the following we will present a completely feasible and continuous mechanism with a minimal-sized message space which fully Nash-implements the Ratio-Lindahl correspondence. For simplicity, we only consider the case of one privately-owned firm, an extension to the more general case is straightforward. Before the formal presentation of this mechanism, we give the following brief description. After all agents report their messages, first determine the consumers' ratios for the cost of productions of state-owned firms and personalized prices for public goods produced by the privately-owned firms. Then define the prices of public goods y_p (produced by privately-owned firms) by summing all the consumers' personalized prices. After that, find the profit maximizing level of output for the public goods y_p in the feasible production set A under given resources such that its marginal cost is the closest to the price vector of the public goods y_p. Then define a feasible choice correspondence, B, for all public goods that can be produced with the total endowments and can be purchased by all agents. The outcome $Y_p(m)$ for public goods produced by privately-owned firms and outcome

[5] A game form is also called a mechanism in the incentives literature. Note that messages are chosen in whole space (not only in the set of stationary messages).

$V_s(m)$ for the input demand function of state-owned firms will be chosen from $B(m)$ such that it is the closest to the profit maximizing level of output for public goods y_p and the sum of contributions that all agents are willing to make to produce public goods by state-owned firms. The outcome function for public goods produced by state-owned firms then is given by $f_s(V_s(m))$. Finally, the outcome for private good $X(m)$ is then determined in such a way that the budget constraint of each agent holds with equality. It will be seen that the allocation $(X(m), Y(m))$ resulting from the message m is individually feasible, balanced, and continuous for all $m \in M$ and almost everywhere differentiable on the message space, further, the mechanism has a message space of minimal dimension.

Now we turn to the formal construction of the mechanism. We first make the following additional assumptions.

Assumption 1. $n \geq 3$.

Assumption 2. Each agent's transformed preferences R'_i can be represented by a twice continuously differentiable (C^2) and strictly quasi-concave utility function u'_i which satisfies the strictly differentiable monotonicity (i.e., $\partial u'_i/\partial x_i > 0$ and $\partial u'_i/\partial z > 0$) on $\mathbb{R}^{1+G}_{++}$, where $G = T + K_p$.

Assumption 3. For all $i \in N$, $(x_i, z)P'_i(x'_i, z')$ for all $(x_i, z) \in \mathbb{R}^{1+G}_{++}$ and $(x'_i, z')\mathbb{R}^{1+G}_{+}$, where $\partial\mathbb{R}^{1+G}_{+}$ is the boundary of $\mathbb{R}^{1+G}_{+}$.

Assumption 4. The technologies of production of privately-owned firms can be represented by a production function f_p: $\mathbb{R}_+ \to \mathbb{R}^H_+$ which displays DRS, and the cost function $C_p(y_p)$ dual to f_p is C^2, strictly increasing, and strictly convex, and $C_p(0) = 0$. Thus, $DC_p > 0$ for all $y_p > 0$ and D^2C_p is positive definite. [6]

Remark 4. Assumption 1 is a necessary condition for a balanced and continuous mechanism (cf. Kwan and Nakamura (1987)). Assumption 3 cannot be dispensed with in Theorem 2 below, for without it, the Ratio-Lindahl correspondence may not be monotone since the Lindahl correspondence defined on the transformed economies may not be monotone (cf. Tian (1988)) and thus cannot be Nash-implemented by a feasible mechanism by the result of Maskin (1977). However, this interiority assumption can be replaced by a weaker assumption used in Tian (1989) to allow the boundary Ratio-Lindahl allocations.

For each $i \in N$, the message domain is of the form

$$M_i = R^T \times R^{K_p}. \tag{6}$$

A generic element of M_i is $m_i = (\gamma_i, \phi_i)$, where γ_i is interpreted as the contribution that agent i is willing to make to produce public goods produced by state-owned firms and ϕ_i is the personalized price vector that agent i is willing to pay for public goods produced by privately-owned firms (a negative m_i means agent i hope to have a subsidy from society).

[6] DC_p and D^2C_p represent the first and second derivatives respectively.

Let the ratio to the i-th agent attached by the state-owned firm t be of the form

$$r_i^t(m) = b_i^t + \sum_{k=1}^{n} a_{ik}^t \gamma_k^t,$$

where $\Sigma_{i=1}^n b_i^t = 1$, $\Sigma_{i=1}^n a_{ik}^t = 0$, $a_{ii}^t = 0$, and $\Sigma_{k=1}^n |a_{ik}^t| > 0$ for $i \in N$ and $t = 1,...,T$. In addition, the coefficients a_{ik}^t of γ_k^t are chosen so that for any given personal ratio vectors r_i the coefficient vectors of γ_k^t are linearly independent of the vector of ones, where r_i is the i-th agent's ratio vector. Notice that by construction $\Sigma_{i=1}^n r_i^t(m) = 1$ for $t = 1,...,T$. Then $r_i(m) = (r_i^1(m), ..., r_i^T(m))$ for all $m \in M$.

Remark 5. $r_i(m)$ in (7) is independent of m_i.

In order to define the personalized prices of y_p for each consumer, we further assume, for simplicity, that $\theta_i \in [0,1)$ for all $i \in N$. [7]

Define the personalized price vector of consumer i by

$$q_i(m) = \theta_i \phi_i + (1 - \theta_{i+1}) \phi_{i+1}, \tag{8}$$

where $n + 1$ is to be read as 1.

Define the producer's price (market price) vector $q(m)$ as the sum of all of the consumers' personalized prices. Then we have

$$q(m) = \sum_{i=1}^{n} q_i(m) = \sum_{i=1}^{n} \phi_i \equiv \tilde{\phi}. \tag{9}$$

Observe that, by construction, consumers with zero profit shares cannot change their personalized prices $q_i(m)$ by changing their own messages, while the personalized prices of consumers with non-zero profit shares depend on their own messages.

Define the feasible production set for the privately-owned firm under given resources by

$$A = \left\{ y_p \in \mathbb{R}_+^{K_p} : 0 \le y_p \le f_p(w) \right\}, \tag{10}$$

where $w = \Sigma_{i=1}^n w_i$.

The profit maximizing level of output for public goods $\tilde{y}_p$ is given by

$$\tilde{y}_p(m) = \arg\min \left\{ \| q(m) - DC_p(y_p) \| : y_p \in A \right\}, \tag{11}$$

whose marginal cost is the closest to the producer's price vector. Note that

$$\tilde{y}_p(m) = (DC_p)^{-1}(q(m)) \tag{12}$$

if $q(m) \in DC_p[A]$.

[7] This assumption rules out the case that a single consumer has the total profit share for any private firms. It can be relaxed (cf. Li, Nakamura and Tian (1990)).

Define the feasible correspondence $B: M \to\to \mathbb{R}^G_+$ by

$$B(m) = \left\{ z \in \mathbb{R}^G_+ : w_i + \theta_i \left[q(m) \cdot y_p - C_p(y_p) \right] - q_i(m) \cdot y_p - r_i(m) \cdot v_s \geq 0, \forall i \in N \right\}, \tag{13}$$

which is clearly nonempty, compact, convex (by noting that $C_p(y_p)$ is convex), and continuous on M. Here $z = (v_s, y_p)$.

Define the outcome function $Z: M \to B$ by

$$Z(m) = \arg\min \left\{ \| z - \tilde{z}(m) \| : z \in B(m) \right\}, \tag{14}$$

where $\tilde{z}(m) = (\tilde{\gamma}, \tilde{y}_p(m))$ with $\tilde{\gamma} = \sum_{i \in N} \gamma_i$. Then $Z(m) = (V_s(m), Y_p(m))$ is single-valued and continuous on M.[8]

Define the outcome function of public goods produced by state-owned firms $Y_s: M \to \mathbb{R}^{K_s}_+$ by $Y_s(m) = f_s(V(m))$. Then $C_s(Y_s(m)) = V_s(m)$.

Define the i-th agent's outcome function $X_i: M \to R_+$ by

$$X_i(m) = w_i + \theta_i \left[q(m) \cdot Y_p(m) - C_p(Y_p(m)) \right] - r_i(m) \cdot C_s(Y_s(m)) - q_i(m) \cdot Y_p(m), \tag{15}$$

which corresponds to the budget constraint. Since $\sum_{i \in N} r_i(m) = \iota_T$, $\sum_{i \in N} \theta_i = 1$ and $\sum_{i \in N} q_i(m) = q(m)$, we have

$$\sum_{i=1}^{n} X_i(m) + \iota_T \cdot C_s(Y_s(m)) + C_p(Y_p(m)) = \sum_{i=1}^{n} w_i, \tag{16}$$

which means the balanced condition holds for all $m \in M$.

Thus the mechanism specified above is individually feasible, balanced, continuous, and almost everywhere differentiable on M.

Remark 6. Since the outcome function of the mechanism depends on economic environments (endowments and profit shares), it is the so-called parametric mechanism (cf. Groves and Ledyard (1987)). This kind of dependence is necessary for a feasible mechanism by the results of Hurwicz et al. (1984).

3.1. Equivalence of Nash and Lindahl allocations

This subsection is devoted to the proof of equivalence between Nash allocations and Lindahl allocations, as stated in Propositions 3 and 4 below. Lemmas 1 and 2 are preliminary results used to prove these propositions.

Lemma 1. If $m^* \in V_{M,h}(e)$, then $(X(m^*), Z(m^*)) \in \mathbb{R}^{n+G}_{++}$.

Proof. We argue by contradiction. Suppose $(X(m^*), Z(m)) \in \mathbb{R}^{n+G}_+$. Then $Z(m^*) \in \mathbb{R}^{T+G}_+$ or $X_i(m^*) = 0$ for some $i \in N$.

[8] This is because $Z(m)$ is an upper semi-continuous correspondence by Berge's Maximum Theorem (see Berge (1963, p. 116)) and single-valued (see Mas-Colell (1985, p. 28)).

Let

$$\delta = \frac{\bar{w}}{2\left(\iota_{K_p} \cdot DC_p(\iota_{K_p}) + 2\hat{\phi}^* + c\right)},$$

where $\bar{w} = \min_{i \in N}|w|$, $\hat{\phi}^* = \Sigma_{h=1}^{K_p}\Sigma_{i=1}^{n}|\phi_i^{*h}|$, $a = \max_{i,k \in N \& t \in \{1,2,\ldots,T\}}|a_{ik}^t|$, $b = \max_{i \in N \& t\{1,2,\ldots,T\}}|b_i^t|$, and $c = b + 2a\Sigma_{t=1}^{T}\Sigma_{i=1}^{n}|\gamma_i^{*t}|$. Let $d = \min\{1, \delta\}$ and $\tilde{z} = (\tilde{v}_s, \tilde{y}_p) = (1/G)(d,\ldots,d)'$.

Now suppose that consumer i chooses his/her message $\phi_i = DC_p(\tilde{y}_p)$-$\Sigma_{l \neq i}^{n}\phi_l^*$ and $\gamma_i = \tilde{v}_s$-$\Sigma_{l \neq i}\gamma_l^*$. Then $q(m^*/m_i, i) = \phi_i + \Sigma_{l \neq i}^{n}\phi_l^* = DC_p(\tilde{y}_p)$ and

$$\begin{aligned}
& w_k + \theta_k\left[q(m^*/m_i,i) \cdot \tilde{y}_p - C_p(\tilde{y}_p)\right] - q_k(m^*/m_i,i) \cdot \tilde{y}_p - r_k(m^*/m_i,i) \\
& \quad \cdot \tilde{v}_s \geq w_k - q_k(m^*/m_i,i) \cdot \tilde{y}_p - r_k(m^*/m_i,i) \cdot \tilde{v}_s \\
& > w_k - \frac{1}{K_p}\sum_{h=1}^{K_p}\left[|\phi_i^h| + \hat{\phi}^*\right]d - \frac{1}{T}\sum_{t=1}^{T}\left[b + 2a\sum_{t=1}^{T}\sum_{i=1}^{n}|\gamma_i^{*t}|\right]d \\
& \geq w_k - \left[\iota_{K_p} \cdot DC_p(\tilde{y}_p) + 2\hat{\phi}^* + c\right]d \\
& \geq w_k - \left[\iota_{K_p} \cdot DC_p(\iota_{T_p}) + 2\hat{\phi}^* + c\right]\delta \text{ (by noting } d \leq 1 \text{ and } d \leq \delta) \\
& = w_k - \frac{\bar{w}}{2} \geq \frac{\bar{w}}{2} > 0
\end{aligned}$$

for all $k \in N$. Therefore $\tilde{z} \in B(m^*/m_i, i)$ and thus

$$Z(m^*/m_i,i) = \tilde{z} > 0,$$

$$\begin{aligned}
X_k(m^*/m_i,i) = {} & w_k + \theta_k\left[q(m^*/m_i,i) \cdot \tilde{y}_p - C_p(\tilde{y}_p)\right] - r_k(m^*/m_i,i) \\
& \cdot \tilde{v}_s - q_k(m^*/m_i,i) \cdot \tilde{y}_p > 0.
\end{aligned}$$

Hence, by Assumption 3, we have

$$\left(X_i(m^*/m_i,i), Z(m^*/m_i,i)\right)P_i'\left(X_i(m^*), Z(m^*)\right)$$

and thus

$$\left(X_i(m^*/m_i,i), Y(m^*/m_i,i)\right)P_i\left(X_i(m^*), Y(m^*)\right),$$

which contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

Lemma 2. If $m^* \in V_{M,h}(e)$, then $Z(m^*)$ int $B(m^*)$ and thus $Y_p(m^*) = \tilde{y}_p(m^*) = (DC_p)^{-1}(q(m^*))$, and $V_s(m^*) = \tilde{\gamma}^*$.

Proof. Suppose, by way of contradiction, that $Z(m^*) \in \partial B(m^*)$. [9] Then either $Z_k(m^*) = 0$ for some k or $0 = w_i + \theta_i[q(m^*) \cdot Y_p(m^*) - C_p(Y_p(m^*))] - p_i(m^*) \cdot Z(m^*) = X_i(m^*)$ for some $i \in N$, where $p(m^*) = (r(m^*), q(m^*))$. But both

[9] $\partial B(m)$ denotes the boundary of $B(m)$.

cases are impossible by Lemma 1. So $Z(m^*) \in$ int $B(m^*)$ and $Y_p(m^*) = \tilde{y}_p(m^*) = DC_p^{-1}(q(m^*))$ and $V_s(m^*) = \tilde{\gamma}^*$. Q.E.D.

Remark 7. From Lemma 2 and the twice differentiability of the cost function, we know that the outcome function is differentiable on some neighbourhood of $m^* \in V_{M,h}(e)$ and thus we can use the differential approach (first order condition) to find Nash equilibrium points. Lemma 2 also shows that the outcome for public goods y_p is equal to the profit maximizing level of output $\hat{y}_p$ at Nash equilibrium.

We now turn to prove the main results of this paper in the following theorems.

Proposition 3. Under Assumptions 1–4, if the mechanism defined above has a Nash equilibrium m^*, then $(X(m^*), Y(m^*))$ is a Ratio-Lindahl allocation with the ratio vector $(r_1(m^*),...,r_n(m^*))$ and personalized price vector $(q_1(m^*),...,q_n(m^*))$, i.e., $N_{M,h}(e) \subseteq RL(e)$.

Proof. Let m^* be a Nash equilibrium. We prove that $(X(m^*), Y(m^*))$ is a Ratio-Lindahl allocation with the ratio vector $(r_1(m^*),...,r_n(m^*))$ and personalized price vector $(q_1(m^*),...,q_n(m^*))$ for economy e. By the TE-Principle, we only need to show that $(X(m^*), Z(m^*))$ is a Lindahl allocation with the price vector $(p_1(m^*),...,p_n(m^*)) \in \mathbb{R}_+^{nG}$ for the transformed economy e', where $p_i(m^*) = (r_i(m^*), q_i(m^*))$. By the construction of the mechanism and Lemma 2, we know that $(X(m^*), Z(m^*))$ is feasible, $Y_p(m^*)$ is the profit maximizing level of output, and $(X_i(m^*), Z(m^*))$ satisfies the budget constraint of agent i. So we only need to show that each individual is maximizing his/her preferences.

By Lemma 1, we know that $(X_i(m^*), Z(m^*)) \in {}_{++}^{1+G}$. By Lemma 2, $Z(m^*) \in$ int $B(m^*)$ and $Z(m^*) = ((DC_p)^{-1}(q(m^*)), \tilde{\gamma}^*)$. Therefore there exists a neighbourhood $\mathscr{O}(m^*)$ of m^* such that $Z(m) \in$ int $B(m^*)$ for all $m \in \mathscr{O}(m^*)$. Hence all the outcome functions $p_i(m)$, $p(m)$, $X_i(m)$, and $Z(m)$ are differentiable on $\mathscr{O}(m^*)$. Thus, by using the first order condition, we have

$$D_{m_i} u_i'(X_i(m^*), Z(m^*)) = 0. \tag{17}$$

Differentiating $X_i(m)$ in (15), we have

$$D_{m_i} X_i = \theta_i\big(Y_p \cdot D_{m_i} q + q \cdot D_{m_i} Y_p - DC(Y_p) \cdot D_{m_i} Y_p\big)$$
$$- p_i \cdot D_{m_i} Z - Z \cdot D_{m_i} p_i. \tag{18}$$

By the constructions of $q(m)$, $q_i(m)$, and $r_i(m)$ and Lemma 2, we know that, for all $m \in \mathscr{O}(m^*)$, $q(m) = DC_p(Y_p(m))$, $D_{m_i} q = I_{K_p}$, and $D_{m_i} q_i = \theta_i I_{K_p}$ and $D_{m_i} r_i(m) = 0$. Here I_H is an $H \times H$ identity matrix. Thus (18) becomes

$$D_{m_i} X_i = -p_i D_{m_i} Z, \tag{19}$$

and therefore the first order condition (17) can be written as

$$\big[(D_x u_i')(-p_i(m^*)) + (D_z u_i')\big] \cdot D_{m_i} Z = 0. \tag{20}$$

Since

$$D_{m_i}Z = D_{m_i}(Y_P, V_s) = \begin{pmatrix} \left(D^2C_p\right)^{-1} & 0 \\ 0 & I_T \end{pmatrix} \tag{21}$$

is a non-singular matrix, we have

$$D_x u_i'(-p_i(m^*)) + D_z u_i' = 0 \tag{22}$$

and thus

$$\frac{1}{D_x u_i'} D_z u_i' = p_i(m^*), \tag{23}$$

which is the first order condition and is also the sufficient condition for the utility maximization since u_i' is strictly quasi-concave. Thus $(X(m^*), Z(m^*))$ is a Lindahl allocation for the transformed economy e' and thus, by TE-Principle, $(X(m^*), Y(m^*))$ is a Ratio-Lindahl allocation for e. Q.E.D.

Proposition 4. Under Assumptions 1–4, if (x^*, y^*) is a Ratio-Lindahl allocation with ratio vector $(r_1^*,...,r_n^*)$ and personalized priced vector $(q_1^*,...,q_n^*)$, then there is a Nash equilibrium m^* of the mechanism such that $Y(m^*) = y^*$, $X_i(m^*) = x_i^*$, $r_i(m^*) = r_i^*$ and $r_i(m^*) = r_i^*$, for all $i \in N$, i.e., $RL(e) \subseteq N_{M,h}(e)$.

Proof. We want to find a message m^* such that (x^*, y^*) is a Nash allocation. Consider the following linear equations system:

$$\Theta m = q^*, \tag{24}$$

where

$$\Theta = \begin{pmatrix} \theta_1 I_{K_p} & (1-\theta_2)I_{K_p} & 0 & \cdots & 0 & 0 \\ 0 & \theta_2 I_{K_p} & (1-\theta_3)I_{K_p} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \theta_{n-1} I_{K_p} & (1-\theta_n)I_{K_p} \\ (1-\theta_1)I_{K_p} & 0 & 0 & \cdots & 0 & \theta_n I_{K_p} \end{pmatrix} \tag{25}$$

and $q^* = (q_1^*,...,q_n^*)'$. Expanding the determinant $|\Theta|$ by the first T_p columns, we have $|\Theta| = \prod_{i=1}^n \theta_i^{T_p} + (-1)^{n+1}\prod_{i=1}^n (1-\theta_i)^{T_p} \neq 0$ since $\theta_i \in [0\ 1)$ for all $i \in N$.[10] Hence the linear equations system (linear) has a unique solution q^*. Also let γ^* be the unique solution of (7) and the equation $\sum_{k=1}^n \gamma_k = C_s(y_s^*)$.We then can find a unique m^* such that $q_i(m^*) = q_i^*$, $q(m^*) = \sum_{i=1}^n q_i(m^*) =$

[10] This is true for the following reasons. If $\theta_i = 0$ for some $i \in N$, then $|\Theta| = (-1)^{n+1}\prod_{i=1}^n (1-\theta_i)^{T_p} \neq 0$. If $\theta_i \neq 0$ for all $i \in N$, then $\theta_i < 1 - \theta_{i+1}$ since $n \geq 3$ and $\sum_{i=1}^n \theta_i = 1$. Thus $\prod_{i=1}^n \theta_i^{T_p} < \prod_{i=1}^n (1-\theta_i)^{T_p}$. So $|\Theta| = \prod_{i=1}^n \theta_i^{T_p} + (-1)^{n+1}\prod_{i=1}^n (1-\theta_i)^{T_p} \neq 0$.

$\sum_{i=1}^{n}\phi_i^*$, $q(m^*)=DC_p(Y_p(m^*))$, $r_i(m^*)=r_i^*$ for all $i\in N$, $X_i(m^*)=x_i^*$, and $Y(m^*)=y^*$, $C(Y(m^*))=\tilde{\gamma}$.

From (20) and (23), we know that the first order condition for the Ratio-Lindahl allocation is the same as the first order condition for the Nash equilibrium m^*. So we only need to show that the first order necessary condition for the Nash equilibrium is actually sufficient for the Nash equilibrium by showing that consumer i's budget constraint

$$X_i(m^*/m_i,i)=w_i+\theta_i\big[q(m^*/m_i,i)\cdot Y_p(m^*/m_i,i)$$
$$-C_p\big(Y_p(m^*/m_i,i)\big)\big]-p_i(m^*/m_i,i)\cdot Z(m^*/m_i,i) \tag{26}$$

is concave in $Z(m^*/m_i,\ i)$ for all $m_i\in M_i$.[11]

We first consider those m_i such that $(X_i(m^*/m_i,\ i),\ Z(m^*/m_i,\ i))\in\mathbb{R}^{1+G}_{++}$. Then $Z(m^*/m_i,\ i)\in$ int $B(m^*/m_i,\ i)$ and thus all outcome functions are differentiable with respect to m_i on some neighbourhood of m_i. Differentiating (26) with respect to m_i, we have

$$D_Z X_i\cdot D_{m_i}Z=\theta_i\big[q\cdot D_{m_i}Y_p+Y_p\cdot D_{m_i}q-DC_p(Y_p)\cdot D_{m_i}Y_P\big]$$
$$-p_i\cdot D_{m_i}Z-Z\cdot D_{m_i}p_i. \tag{27}$$

By repeating the arguments that led to equation (19), one gets

$$D_Z X_i=-p_i(m). \tag{28}$$

The second order derivative matrix is

$$\big[D_Z^2 X_i\big]\big[D_{m_i}Z\big]=-D_{mi}p_i(m)=\begin{pmatrix}-\theta_i I_{K_p} & 0\\ 0 & 0\end{pmatrix}. \tag{29}$$

Then

$$D_Z^2 X_i=\begin{pmatrix}-\theta_i I_{K_p} & 0\\ 0 & 0\end{pmatrix}\big[D_{m_i}Z\big]^{-1}=\begin{pmatrix}-\theta_i I_{K_p} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}D^2C_p & 0\\ 0 & I_T\end{pmatrix}$$
$$=\begin{pmatrix}-\theta_i D^2C_p & 0\\ 0 & 0\end{pmatrix}, \tag{30}$$

which is negative semi-definite. Thus $X_i(m^*/m_i,\ i)$ is concave in $Z(m^*/m_i,\ i)$ when $(X_i(m^*/m_i,\ i),\ Z(m^*/m_i,\ i))$ is an interior point of $\mathbb{R}^{1+G}_+$. Now consider those m_i such that outcomes $(X_i(m^*/m_i,\ i),\ Z(m^*/m_i,\ i))$ are boundary points of $\mathbb{R}^{1+G}_+$. Since these outcomes are also boundary points of the curve specified by

[11] Note that choosing m_i is equivalent to choosing $Z(m^*/m_i,\ i)$ for consumer i when others' messages are given.

(26) which are continuous in m_i and concave in Z in $\mathbb{R}^{1+G}_{++}$, $X_i(m^*/m_i, i)$ is concave in $Z(m^*/m_i, i)$ for all $m_i \in M_i$. Thus consumer i maximizes a strictly quasi-concave utility function subject to a convex constraint when others' messages are given. Hence the first order condition is sufficient (cf. Arrow and Enthoren (1961)). Thus $(X(m^*), Y(m^*)) = (x^*, y^*)$ is a Nash allocation. Q.E.D.

It may be remarked that since Ratio-Lindahl allocations are Pareto optimal and individually rational, the mechanism yields Pareto-efficient and individually rational allocations. Also, from Theorem 2.4 of Reichelstein and Reiter (1988), we know that Nash implementation is always at least as costly, in message space size, as decentralized realization. Since the minimal dimension required for realization of the Lindahl correspondence on E' is nG (cf. Sato (1981)), the mechanism has a message space of minimal dimension and thus the mechanism is informationally efficient.

Summarizing the above discussion, we have

Theorem 2. Let $\tilde{E}$ be a class of economies with one private good and K public goods which satisfy the assumptions given in this paper. Then there exists a completely feasible and continuous mechanism with a message space of minimal dimension which fully Nash-implements the Ratio-Lindahl correspondence.

Remark 8. From the above proof, we can see that $(X(m^*), V_s(m^*), Y_p(m^*))$ is, in fact, a mechanism which implements the Lindahl correspondence and allows some firms' technologies to display CRS and some firms' technologies to display DRS. To the best of our knowledge, no such mechanisms exist in the literature and thus this is a new mechanism for the Lindahl correspondence. Also, the mechanism reduces to the mechanism presented in Tian and Li (1994a) which implements the Generalized Ratio correspondence when there are no privately-owned firms, it reduces to the mechanism presented in Li, Nakamura, Tian (1995) which implements the Lindahl correspondence when there are no state-owned firms and all production functions of firms exhibit DRS, and it reduces to the mechanism presented in Tian (1991) which implements the Lindahl correspondence when all firms' technologies exhibit CRS.

4. Concluding remarks

In the above sections we used the Ratio-Lindahl equilibrium as a solution concept for a mixed-ownership system which yields Pareto-efficient and individually rational allocations. We present a mechanism which is well-behaved in the sense that the mechanism is individually feasible, balanced, continuous, and has a message space of minimal dimension. Moreover, the mechanism is almost everywhere differentiable on the message space and differentiable on some neighbourhood of every Nash equilibrium so that we can use, as we have done in the paper, the differential approach. We also allow the production technologies of state-owned firms to display CRS, DRS, or IRS and the mechanism given in this paper is valid for all these returns. We have noted that when the technologies producing the public goods are all of CRS and each firm produces only one public good, the differences in the mechanisms among mixed-ownership, state-ownership, and private-ownership disappear since profits are all zero in this case. Thus the results obtained in this paper, like those in Tian and Li (1994a) and Tian (1994), give a somewhat positive answer to the theoretical feasibility of socialist and mixed-ownership systems in the Mises-Hayek-Lange-Lerner debate.

In ending this paper, we want to mention some of the possible extensions of our results. First, the model and results presented in this paper deal only with public goods economies with one private good. In Tian and Li (1994b), we generalized the Ratio-Lindahl equilibrium to include any number of private goods. One can also design a mechanism by using similar techniques given in Tian and Li (1991). The mechanism could be a combination of the present mechanism and a mechanism given in Tian (1992). In this case, it would be completely feasible and continuous and implements the Ratio-Lindahl correspondence.

Second, in the mechanism presented here, we assume that firms report cost functions truthfully. The mechanism does not discuss the problem that firms have incentives to overstate costs (i.e., understate productivity), which commonly exists in multi-divisional or socialist enterprises. However we can modify our mechanism to deal with the problem of understating productivity when the labour efforts of agents are unknown to the designer.

Third, similar to those mechanisms proposed by Hurwicz, Maskin, and Postlewaite (1984) and Tian (1989, 1993), the mechanism presented above can be extended to allow endowments to be unknown to the designer. This case, of course, certainly increases the size of message space but reduces the informational requirements on the designer.

Lastly, when the production technology of a firm displays IRS, one can not use the Lindahl allocation as a social choice rule, so instead we use the Ratio-Lindahl correspondence as a social choice rule. The other possible candidate for social choice rule under such a situation might be a marginal cost pricing rule or average cost pricing rule, which is the decision of the choosing institutions. However, the average pricing rule or even marginal pricing rule generally cannot result in Pareto-efficient allocations (see, Guesnerie (1975)).

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