JOURNAL OF ECONOMIC THEORY 64, 286–297 (1994)

An Implementable State-Ownership System with General Variable Returns*

GUOQIANG TIAN

Department of Economics,
Texas A & M University, College Station, Texas 77843

AND

QI LI[†]

Department of Economics,
Indiana University, Bloomington, Indiana 47405

Received January 31, 1992; revised July 16, 1993

In this paper, we formalize an implementable state-ownership institutional framework for public goods economies by using the Generalized Ratio equilibrium that yields Pareto-efficient and individually rational allocations and that allows for general variable returns. We then address the problems created by manipulative behavior on the part of individuals and study the implementability of this social choice rule by giving an incentive compatible and well-behaved mechanism whose Nash allocations coincide with Generalized Ratio allocations. *Journal of Economic Literature* Classification Numbers: C72, D61, D78, H41.

1. INTRODUCTION

The implementation theory (mechanism design theory) originated by Hurwicz [4, 5] is to design a game form such that a prespecified social choice correspondence can be achieved by the game across a large domain of possible economic environments. For a private-ownership economy with

* We wish to thank M. Baye, L. Hurwicz, J. Green, J. Jordan, M. Kaneko, A. Mas–Colell, S. Reichelstein, and associate editor, an anonymous reference, and participants in both the NSF–NBER Conference on Decentralization and the 1990 Winter Meeting of the Econometric Society for many helpful comments and suggestions. Qi Li thanks the SSHRC of Canada for research support.

[†] Present address: University of Guelph, Guelph, Ontario NIG 2W1, Canada.

0022-0531/94 $6.00

public good, a commonly used social choice rule is the Lindahl correspondence. Groves and Ledyard [2] were the first to propose a mechanism that yields Pareto-efficient allocations through Nash equilibria. Since then, there have been many mechanisms which implement Lindahl allocations such as those in Hurwicz [6], Walker [21], Hurwicz *et al.* [7], Groves and Ledyard [3], Tian [14–16], Li *et al.* [10], and Tian and Li [18], among others. However, in all of these mechanisms, the economic systems are assumed to be of private-ownership with an exogenous profit distribution and preclude the presence of increasing returns to scale (IRS).[1] Thus a natural question is whether or not there is some kind of state-ownership system[2] in which Pareto-efficient allocations can be reached even for the IRS economies and without considering profit distribution. If the answer is yes, one needs to solve the "free rider" problem which commonly exists in state-ownership systems, by designing some "incentive-compatible" and "informationally decentralized" mechanisms with some desirable properties which implement this social choice correspondence.

In this paper, we formalize an implementable state-ownership institutional framework for public goods economies by using the Generalized Ratio equilibrium that yields Pareto-efficient and individually rational allocations. This equilibrium concept is a slight generalization of the Ratio equilibrium introduced by Kaneko [8]. It allows the presence of by-products (i.e., joint-production) and general variable returns (DRS, IRS, or CRS) in the production of public goods at different scales. In our model, the government gives each firm an exclusive franchise to produce some public goods. Unlike the private-ownership system, the government asks agents (consumers) to share the costs of these public goods but not the profits. Agents are self-interested and maximize their utilities given their budget constraints so that the sum of the expenditure on private goods and the cost shares for public goods does not exceed their wealth.

We then consider the implementability of the Generalized Ratio correspondence by giving a specific mechanism whose Nash allocations coincide with Generalized Ratio allocations and which is valid for general variable returns.[3] The mechanism presented here uses outcome functions

[1] This is because production must take place at a profit maximizing point. When the production technology displays IRS, there does not exist such a point.

[2] Here, state-ownership means the government owns the firms and individuals do not share profits (losses) but probably need to share costs of production if they consume these commodities.

[3] This is a very remarkable property since returns in production, in general, vary with the level of output which results in the so-called U-shaped cost functions. For implementation of the Lindahl correspondene, one has to give different mechanisms for different returns (cf. Li *et al.* [10]). But, for the implementation of the Generalized Ratio correspondance, we only need to give one mechanism to deal with all kinds of returns.

that are completely feasible (i.e., individually feasible and balanced) and continuous.[4]

The plan of this paper is as follows. In Section 2, we give a formal description of a state-ownership model and the definition of Generalized Ratio allocations. Section 3 presents a mechanism which has the desirable properties mentioned above and which fully implements Generalized Ratio allocations.

2. Model and Generalized Ratio Allocations

2.1. *Economic Environments*

We consider economies with public goods where there are n agents who consume one private good and K public goods, x being private (as a numeraire) and y public. The single private good x can, and probably should, be thought of as a Hicksian composite commodity or money, and public goods y can be thought of as K projects. Denote by $N=\{1, 2, ..., n\}$ the set of agents. Each agent's characteristic is denoted by $e_i=(w_i, R_i)$, where $w_i>0$ is the initial endowment of the private good and R_i is the preference ordering (P_i denotes the asymmetric part of the preference R_i) defined on $\mathbb{R}_+^{1+K}$. We assume that there are no initial endowments of the public goods, but that the public goods can be produced from the private good by T state-owned firms that may have CRS, DRS, or IRS technologies. Each firm t is given an exclusive franchise to produce K_t public goods, denoted by $y_t \in R_+^{K_t}$. Then $\sum_{t=1}^{T} K_t = K$. Further we assume that the technology of production of each firm t is represented by production function $f_t\colon R_+ \to R_+^{K_t}$ and thus $y_t=f_t(v_t)$, where v_t is the input used by firm t. Thus we allow the presence of by-products (joint-production).[5] Let $f(v)=(f_1(v_1), ..., f_T(v_T))$. The cost function (the input demand function) of firm t is denoted by $C_t(y_t)$.[6] We assume throughout that $C_t(0)=0$, and that C_t is continuous and increasing (i.e., $C_t(y_t)>C_t(y_t')$ if $y_t>y_t'$).[7] Let $C(y)=(C_1(y_1), ..., C_T(y_T))$.

[4] Note that the mechanism also has a message space of minimal dimension required for implementation of the Generalized Ratio correspondence.

[5] That is, the output level of each public good produced by a firm is fixed for each given level of input. In other words, there is no substitution of production between public good. This assumption is more realistic and reasonable than the assumption that each firm produces only one public good.

[6] The cost function coincides with the input demand function since the private good is a numeraire good.

[7] As usual, vector inequalities are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s=1, ..., m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a>b$ means $a_s>b_s$ for all $s=1, ..., m$.

An economy is the full vector $e = (e_1, ..., e_n, C^1(\cdot), ..., C^T(\cdot))$ and the set of all such economies is denoted by E.

2.2. *Generalized Ratio Allocations*

An allocation $(x, y) = (x_1, ..., x_n, y)$ is *feasible* for an economy e if $(x, y) \in \mathbb{R}^{n+K}_+$ and

$$\sum_{i=1}^{n} x_i + \iota \cdot C(y) \leqslant \sum_{i=1}^{n} w_i, \tag{1}$$

where ι is a vector of ones of dimension T. An allocation (x, y) is *Pareto-efficient* for an economy e if it is feasible and there is no other feasible allocation (x', y') such that $(x'_i, y')\, R_i(x_i, y)$ for all $i \in N$ and (x'_j, y') $P_j(x_j, y)$ for some $j \in N$. An allocation (x, y) is *individually rational* for an economy e if $(x_i, y)\, R_i(w_i, 0)$ for all $i \in N$.

An allocation (x^*, y^*) is a *Generalized Ratio* (in short, GR) *allocation* for an economy e in a state-ownership system if it is feasible and there are ratio vectors $r_i^* \in \mathbb{R}^T_+$, one for each i, such that:

(1) $x_i^* + r_i^* \cdot C(y^*) \leqslant w_i$ for all $i \in N$;

(2) for all $i \in N$, there does not exist (x_i, y) such that (x_i, y) $P_i(x_i^*, y^*)$ and $x_i + r_i^* \cdot C(y) \leqslant w_i$;

(3) $\sum_{i=1}^{n} r_i^* = \iota$.

Denote by $GR(e)$ the set of all such allocations.

Remark 1. Observe that the Generalized Ratio equilibrium reduces to the Ratio equilibrium (RE) of Kaneko [8] when $K_t = 1$ for all t's. Further, when $K_t = 1$ and the production functions display CRS, the Generalized Ratio allocation reduces to the Lindahl allocation with personalized price system $(r_1^* C(\iota), ..., r_N^* C(\iota))$. Also, in the case of one firm, the Generalized Ratio equilibrium is a Balanced Linear Cost Share equilibrium of Mas-Colell and Silvestre [12] (by taking their a vectors to be 0).[8]

Remark 2. It is clear that every Generalized Ratio allocation is individually rational. One can also show in a standard way that every Generalized Ratio allocation is Pareto-efficient. The proof of the existence of a Generalized Ratio equilibrium under general conditions can be found in Tian and Li [20], who used a transformation approach.[9] The conditions

[8] We thank Mas-Colell for pointing out this fact to us.

[9] For an economy e with $e_i = (w_i, R_i)$ and production functions $f_t: R_+ \to R^{K_t}_+$ which displays CRS, DRS, or IRS, one can define a transformed economy e' with $e'_i = (R'_i, w_i)$, x being a private good, $v \in R^T_+$ being public goods, and preferences R'_i defined by $(x_i, v)\, R'_i\, (\bar{x}_i, \bar{v})$ if and only if $(x_i, f(v))\, R_i(\bar{x}_i, f(\bar{v}))$. Note that when preference R_i can be represented by utility functions $u_i(x_i, y)$, R'_i are defined by utility functions $u'_i(x_i, v) \equiv u(x, f(v))$.

they use to prove the existence of the General Ratio equilibrium allow for production technologies to display some types of IRS, and thus are weaker than those used to prove the existence of Lindahl equilibrium. For instance, when preferences R_i can be represented by utility functions $u_i(x_i, y)$, there exists a Generalized Ratio equilibrium if the composite function $u(x, f(v))$ is continuous and quasi-concave.[10]

3. Implementation of Generalized Ratio Allocations

In this section we consider the issue of incentives of the Generalized Ratio allocation process. The Generalized Ratio allocation process, like the Lindahl allocation process, has a "free rider" problem. So we need to design some mechanism which implements the Generalized Ratio correspondence. Below, we give a completely feasible and continuous mechanism which fully implements the Generalized Ratio correspondence through Nash equilibria.

Let F be a social choice rule, i.e., a correspondence from E to the set Z of resource allocations. Let $\mathcal{M}_i$ denote the ith message (strategy) domain. Its elements are written as m_i and are called messages. Let $\mathcal{M} = \prod_{i=1}^{n} \mathcal{M}_i$ denote the message (strategy) space. Let $h : \mathcal{M} \to Z$ denote the outcome function, or more explicitly, $h_i(m) = (X_i(m), Y(m), V(m))$, where $X_i(m)$ is the ith agent's outcome function for the private good, $V(m)$ is the input demand outcome function, and $Y(m)$ is the outcome function for the public goods which satisfies $Y(m) = f(V(m))$.

A message $m^* = (m_1^*, ..., m_n^*) \in \mathcal{M}$ is a *Nash equilibrium* (NE) of the mechanism (game form) $\langle \mathcal{M}, h \rangle$ for an economy e if for any $i \in N$ and for all $m_i \in \mathcal{M}_i$,

$$(X_i(m^*), Y_i(m^*)) \, R_i (X_i(m^*/m_i, i), Y_i(m^*/m_i, i)), \tag{2}$$

where $(m^*/m_i, i) = (m_1^*, ..., m_{i-1}^*, m_i, m_{i+1}^*, ..., m_n^*)$. The $h(m^*)$ is then called a *Nash (equilibrium) allocation*. Denote by $V_{\mathcal{M},h}(e)$ the set of all such Nash equilibria and by $N_{\mathcal{M},h}(e)$ the set of all such Nash (equilibrium) allocations.

A mechanism $\langle \mathcal{M}, h \rangle$ fully *Nash-implements* the social choice rule F on E, if for all $e \in E$, $N_{\mathcal{M},h}(e) = F(e)$. A mechanism $\langle \mathcal{M}, h \rangle$ is *individually feasible* if $(X_i(m), Y(m)) \in \mathbb{R}_+^{1+K}$ for all $i \in N$ and all $m \in \mathcal{M}$. A mechanism $\langle \mathcal{M}, h \rangle$ is *balanced* if for all $m \in \mathcal{M}$,

$$\sum_{j=1}^{N} X_j(m) + \imath \cdot C(Y(m)) = \sum_{j=1}^{n} w_j. \tag{3}$$

[10] This is true if utility and production functions are both of the Cobb–Douglas type with homogeneity of any degree.

A mechanism $\langle \mathcal{M}, h \rangle$ is *completely feasible* if it is individually feasible and balanced.

The mechanism that will be constructed can be simply described as follows. The designer first determines the ratios of input costs based upon the contribution that agents are willing to pay, and then defines a feasible choice correspondence B for inputs that can be contributed by all the agents under the given ratio vectors. The outcome $V(m)$ for inputs will be chosen from $B(m)$ such that it is the closest to the sum of the contributions that each agent is willing to pay. The outcome function $Y(m)$ of public good is determined by $Y(m) = f(V(m))$. The outcome $X_i(m)$ for the private good is chosen in such a way that the budget constraint holds with equality.

We now turn to the formal construction of the mechanism. We first make the following additional assumptions.

ASSUMPTION 1. $n \geqq 3$.

ASSUMPTION 2. *The transformed preference orderings R_i' defined in Footnote 9 are convex and strictly monotonicly increasing on $\mathbb{R}^{1+K}_{++}$.*

ASSUMPTION 3. *For all $i \in N$, $(x_i, y)\, P_i(x_i', y')$ for all $(x_i, y) \in \mathbb{R}^{1+K}_{++}$ and $(x_i', y') \in \partial\mathbb{R}^{1+K}_{+}$, where $\partial\mathbb{R}^{1+K}_{+}$ is the boundary of $\mathbb{R}^{1+K}_{+}$.*

Remark 3. Assumption 1 is a necessary condition for a balanced and continuous implementation (cf. Kwan and Nakamura [9]). Assumption 3 cannot be dispensed with in Theorem 1 below. Otherwise the Generalized Ratio correspondence may not be monotonic and thus cannot be Nash-implemented by a feasible meachanism by the results of Maskin [13]. However, this interiority assumption can be replaced by a weaker assumption,[11] which is called the "indispensability of money" by Mas-Colell [11], at the cost of increasing the size of the message space.

For each $i \in N$, let $\mathcal{M}_i = \mathbb{R}^T$ whose elements m_i are interpreted as the contribution that agent i is willing to make. Let the ratio to the ith agent associated with firm t be of the form

$$r_i^t(m) = \frac{1}{n} + m_{i+2}^t - m_{i+1}^t, \tag{4}$$

where $n+1$ and $n+2$ are to be read as 1 and 2, respectively. Note that by construction $\sum_{i=1}^n r_i^t(m) = 1$ for $t = \{1, ..., T\}$ and $r_i^t(m)$ in (4) is independent of m_i.

[11] That is, for all $i \in N$, $(x_i, y)\, P_i(x_i', y')$ for all $x_i \in \mathbb{R}_{++}$, $x_i' \in \partial\mathbb{R}_{+}$, and $y, y' \in \mathbb{R}^K_{+}$.

Define the feasible choice correspondence $B : \mathcal{M} \twoheadrightarrow \mathbb{R}^T_+$ for input demand by

$$B(m) = \{z \in \mathbb{R}^T_+ : w_i - r_i(m) \cdot z \geqq 0 \text{ for all } i \in N\}, \tag{5}$$

which is a continuous correspondence with nonempty, compact, and convex values. The outcome function of the inputs demand $V(m)$: $\mathcal{M} \to \mathbb{R}^T_+$ is given by

$$V(m) = \{z : \min_{z \in B(m)} \|z - \tilde{m}\|\}, \tag{6}$$

which is the closest to $\tilde{m}$. Here $\tilde{m} = \sum_{i=1}^n m_i$ and $\|\cdot\|$ is the Euclidian norm. Then $V(m)$ is single-valued and continuous on $\mathcal{M}$.

Define the outcome function of public goods as $Y : \mathcal{M} \to \mathbb{R}^K_+$ by $Y(m) = f(V(m))$. Then $C(Y(m)) = V(m)$.

Define agent i's cost share outcome function $g_i(m) : \mathcal{M} \to \mathbb{R}$ by

$$q_i(m) = r_i(m) \cdot C(Y(m)). \tag{7}$$

By $\sum_{i \in N} r_i(m) = \iota$, the total cost share is then equal to the cost of production, i.e.,

$$\sum_{i=1}^n q_i(m) = \iota \cdot C(Y(m)). \tag{8}$$

The ith agent's outcome function for the private good $X_i(m) : \mathcal{M} \to \mathbb{R}_+$ is given by

$$X_i(m) = w_i - r_i(m) \cdot C(Y(m)). \tag{9}$$

The outcome function is clearly continuous on $\mathcal{M}$. Also, since $(X_i(m), Y(m)) \in \mathbb{R}^{1+K}_+$ and $\sum_{i \in N} r_i = \iota$, we have

$$\sum_{i=1}^n X_i(m) + \iota \cdot C(Y(m)) = \sum_{i=1}^n w_i. \tag{10}$$

Hence the mechanism is balanced. Thus, the mechanism specified above is single-valued, completely feasible, and continuous for all $m \in \mathcal{M}$.

Before proving the equivalence between Nash allocations and Generalized Ratio allocations, we first prove two Lemmas which are then used to prove that every Nash allocation is a Generalized Ratio allocation.

LEMMA 1. *If* $(X(m^*), Y(m^*)) \in N_{\mathcal{M},h}(e)$, *then* $(X_i(m^*), Y(m^*)) \in \mathbb{R}^{1+K}_{++}$ *for all* $i \in N$.

Proof. Suppose, by way of contradiction, that $(X_i(m^*), Y(m^*)) \in \partial\mathbb{R}^{1+K}_+$ for some $i \in N$. Then $(X_i(m^*), V(m^*)) \in \partial\mathbb{R}^{1+T}_+$ for some $i \in N$ by monotonicities of production functions and preferences. Let $\bar{\delta}$ be the larger root of the quadratic equation

$$\delta = \frac{\tilde{w}}{2(\delta + c)}, \tag{11}$$

where $\tilde{w} = \min_{i \in N} w_i$ and $c = 1/n + 2\sum_{t=1}^{T} \sum_{i=1}^{n} |m_i^{*t}|$. Then $\bar{\delta} > 0$. Let $\tilde{m} = (1/T)(\bar{\delta}, \bar{\delta}, ..., \bar{\delta})'$. Suppose that player i chooses his/her message $m_i = \tilde{m} - \sum_{j \neq i}^{n} m_j^*$. Then $\tilde{m} = m_i + \sum_{j \neq i}^{n} m_j^*$ and

$$\begin{aligned} w_j - r_j(m^*/m_i, i) \cdot \tilde{m} &\geqq w_j - \left(\bar{\delta} + \frac{1}{n} + 2 \sum_{t=1}^{T} \sum_{s=1}^{n} |m_s^{*t}|\right) \bar{\delta} \\ &= w_j - \tilde{w}/2 \geqq w_j/2 > 0 \end{aligned}$$

for all $j \in N$. So $\tilde{m} \in B(m^*/m_i, i)$ and thus $V(m^*/m_i) = \tilde{m} > 0$. Since $Y(m^*/m_i, i) = f(V(m^*/m_i, i)) = f(\tilde{m}) > 0$ and $X_j(m^*/m_i, i) = w_j - r_j(m^*/m_i, i) \cdot V(m^*/m_i, i) = w_j - r_j(m^*/m_i, i) \cdot \tilde{m} > 0$ for all $j \in N$, we have

$$(X_i(m^*/m_i, i), Y(m^*/m_i, i)) \, P_i(X_i(m^*), Y(m^*))$$

by Assumption 3. This contradicts the hypothesis $(X(m^*), Y(m^*)) \in N_{\mathcal{M},h}(e)$. Q.E.D.

LEMMA 2. *If* $(X(m^*), Y(m^*)) \in N_{\mathcal{M},h}(e)$, *then* $\tilde{m}^* = \sum_{i=1}^{n} m_i^* \in \operatorname{Int} B(m^*)$ *and thus* $C(Y(m^*)) = V(m^*) = \sum_{i=1}^{n} m_i^*$. *Here* $\operatorname{Int} B(m^*)$ *is the interior of* $B(m^*)$.

Proof. Suppose, by way of contradiction, that $\tilde{m}^* \notin \operatorname{Int} B(m^*)$. Then $C(Y(m^*)) = V(m^*) \in \partial B(m^*)$ and thus $Y(m^*) \in \partial \mathbb{R}_+^K$ or $X_j(m^*) = w_j - r_j(m^*) \cdot V(m^*) = 0$ for some $j \in N$. But both cases contradict $(X(m^*), Y(m^*)) \in \mathbb{R}_{++}^{1+K}$ by Lemma 1. Q.E.D.

We now prove the main results of this section in the following propositions.

PROPOSITION 1. *If the mechanism has a Nash equilibrium* m^*, *then* $(X(m^*), Y(m^*))$ *is a Generalized Ratio allocation with* $(r_1(m^*), ..., r_n(m^*))$ *as the ratio vectors, i.e.,* $N_{\mathcal{M},h} \subseteq GR(e)$.

Proof. Let m^* be a Nash equilibrium. Now we prove that $(X(m^*), Y(m^*))$ is a Generalized Ratio allocation with $(r_1(m^*), ..., r_n(m^*))$ as the ratio vectors. Since the mechanism is completely feasible, $\sum_{i=1}^{n} r_i(m^*) = \imath$, and $X_i(m^*) + r_i(m^*) \cdot C(Y(m^*)) = w_i$ for all $i \in N$, we only need to show that each individual is maximizing his/her preference. Suppose, by way of contradiction, that there is some $(x_i, y) \in \mathbb{R}_+^{1+K}$ such that $(x_i, y)\, P_i(X_i(m^*), Y(m^*))$ and $x_i + r_i(m^*) \cdot C(y) \leqq w_i$. Because of monotonicity of preferences, it will be enough to confine ourselves to the case of $x_i + r_i(m^*) \cdot C(y) = w_i$. Since $f(C(y)) \geqslant y$, then $(x_i, f(C(y)))\, P_i(x_i^*, f(V(m^*)))$. Let $v = C(y)$. Then we have $(x_i, f(v))\, P_i(x_i^*, f(V(m^*)))$ and $x_i + r_i^* \cdot v = w_i$.

Let $(x_{\lambda i}, v_\lambda) = (\lambda x_i + (1-\lambda) X_i(m^*), \lambda v + (1-\lambda) V(m^*))$. Then by convexity of preferences R_i' we have $(x_{\lambda i}, y_\lambda)\, P_i(X_i(m^*), Y(m^*))$ for any $0 < \lambda < 1$, where $y_\lambda = f(v_\lambda)$. Also $(x_{\lambda i}, y_\lambda) \in \mathbb{R}_+^{1+K}$ and $x_{\lambda i} + r_i(m^*) \cdot v_\lambda = w_i$.

Suppose that player i chooses his/her message $m_i = v_\lambda - \sum_{j \neq i}^n m_j^*$. Since $C(Y(m^*)) = \sum_{j=1}^n m_j^*$ by Lemma 2, $m_i = v_\lambda - C(Y(m^*)) + m_i^*$. Thus as $\lambda \to 0$, $v_\lambda \to C(Y(m^*))$, and therefore $m_i \to m_i^*$. Since $X_j(m^*) = w_j - r_j(m^*) \cdot C(Y(m^*)) > 0$ for all $j \in N$ by Lemma 1, we have $w_j - r_j(m^*/m_i, i) \cdot v_\lambda > 0$ for all $j \in N$ as λ is a sufficiently small positive number. Therefore, $V(m^*/m_i, i) = v_\lambda$, $Y(m^*/m_i, i) = f(V(m^*/m_i, i)) = f(v_\lambda) = y_\lambda$, and $X_i(m^*/m_i, i) = w_i - r_i(m^*) \cdot C(Y(m^*/m_i, i)) = w_i - r_i(m^*) \cdot v_\lambda = x_{i\lambda}$. From $(x_{i\lambda}, y_\lambda)\, P_i(X_i(m^*), Y(m^*))$, we have $(X_i(m^*/m_i, i), Y(m^*/m_i, i))\, P_i(X_i(m^*), Y(m^*))$. This contradicts the hypothesis that $(X(m^*), Y(m^*)) \in N_{\mathcal{M},h}(e)$. Q.E.D.

PROPOSITION 2. *If (x^*, y^*) is a Generalized Ratio allocation with the ratio vector $r^* = (r_1^*, ..., r_n^*)$, then there is a Nash equilibrium m^* of the mechanism such that $X_i(m^*) = x_i^*$, $r_i(m^*) = r_i^*$, for all $i \in N$, $Y(m^*) = y^*$, i.e., $GR(e) \subseteq N_{\mathcal{M},h}(e)$.*

Proof. We need to show that there is a message m^* such that (x^*, y^*) is a Nash allocation. Let $m^* = (m_1^*, ..., m_n^*)$ be the solution of the following linear equations system:

$$\sum_{i=1}^n m_i = C(y^*),$$
$$m_{i+2}^t - m_{i+1}^t = r_i^{*t} - \frac{1}{n} \qquad (i = 1, ..., n;\ t = 1, ..., T). \tag{12}$$

Then $X_i(m^*) = x_i^*$, $V(m^*) = C(y^*)$, $Y(m^*) = y^*$ and $r_i(m^*) = r_i^*$ for all $i \in N$. Thus, from $(X(m^*/m_i, i), Y(m^*/m_i, i)) \in \mathbb{R}_+^{1+K}$ and $X_i(m^*/m_i, i) + r_i(m^*) \cdot C(Y(m^*/m_i, i) = w_i$ for all $i \in N$ and $m_i \in \mathcal{M}_i$, we know that

$$(X_i(m^*), Y(m^*))\, R_i(X_i(m^*/m_i, i), Y(m^*/m_i, i)),$$

for otherwise it contradicts the fact that $(X_i(m^*), Y(m^*))$ is a Generalized Ratio allocation. Q.E.D.

Note that since Generalized Ratio allocations are Pareto-efficient and individually rational, the mechanism yields Pareto-efficient and individually rational allocations.

Summarizing the above discussion, we have:

THEOREM 1. *Let E be a class of economies which satisfy the assumptions given in this paper. Then there exists a completely feasible and continuous mechanism which fully Nash-implements the Generalized Ratio correspondence.*

4. CONCLUDING REMARKS

In this paper we use the Generalized Ratio correspondence as a desirable social choice rule for the state-ownership system. We then give a completely feasible and continuous mechanism which fully implements the Generalized Ratio correspondence through Nash-equilibria on a large class of economies whose production technologies may display CRS, DRS, or IRS. Thus we can reach Pareto-efficient and individually rational allocations by implementing the Generalized Ratio correspondence even for state-ownership and IRS technologies.

When the production technology does not display CRS (in particular, when it displays DRS), Li *et al.* [10] show that the mechanisms which implement the Lindahl correspondence (viewed as a private-ownership system) are quite different from those mechanisms for CRS technology. However, the mechanism given in this paper is valid for all kinds of returns when the social choice rule is the Generalized Ratio correspondence. We have also noted that when all the technologies producing the public goods display CRS and each firm only produces one public good, there is no difference between the mechanisms for state-ownership and those for private-ownership.

We now mention some of the possible extensions. First, the model and results presented here deal only with public good economies with one private good. Recently, Diamantaras and Wilkie [1] and Tian and Li [19] independently extended the notion of Ratio equilibrium to the case of public goods economies with any number of private goods, but without joint-production.[12] By using similar techniques given in Tian and Li [18], one can give a mechanism for this general model. The mechanism could be a combination of the above mechanism and a mechanism given in Tian [17]. Second, since almost all economic systems in the real-world are a mixture of private-ownership and state-ownership, Tian and Li [19] also gave an equilibrium concept which we call the Ratio-Lindahl equilibrium

[12] The approach that Diamantaras and Wilkie [1] used to prove the existence of an equilibrium requires that the state-owned firms' technologies for producing public goods be convex, while Tian and Li [19] did not need this requirement.

to formalize the mixed-ownership system. A desirable property of this equilibrium notion is that the Ratio-Lindahl allocations always yield Pareto-efficient and individually rational allocations. Finally, similar to Tian [16], the mechanism presented in the paper can be extented to allow for preferences to be nontotal–nontransitive.

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