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Implementation of Linear Cost Share Equilibrium Allocations*

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In this paper we investigate incentive aspects of the Linear Cost Share Equilibrium principle by designing mechanisms whose Nash allocations coincide with Linear Cost Share equilibrium allocations. The mechanisms presented here allow not only preferences but also initial indowments to be privately observed, a feature missing from much recent work in implementation theory. We give mechanisms for both cases of withholding and destruction of endowments. If one reinterprets the commodity space, our mechanisms also implements Walransian allocations to economics with purely private goods. Thus, our mechanisms are sufficiently general to cover Walrasian equilibria, Lindahl equilibria, and cost share equilibria. In other words, the mechanisms in this paper appear to represent "generic" mechanisms to implement competitive-type allocations in private and public goods economies. *Journal of Economic Literature* Classification Numbers: C72, D61, D78.

1. INTRODUCTION

In their recent paper Mas-Colell and Silvestre [12] introduced the notion of Linear Cost Share Equilibrium for public goods economies which yields Pareto-efficient allocations even in the increasing returns case and in the presence of externalities (if the equilibrium concept is applied to economies with pure private goods). Weber and Wiesmeth [28] then gave an equivalence result for the core and Linear Cost Share Equilibrium. This paper considers incentive aspects (implementation) of the Linear Cost Share Equilibrium principle which was recently introduced by Mas-Colell

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and Silvestre [12]. Until recently, for the general equilibrium approach to the efficiency of resource allocation of public goods, the most commonly used general equilibrium notion was the Lindahl equilibrium principle. The equilibrium principle is a generalization of the conventional concept of Walrasian equilibrium and shares some of the same properties as the Walrasian equilibrium: Both result in Pareto-efficient allocations and are solution concepts for private-ownership institutions. As is well known, the Lindahl mechanism is not incentive-compatible, in the sense that it has the free rider problem. Many mechanisms have been proposed which solve the incentive problems in the sense that they result in Pareto-efficient allocations for public goods economies when individual self-interested behavior is characterized by Nash-equilibrium strategies. Groves and Ledyard [3] were the first to propose a mechanism that yields Pareto-efficient allocations through Nash equilibria. Since then, there have been many mechanisms which implement Lindahl allocations, such as those in Hurwicz [5], Walker [27], Hurwicz *et al.* [6], Groves and Ledyard [4], Tian [20–23], Li *et al.* [10], and Tian and Li [25], among others. Among these, Hurwicz *et al.* [6] were the first to consider the case where the initial endownments are private information. This situation would certainly increase the size of the message space, but would reduce the information requirements on the designer. They considered cases both of withholding and of destroying unreported endowments.

However, for the general variable returns case, the Lindahl equilibrium principle has at least four drawbacks. (1) Production must take place at a price-taking, profit-maximizing point. This precludes the existence of an equilibrium if increasing returns to scale (IRS) are present. (2) If profits are positive, they must be distributed in accordance with some exogenously given profit distributions. (3) When some firms in a society are owned by the government and the technologies of these firms do not display constant returns to scale (CRS), the conventional Lindahl mechanism is problematic since it is not clear how the profits or losses should be distributed. (4) Unlike the Walrasian equilibrium principle, the core equivalence result for the Lindahl equilibrium does not hold even in the case of a continuum of consumers; the core of a public goods economy can be much larger than the set of Lindahl allocations (cf. Muench [15]). Thus, Edgeworth's conjecture, as interpreted through the core, cannot be extended to public goods economies with respect to the Lindahl solution.

To overcome the above drawbacks of the Lindahl equilibrium solution, some alternative solution concepts for public goods economies have been proposed in the literature, including the Ratio equilibrium notion of Kaneko [8] and the Generalized Ratio equilibrium notion of Tian [24] and Tian and Li [26], as well as the more general solution notions of the various Cost Share equilibria of Mas-Colell and Silvestre [12]. The Cost

Share equilibrium notion is not radically different from the Lindahl equilibrium solution. In fact, it coincides with the Lindahl equilibrium notion for the case of convex economies with constant returns to scale. With this equilibrium notion, each consumer is assigned a cost share function, and given these, there is unanimity on the desired level of public goods. The allocation process is therefore designed in a decentralized way, yields efficient outcomes, and guarantees that individual contributions are in line with individual benefits. Note that, as Mas-Colell and Silvestre [12] pointed out, there is no free lunch in theoretical analysis. The need for a profit maximization condition is replaced, relative to the Lindahl equilibrium principle, by a stronger informational requirement: individual agents must know the cost function. However, the assumption may be reasonable if we, as we did for the (Generalized) Ratio equilibrium in Tian [24], view the Cost Share Equilibrium as a solution concept for state ownership. In this case, we can reasonably assume that technologies of producing public goods are common knowledge. Nevertheless, as Mas-Colell and Silvestre [12] mentioned, they neglected incentive aspects of the equilibrium principle. Weber and Wiesmeth [28] then studied the equivalence of core and (Linear) Cost Share equilibria. Thus the fact that the Cost Share Equilibria are the only core allocations provides a strong support for the concept of the Cost Share Equilibrium. One can argue that, no matter how goods are actually allocated, if we allow individuals to defend their interests by rejecting disadvantageous proposals, then any acceptable allocation can be supported by a suitably chosen cost sharing mechanism. But the core is only a concept for large economies. When the number of agents is small, as for the Lindahl mechanism, the Cost Share Equilibrium mechanism is not incentive-compatible either. So one needs to design incentive-compatible mechnisms which implement the Cost Share Equilibrium allocations under some solution concepts of self-interested behavior of individuals such as the Nash-equilibrium strategy.

In this paper we use the non-cooperative game theoretical approach to consider the problem of designing incentive-compatible and informationally decentralized mechanisms whose Nash allocations coincide with Linear Cost Share Equilibrium allocations under minimal possible assumptions on preferences and initial endowments which are both private information and unknown to the designer. We given mechanisms for both cases of destroying and withholding unreported endowments. These mechanisms allow each agent to reveal information about his own endowment in such a way that he can understate, but not overstate his own endowment,[1] which is necessary to guarantee feasibility even at disequilibrium points.

[1] When goods are physical goods, this requirement can be guaranteed by asking agents to *exhibit* their reported endowments to the designer.

The intuition here is straightforward: if a mechanism allows agents to overstate their endowments, then it allows for infeasible outcomes—it will sometimes attempt to allocate more than is possible, given the true aggregate endowment. As well be noted, these mechanisms are well-behaved in the sense that they are individually feasible, balanced (weakly balanced for the destruction mechanism), and continuous. The "better" mechanism design provides at least a partial response to a common concern of much of the implementation literature, namely that the implementing mechanisms are highly unrealistic and impossible for a real player to use. Also, we allow preferences of agents to be discontinuous and nontotal–nontransitive. As Tian [22] pointed out, the situation where preferences of agents are nontotal–nontransitive is potentially very important, since in many cases, in particular where economic entities are composed of more than one individual, it is natural that the preferences for such agents would be nontransitive or nontotal due to the problem of aggregating the individuals' preferences. This is particularly true for public goods (projects) since choices of public goods are likely to be determined by communities. Because of well known problems in aggregating preferences of individuals, it may be necessary (or desirable) to represent the preference of groups (communities) as nontransitive or nontotal. It should be noted that the mechanism dealing with the withholding of unreported endowments in this paper has the advantage that agents are not required to report their true endowments even at equilibrium. Another advantage of our mechanisms is that each agent is required to announce only his own endowment but not others' endowments, in contrast to the mechanisms of Hurwicz *et al.* [6]

The plan of this paper is as follows. Section 2 sets forth a public goods model and gives the definition of Linear Cost Share Equilibrium allocations. Sections 3 and 4 present mechanisms which have the desirable properties mentioned above and which fully implement Linear Cost Share Equilibrium allocations in the presence of withholding and destruction respectively. Concluding remarks will be offered in Section 5.

2. A Public Goods Model and Linear Cost Share Equilibria

2.1. *A Public Goods Model*

We will study a model with n consumers, K public goods, and one private good, x being private (as a numeraire) and y public. Denote by $N = \{1, 2, ..., n\}$ the set of consumers. The single private good x can, and probably should, be though of as a Hicksian composite commodity or money, and public goods y can be thought of as K public projects and are producible from the private good. The technology is given to us as a single cost function $C(y)$. We assume throughout that $C(0)=0$ and that C is increasing[2] (i.e., $C(y')>C(y)$ if $y'>y$),[3] continuous, and convex.

Each consumer's characteristic is denoted by $e_i=(\mathring{w}_i, P_i)$, where $\mathring{w}_i$ is the true initial endowment of the private good and P_i is the strict (irreflexive) preference defined on $\mathbb{R}^{1+K}_+$, which may be nontotal or nontransitive.[4] We assume that $n>2$, $\mathring{w}_i>0$, and preference P_i is convex[5] and strictly monotonically increasing in the private good, which is indispensable (i.e., for all $i\in N$, $(x_i, y)\ P_i(0, y')$ for all $x_i\in\mathbb{R}_{++}$, and $y, y'\in\mathbb{R}^K_+$). An economy is the full vector $e=(e_1, ..., e_n, C(y))$ and the set of all such economies is denoted by E.

A state of economy e is a vector $(x, y)\in\mathbb{R}^n_+\times\mathbb{R}^K_+$. A state is *feasible* if

$$\sum_{i=1}^{n} x_i + C(y) \leqq \sum_{i=1}^{n} \mathring{w}_i. \tag{1}$$

An allocation (x, y) is *Pareto-optimal* with respect to the strict preference profile $P=(P_1, ..., P_n)$ if it is feasible and there does not exist another feasible allocation (x', y') such that $(x'_i, y')\ P_i(x_i, y)$ for all $i\in N$. An allocation (x, y) is *individually rational* with respect to P if $\neg(w_i, 0)\ P_i(x_i, y)$ for all $i\in N$.

2.2. *Linear Cost Share Equilibria*

An allocation (x^*, y^*) is a *Linear Cost Share Equilibrium* (*LCSE*) *allocation* for an economy e if it is feasible and there are $(a^*_1, ..., a^*_n)\in\mathbb{R}^{nK}$ and $(b^*_1, ..., b^*_n)\in\mathbb{R}^n_+$ such that

(1) $x^*_i + a^*_i\cdot y^* + b^*_i C(y^*) \leqq \mathring{w}_i$ for all $i\in N$;

(2) for all $i\in N$, there does not exist (x_i, y) such that $(x_i, y)\ P_i(x^*_i, y^*)$ and $x_i + a^*_i\cdot y + b^*_i C(y)\leqq \mathring{w}_i$;

(3) $\sum_{i=1}^n a^*_i=0$ and $\sum_{i=1}^n b^*_i=1$.

Denote by $LCSE(e)$ the set of all such allocations. Note that the Linear Cost Share Equilibrium solution concept does not preclude the presence of increasing returns to scale (IRS). The interpretations of the parameters of

[2] This is true if there are by-product technologies.

[3] As usual, vector inequalities are defined as follows: Let $a, b\in\mathbb{R}^m$. Then $a\geqq b$ means $a_s\geq b_s$ for all $s=1, ..., m$; $a\geq b$ means $a\geqq b$ but $a\neq b$; $a>b$ means $a_s>b_s$ for all $s=1, ..., m$.

[4] If we define the binary relation P^*_i in the way that $a\ P^*_i\ b$ if and only if $\neg b\ P_i\ a$, where $\neg$ stands for "it is not the case that," then P^*_i is the weak (reflexive) preference. Let concepts used in this paper such as Nash equilibrium and Linear Cost Share equilibrium allocations be interpreted in terms of the P^*_i. Then the results obtained in this paper for P_i are in particular valid for the P^*_i.

[5] P_i is convex if for bundles a, b, c with $0<\lambda\leq 1$ and $c=\lambda a+(1-\lambda)b$, the relation $a\ P_i\ b$ implies $c\ P_i\ b$.

the linear cost share system are clear. The b_i parameters are direct cost share parameters while the a_{ij}, which can be positive or negative, are side compensations based on consumption of public goods.

It is clear that every LCSE allocation is individually rational. Mas-Colell and Silvestre [12] showed that every LCSE allocation is Pareto optimal even in the increasing returns case. Further they showed that, if one reinterpretes the commodity space, the equilibrium concept can be applied to economies with purely private goods or with externalities. Thus, the equilibrium concept can also be viewed as optimality-guaranteeing equilibrium concepts. Note that, in contrast to the Lindahl equilibrium, production of public goods at the Linear Cost Share Equilibrium does not take place at a price-taking, profit-maximizing point.

Let $\Delta^{n-1} = \{\theta \in \mathbb{R}^n_+ : \sum_{i=1}^n \theta_i = 1\}$ be the $n-1$ dimensional unit simplex. We now define the Lindahl equilibrium.

Given profit share vector $\theta \in \Delta^{n-1}$, an allocation (x^*, y^*) is a *θ-Lindahl equilibrium allocation* for an economy e if it is feasible and there are personalized price vectors, $q_i^* \in \mathbb{R}^K$, one for each, such that

(1) y^* maximizes profits $q^* \cdot y - C(y)$;

(2) $x_i^* + q_i^* \cdot y^* \leqq \mathring{w}_i + \theta_i[q^* \cdot y^* - C(y^*)]$ for all $i \in N$;

(3) for all $i \in N$, there does not exist (x_i, y) such that $(x_i, y)\, P_i(x_i^*, y^*)$ and $x_i + q_i^* \cdot y \leqq \mathring{w}_i + \theta_i[q^* \cdot y^* - C(y^*)]$;

(4) $\sum_{i=1}^n q_i^* = q^*$.

Denote by $L(e; \theta)$ the set of all such allocations.

Mas-Colell and Silvestre [12, Propositions 2 and 3] showed that in the convex technology case the LCSE allocations are in one-to-one correspondence with the Lindahl equilibrium allocations. The correspondence is established by varying the profit share parameters which characterize the Lindahl equilibrium allocations; i.e., $LCSE(e) = \bigcup_{\theta \in \Delta^{n-1}} L(e; \theta)$. Thus, the existence of the LCSE is guaranteed under the same conditions which guarantee the existence of Lindahl equilibria (cf. Foley [2], Milleron [14], and Roberts [18]). Note that in the constant returns case, the Linear Cost Share equilibrium allocation reduces to the Lindahl allocation.

Remark 1. Even though the assumptions of $n > 2$ and "indispensability of the private good," stated by Mas-Colell [11], are not necessary for existence, they are necessary for continuous and feasible (i.e., individually feasible and balanced) implementation. Indeed, the first one is a necessary condition for balanced and continuous implementation. Kwan and Kakamura [9] proved that, for two-agent economies with constant returns technologies, there are no balanced and continuous mechanisms which implement the Lindahl correspondence. The assumption of "indispensability of the private good" cannot be dispensed with either. Tian [19] showed that the Lindahl correspondence violates Maskin's [13] monotonicity condition without this assumption and thus cannot be Nash-implemented by an individually feasible and balanced mechanism. Since the Linear Cost Share equilibrium allocation reduces to the Lindahl equilibrium allocation for constant returns economies, these are also necessary conditions for the balanced and continuous implementation of Linear Cost Share equilibrium allocations.

3. Implementation: Withholding of Endowments

In the following we will present an individually feasible, balanced (not merely weakly balanced), and continuous mechanism which fully Nash-implements the LCSE correspondence when the withheld endowments are consumed but not destroyed.

Let M_i denote the ith agent's message domain. Its elements are written as m_i and called messages. Let $M = \prod_{i=1}^{n} M_i$ denote the message space. The message spaces of consumers are defined as follows.

For each $i \in N$, his/her message domain is of the form

$$M_i = (0, 1] \times (0, \mathring{w}_i] \times \mathbb{R}^K \times Q^n_+ \times \mathbb{R}^K, \tag{2}$$

where $Q^n_+ = \mathbb{R}^n_+ \setminus \{0\}$. A generic element of M_i is $(\delta_i, w_i, \alpha_i, (\beta_{1i}, ..., \beta_{ni}), y_i)$, whose components have the following interpretations. The component δ_i denotes the degree of desirability of the private good for consumer i. In particular, when $\delta_i = 1$, consumer i wishes that public goods not be produced. The designer will use the smallest δ_i of consumers to determine the level of public goods (see Eq. (5) below). The component w_i denotes a profession of consumer i's endowment, and the inequality $0 < w_i \leqq \mathring{w}_i$ means that the consumer cannot overstate his own endowment; on the other hand, the endowment can be understated, but the claimed endowment w_i must be positive. The component α_i denotes the side compensation vector proposed by consumer i for use in other consumers' budget constraints, and β_{ji} is the direct cost share of consumer j proposed by consumer i which is used to construct the normalized average direct cost shares of consumers. The component y_i denotes the proposed level of public goods that consumer i is willing to contribute (a negative y_i means the consumer wants to receive a compensation from the society).

Define the side compensations for consumption of public goods for the ith consumer by

$$a_i(m) = \alpha_{i+1} - \alpha_{i+2}, \tag{3}$$

where $n+1$ and $n+2$ are to be read as 1 and 2, respectively. Note that even though $a_i(m)$ is only a function of the α-component of the message m, we can write it as a function of m without loss of generality. Observe that, by construction, each consumer's side compensation is independent of this own messages and $\sum_{i=1}^{n} a_i(m) = 0$ for all $m \in M$.

Based on the β_{ij}'s, the average direct cost share of consumer i, $b_i(m)$, can be constructed as follows. Define

$$\gamma_{ij} = \sum_{l,s \neq j}^{n} |\beta_{il} - \beta_{is}|, \qquad \gamma_i = \sum_{j=1}^{n} \gamma_{ij},$$

$$\bar{b}_i(m) = \begin{cases} \sum_{j=1}^{n} \frac{\gamma_{ij}}{\gamma_i} \beta_{ij} & \text{if } \gamma_i > 0 \\ \sum_{j=1}^{n} \frac{1}{n} \beta_{ij} & \text{if } \gamma_i = 0, \end{cases}$$

$$\bar{b} = \sum_{l=1}^{n} \bar{b}_l,$$

and finally,

$$b_i(m) = \frac{\bar{b}_i}{\bar{b}}. \tag{4}$$

Note that $\bar{b} > 0$ since $(\beta_{1i}, ..., \beta_{ni}) \in Q^n_+$ for all i. Thus, by construction, b_i is continuous.[6]

Define a feasible correspondence $B: M \to 2^{\mathbb{R}^K_+}$ by

$$B(m) = \{ y \in \mathbb{R}^K_+ : (1 - \delta(m))\, w_i - a_i(m) \cdot y - b_i(m)\, C(y) \geqq 0 \ \forall i \in N \}, \tag{5}$$

which is clearly a continuous correspondence with nonempty compact convex values by noting that $b_i(m)$ is nonnegative and C is continuous and convex. Here $\delta(m) = \min\{\delta_1, ..., \delta_n\}$.

Define the outcome function for public goods, $Y: M \to B$, by

$$Y(m) = \{ y : \min_{y \in B(m)} \| y - \tilde{y} \| \}, \tag{6}$$

which is the closest point to $\tilde{y}$. Here $\tilde{y} = \sum_{i=1}^{n} y_i$. Then $Y(m)$ is single-valued and continuous on M.

[6] We only need to show that $\bar{b}_i$ is continuous. $\bar{b}_i(m)$ is clearly continuous at m with $\gamma_i > 0$. When $\gamma_i = 0$, $\beta_{i1} = \cdots = \beta_{in} = \beta_{ii}$ and thus, for any $\varepsilon > 0$, $|\bar{b}_i(m') - \bar{b}_i(m)| = |\sum_{j=1}^{n} (\gamma'_{ij}/\gamma'_i)\, \beta'_{ij} - \beta_{ii}| = |\sum_{j=1}^{n} (\gamma'_{ij}/\gamma'_i)(\beta'_{ij} - \beta_{ii})| \leqq \sum_{j=1}^{n} |\beta'_{ij} - \beta_{ii}| < \varepsilon$ as long as β'_{ij} is sufficiently close to β_{ii} for all $j \in N$. Hence, $\bar{b}_i(m)$ is also continuous at m with $\gamma_i = 0$.

Define the linear cost share function $g_i: M \to \mathbb{R}$ by

$$g_i(m) = a_i(m) \cdot Y(m) + b_i(m)\, C(Y(m)). \tag{7}$$

Then, by $\sum_{i=1}^n a_i(m) = 0$ and $\sum_{i=1}^n b_i(m) = 1$, the total cost share is then equal to the cost of production; i.e.,

$$\sum_{i=1}^{n} g_i(m) = C(Y(m)). \tag{8}$$

The ith consumer's outcome function for the private good $X_i(m)$: $M \to \mathbb{R}_+$ is given by

$$X_i(m) = w_i - a_i(m) \cdot Y(m) - b_i(m)\, C(Y(m)). \tag{9}$$

Note that $X_i(m) > 0$ by the definition of the constrained correspondence B, and the total (final) consumption of consumer i for the private good is the sum of $X_i(m)$ and $(\mathring{w}_i - w_i)$. That is, it is the sum of the amount of private good allocated by the mechanism and the unreported amount of his/her own endowment.

Thus the outcome function is continuous on M, $(X(m), Y(m)) \in \mathbb{R}_+^{n+K}$, and

$$\sum_{i=1}^{n} [X_i(m) + \mathring{w}_i - w_i] + C(Y(m)) = \sum_{i=1}^{n} \mathring{w}_i \tag{10}$$

for all $m \in M$.

From (10), we have

$$\sum_{i=1}^{n} X_i(m) + C(Y(m)) = \sum_{i=1}^{n} w_i, \tag{11}$$

which means that the aggregate consumption of the private good allocated by the mechanism is equal to the aggregate endowments reported by consumers for all $m \in M$.

Denote by $h: M \to \mathbb{R}_+^{n+K}$ the outcome function, or more explicitly, $h_i(m) = (X_i(m), Y(m))$. Then the mechanism consists of $\langle M, h \rangle$ defined on E. By the constructions of the mechanism, the mechanism $\langle M, h \rangle$ is *individually feasible* (i.e., $(X_i(m) + \mathring{w}_i - w_i, Y(m)) \in \mathbb{R}_+^{1+K}$ for all $i \in N$ and all $m \in M$), *balanced* (i.e., (10) holds for all $m \in M$), and continuous.

A message $m^* = (m_1^*, ..., m_n^*) \in M$ is said to be a *Nash equilibrium* of the mechanism $\langle M, h \rangle$ in the presence of withholding for an economy e if for any $i \in N$ and for all $m_i \in M_i$,

$$\neg (X_i(m^*_{-i}, m_i) + \mathring{w}_i - w_i, Y(m^*_{-i}, m_i))\, P_i (X_i(m^*) + \mathring{w}_i - w_i^*, Y(m^*)), \tag{12}$$

where $(m^*_{-i}, m_i) = (m^*_1, ..., m^*_{i-1}, m_i, m^*_{i+1}, ..., m^*_n)$. $(X(m^*) + \mathring{w} - w^*, Y(m^*))$ is then called a *Nash (equilibrium) allocation* of the mechanism for the economy e. Denote by $V_{M,h}(e)$ the set of all such Nash equilibria and by $N_{M,h}(e)$ the set of all such Nash (equilibrium) allocations. The mechanism $\langle M, h \rangle$ is said to fully *Nash-implement* the Linear Cost Share equilibrium allocations on E, if, for all $e \in E$, $N_{M,h}(e) = LCSE(e)$.

The remainder of this section is devoted to proving the following theorem.

THEOREM 1. *For the class of public goods economies specified in the last section, E, the above withholding mechanism, which is individually feasible, balanced, and continuous, fully Nash-implements the LCSE correspondence on E.*

Proof. The proof of Theorem 1 consists of the following two propositions, which show the equivalence between Nash allocations of the mechanism and LCSE allocations. Proposition 1 below proves that every Nash allocation is a LCSE allocation. Proposition 2 below proves that every LCSE allocation is a Nash allocation.

PROPOSITION 1. *If the withholding mechanism defined above has a Nash equilibrium m^*, then the Nash allocation $(X(m^*) + \mathring{w} - w^*, Y(m^*))$ is a Linear Cost Share equilibrium allocation with $(a_1(m^*), ..., a_n(m^*))$ and $(b_1(m^*), ..., b_n(m^*))$ as the parameters of the linear cost share system, i.e., $N_{M,h}(e) \subseteq LCSE(e)$.*

Proof. Let m^* be a Nash equilibrium. We need to prove that $(X(m^*) + \mathring{w} - w^*, Y(m^*))$ is a LCSE allocation with $(a_1(m^*), ..., a_n(m^*))$ and $(b_1(m^*), ..., b_n(m^*))$ as the parameters of a linear cost share system. Since the mechanism is individually feasible and balanced, $\sum_{i=1}^n a_i(m^*) = 0$, $\sum_{i=1}^n b_i(m^*) = 1$, and $[X_i(m^*) + \mathring{w}_i - w_i^*] + a_i(m^*) \cdot Y(m^*) + b_i(m^*)\, C(Y(m^*)) = \mathring{w}_i$ for all $i \in N$, we only need to show that each individual is maximizing his/her preferences. Suppose, by way of contradiction, that there is some $(x_i, y) \in \mathbb{R}^{1+K}_+$ such that $(x_i, y)\, P_i(X_i(m^*) + \mathring{w}_i - w_i^*, Y(m^*))$ and $x_i + a_i(m^*) \cdot y + b_i(m^*)\, C(y) \leqq \mathring{w}_i$. Let

$$x_{\lambda i} = \lambda x_i + (1 - \lambda)[X_i(m^*) + \mathring{w}_i - w_i^*]$$

$$y_\lambda = \lambda y + (1 - \lambda)\, Y(m^*).$$

Then by the convexity of preference we have $(x_{\lambda i}, y_\lambda)\, P_i(X_i(m^*) + \mathring{w}_i - w_i^*, Y(m^*))$ for any $0 < \lambda < 1$. Also $(x_{\lambda i}, y_\lambda) \in \mathbb{R}^{1+K}_+$ and $x_{\lambda i} + a_i(m^*) \cdot y_\lambda + b_i(m^*)\, C(y_\lambda) \leqq \mathring{w}_i$ by convexity of the cost function and nonnegativity of $b_i(m^*)$. Now suppose that player i chooses δ_i so that $\delta_i < \delta(m^*)$, $y_i = y_\lambda - \sum_{j \neq i}^n y_j^*$, and keeps w_i^*, α_i^*, and $(\beta_{1i}^*, ..., \beta_{ni}^*)$ unchanged. Then

$\delta(m^*_{-i}, m_i) = \delta_i < \delta(m^*)$ and thus $(1-\delta(m^*_{-i}, m_i))\, w_j^* - a_j(m^*) \cdot Y(m^*) - b_j(m^*)\, C(Y(m^*)) > (1-\delta(m^*))\, w_j^* - a_j(m^*) \cdot Y(m^*) - b_j(m^*)\, C(Y(m^*)) \geqq 0$ for all $j \in N$ by the construction of the mechanism. Thus, by the continuity of the cost function and outcome functions, we have $(1-\delta(m^*_{-i}, m_i))\, w_j^* - a_j(m^*_{-i}, m_i) \cdot y_\lambda - b_j(m^*_{-i}, m_i)\, C(y_\lambda) > 0$ for all $j \in N$ as λ is sufficiently small. Hence $y_\lambda \in B(m^*_{-i}, m_i)$ and therefore $Y(m^*_{-i}, m_i) = y_\lambda$ as well as $X_i(m^*_{-i}, m_i) = w_i^* - a_i(m^*) \cdot Y(m^*_{-i}, m_i) - b_i(m^*)\, C(Y(m^*_{-i}, m_i)) = w_i^* - a_i(m^*) \cdot y_\lambda - b_i(m^*)\, C(y_\lambda) \geqq x_{i\lambda} - [\mathring{w}_i - w_i^*]$. Then $X_i(m^*_{-i}, m_i) + \mathring{w}_i - w_i^* \geqq x_{i\lambda}$. From $(x_{i\lambda}, y_\lambda)\, P_i(X_i(m^*) + \mathring{w}_i - w_i^*, Y(m^*))$, we have

$$(X_i(m^*_{-i}, m_i) + \mathring{w}_i - w_i^*, Y(m^*_{-i}, m_i))\, P_i(X_i(m^*) + \mathring{w}_i - w_i^*, Y(m^*)).$$

This contradicts the hypothesis that $(X(m^*) + \mathring{w}_i - w_i^*, Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

PROPOSITION 2. *If (x^*, y^*) is a LCSE allocation with $(a_1^*, ..., a_n^*)$ and $(b_1^*, ..., b_n^*)$ as the parameters of the linear cost share system, then there is a Nash equilibrium m^* for the withholding mechanism defined above such that $X_i(m^*) + \mathring{w}_i - w_i^* = x_i^*$, $a_i(m^*) = a_i^*$, and $b_i(m^*) = b_i^*$, for all $i \in N$, $Y(m^*) = y^*$, i.e., $LCSE(e) \subseteq N_{M,h}(e)$.*

Proof. We first note that $x^* \in \mathbb{R}^n_{++}$ by the assumption that the private good is indispensable. We need to show that there is a message m^* such that (x^*, y^*) is a Nash equilibrium allocation. Let $\alpha_1^* = 0$, $\alpha_2^* = -a_n^*$, $\alpha_i^* = \alpha_{i-1}^* - \alpha_{i-2}^*$ for $i = 3, ..., n$. Let $w_i^* = \mathring{w}_i$, $\beta_{ij}^* = b_j^*$ $(j = 1, ..., n)$, $y_i^* = y^*/n$, and let δ_i be sufficiently small so that $(1-\delta(m))\, \mathring{w}_i - a_i(m) \cdot y^* - b_i(m)\, C(y^*) > 0$ for all $i \in N$. Then, $a_i(m^*) = a_i^*$, $b_i(m^*) = b_i^*$, $Y(m^*) = y^*$, and $X_i(m^*) = x_i^*$, for all $i \in N$. Note that consumer i cannot change $b_i(m^*)$ by changing his proposed $(\beta_{1i}, ... \beta_{ni})$ since changing (β_{1i}, β_{ni}) yields $\gamma_l > 0$ and $\gamma_{li} = 0$ $(l = 1, ..., n)$ so that the new $(\beta_{1i}, ..., \beta_{ni})$ cannot change $b_l(m^*)$ for all $l \in N$, and thus cannot change $b_i(m^*)$. Thus, $(a_i(m^*_{-i}, m_i),\ b_i(m^*_{-i}, m_i)) = (a_i(m^*), b_i(m^*))$ for all $m_i \in M_i$. Also, $(X(m^*_{-i}, m_i) + \mathring{w}_i - w_i, Y(m^*_{-i}, m_i)) \in \mathbb{R}^{1+K}_+$ and $[X_i(m^*_{-i}, m_i) + \mathring{w}_i - w_i] + a_i(m^*) \cdot Y(m^*_{-i}, m_i) + b_i(m^*)\, C(Y(m^*_{-i}, m_i)) = \mathring{w}_i$ for all $i \in N$ and $m_i \in M_i$. Therefore, we know that

$$\neg (X_i(m^*_{-i}, m_i) + \mathring{w}_i - w_i, Y(m^*_{-i}, m_i))\, P_i(X_i(m^*), Y(m^*)),$$

for otherwise it contradicts the fact that $(X_i(m^*), Y(m^*))$ is a LCSE allocation. Q.E.D.

From Proposition 1–2, we know that $N_{M,h}(e) = LCSE(e)$ for all $e \in E$ and thus the proof of Theorem 1 is completed. Q.E.D.

Remark 2. From Proposition 1, we can see that even at Nash equilibria consumers are not necessarily reporting their true endowments since we may have $w_i^* \neq \mathring{w}_i$. Indeed, we can modify the proof of Proposition 2 so that the underreported endowment is a Nash equilibrium strategy as long as the reported endowments are close enough to the true endowments. Thus every LCSE allocation can be supported by a Nash equilibrium even with the false announcement about endowments.

Since LCSE allocations are Pareto optimal and individually rational, the mechanism yields Pareto optimal and individually rational allocations.

4. Implementation: Destruction of Endowments

In this section, we give an individually feasible, weakly balanced, and continuous mechanism which fully Nash-implements the LCSE correspondence when the consumers may destroy a part of their endowments, but cannot withhold any of the unreported endowment. Note that when the unreported endowments are canceled (destroyed) rather than consumed, any mechanism dealing with this case is merely weakly balanced, but not balanced. The destruction mechanism is defined analogously to the withholding mechanism given in the above section. So we briefly describe it in the following.

The message spaces of consumers are defined as follows: For each $i \in N$, his/her message domain is of the form

$$M_i = (0, \mathring{w}_i] \times \mathbb{R}^K \times Q^n_+ \times \mathbb{R}^K. \tag{13}$$

All of the components of messages have the same interpretations as before. Note that this modified message domain does not contain a component for the degree of desirability of the private good, so the dimension of the message space for each consumer is reduced by one.

The side compensations function $a_i(m)$ and the direct cost share function $b_i(m)$ remain the same as before.

Define the feasible correspondence $B' \colon M \to 2^{\mathbb{R}^K_+}$ by

$$B'(m) = \{y \in \mathbb{R}^K_+ : w_i - a_i(m) \cdot y - b_i(m)\, C(y) \geqq 0\ \forall i \in N\}, \tag{14}$$

which is clearly a continuous correspondence with non-empty compact convex values.

Define the outcome function for public goods $Y \colon M \to B$ by

$$Y(m) = \{y : \min_{y \in B'(m)} \|y - \tilde{y}\|\}. \tag{15}$$

Define the linear cost share function $g_i\colon M \to \mathbb{R}$ by

$$g_i(m) = a_i(m) \cdot Y(m) + b_i(m)\, C(Y(m)). \tag{16}$$

The total cost share is then equal to the cost of production; i.e.,

$$\sum_{i=1}^{n} g_i(m) = C(Y(m)). \tag{17}$$

The ith consumer's outcome function for the private good $X_i(m)\colon M \to \mathbb{R}_+$ is given by

$$X_i(m) = w_i - a_i(m) \cdot Y(m) - b_i(m)\, C(Y(m)). \tag{18}$$

Thus the outcome function is continuous on M, $(X(m), Y(m)) \in \mathbb{R}^{n+K}_+$, and satisfies

$$\sum_{i=1}^{n} X_i(m) + C(Y(m)) = \sum_{i=1}^{n} w_i, \tag{19}$$

which means the aggregate consumption of the private good allocated by the mechanism is equal to the aggregate endowments reported by consumers for all $m \in M$.

Thus, by the construction of the mechanism $\langle M, h\rangle$ is individually feasible; i.e., $(X_i(m), Y(m)) \in \mathbb{R}^{1+K}_+$ for all $i \in N$ and all $m \in M$; weakly balanced, i.e., $\sum_{i=1}^{n} X_i(m) + C(Y(m)) \leqq \sum_{i=1}^{n} \mathring{w}_i$ for all $m \in M$; and continuous.

A message $m^* = (m_1^*, ..., m_n^*) \in M$ is said to be a *Nash equilibrium* of the mechanism $\langle M, h\rangle$ in the presence of destruction for an economy e if for any $i \in N$ and for all $m_i \in M_i$,

$$\neg (X_i(m^*_{-i}, m_i), Y(m^*_{-i}, m_i))\; P_i (X_i(m^*), Y(m^*)). \tag{20}$$

The destruction mechanism $\langle M, h\rangle$ is said to fully *Nash-implement* the Linear Cost Share equilibrium allocation on E, if, for all $e \in E$, $N_{M,h}(e) = LCSE(e)$.

Similarly, we have the following theorem.

THEOREM 2. *For the class of public goods economies specified in Section 2, E, the above destruction mechanism, which is individually feasible, weakly balanced, and continuous, fully Nash-implements the LCSE correspondence.*

Proof. The proof of Theorem 2 consists of the following two propositions. Proposition 3 below proves that every Nash allocation of the mechanism is a LCSE allocation. Proposition 4 below proves that every LCSE allocation is a Nash allocation. Since the proofs are very similar to those of Propositions 1–2, we just outline the proofs of the propositions.

PROPOSITIONS 3. *If the destruction mechanism defined above has a Nash equilibrium m^*, then the Nash allocation $(X(m^*), Y(m^*))$ is a Linear Cost Share equilibrium allocation with $(a_1(m^*), ..., a_n(m^*))$ and $(b_1(m^*), ..., b_n(m^*))$ as the parameters of the linear cost share system; i.e., $N_{M,h}(e) \subseteq LCSE(e)$.*

Proof. We first prove that $X_i(m^*) > 0$ for all $i \in N$. Suppose, by way of contradiction, that $X_i(m^*) = 0$ for some consumer i. Then $(w_i^*, 0)$ $P_i(X_i(m^*), Y(m^*))$ by the assumption of indispensability of the private good. Now suppose that agent i chooses $y_i = -\sum_{j \neq i}^n y_j^*$, and keeps other messages unchanged. Then, $0 \in B^i(m^*_{-i}, m_i)$. Hence, we have $Y(m^*_{-i}, m_i) = 0$ and $X_i(m^*_{-i}, m_i) = w_i^*$ and thus $(X_i(m^*_{-i}, m_i), 0)$ $P_i(X_i(m^*), Y(m^*))$. This contradicts $(X(m^*), (Y(m^*)) \in N_{M,h}(e)$.

We now prove that $(X(m^*), Y(m^*))$ is a Linear Cost Share equilibrium allocation with $(a_1(m^*), ..., a_n(m^*))$ and $(b_1(m^*), ..., b_n(m^*))$ as the parameters of the linear cost share system. By the same reasons as before, we only need to show that each individual is maximizing his/her preferences. Suppose, by way of contradiction, that there is some $(x_i, y) \in \mathbb{R}_+^{1+K}$ such that (x_i, y) $P_i(X_i(m^*), Y(m^*))$ and $x_i + a_i(m^*) \cdot y + b_i(m^*)\, C(y) \leqq \mathring{w}_i$. Let

$$x_{\lambda i} = \lambda x_i + (1-\lambda)\, X_i(m^*)$$

$$y_\lambda = \lambda y + (1-\lambda)\, Y(m^*).$$

Then we have $(x_{\lambda i}, y_\lambda)$ $P_i(X_i(m^*) + \mathring{w}_i + w_i^*, Y(m^*))$ for any $0 < \lambda < 1$. Also $(x_{\lambda i}, y_\lambda) \in \mathbb{R}_+^{1+K}$ and $x_{\lambda i} + a_i(m^*) \cdot y_\lambda + b_i(m^*)\, C(y_\lambda) \leqq \mathring{w}_i$. Now suppose that player i chooses $y_i = y_\lambda - \sum_{j \neq i}^n y_j^*$, $w_i = \mathring{w}_i$, and keeps α_i^* and $(\beta_{1i}^*, ..., \beta_{ni}^*)$ unchanged. Then $\mathring{w}_j - a_j(m^*) \cdot Y(m^*) - b_j(m^*)\, C(Y(m^*)) \geqq w_j^* - a_j(m^*) \cdot Y(m^*) - b_j(m^*)\, C(Y(m^*)) = X_j(m^*) > 0$ for all $j \in N$. Thus, we have $\mathring{w}_j - a_j(m^*_{-i}, m_i) \cdot y_\lambda - b_j(m^*_{-i}, m_i)\, C(y_\lambda) > 0$ for all $j \in N$ as λ is sufficiently small. Hence $y_\lambda \in B(m^*_{-i}, m_i)$ and therefore $Y(m^*_{-i}, m_i) = y_\lambda$ as well as $X_i(m^*_{-i}, m_i) = \mathring{w}_i - a_i(m^*) \cdot Y(m^*_{-i}, m_i) - b_i(m^*)\, C(Y(m^*_{-i}, m_i)) \geqq x_{i\lambda}$. From $(x_{i\lambda}, y_\lambda)$ $P_i(X_i(m^*), Y(m^*))$, we have

$$(X_i(m^*_{-i}, m_i), Y(m^*_{-i}, m_i))\ P_i(X_i(m^*), Y(m^*)).$$

This contradicts the hypothesis that $(X(m^*) + \mathring{w}_i - w_i^*, Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

PROPOSITION 4. *If (x^*, y^*) is a LCSE allocation with $(a_1^*, ..., a_n^*)$ and $(b_1^*, ..., b_n^*)$ as the parameters of the linear cost share system, then there is a Nash equilibrium m^* for the destruction mechanism defined above such that $X_i(m^*) = x_i^*$, $a_i(m^*) = a_i^*$, and $b_i(m^*) = b_i^*$, for all $i \in N$, $Y(m^*) = y^*$; i.e., $LCSE(e) \subseteq N_{M,h}(e)$.*

Proof. We need to show that there is a message m^* such that (x^*, y^*) is a Nash equilibrium allocation. Let $\alpha_1^* = 0$, $\alpha_2^* = -a_n^*$, $\alpha_i^* = \alpha_{i-1}^* - a_{i-2}^*$ for $i = 3, ..., n$. Let $w_i^* = \mathring{w}_i$, $\beta_{ij}^* = b_i^*$ $(j = 1, ..., n)$, and $y_i^* = y^*/n$. Then, $a_i(m^*) = a_i^*$, $b_i(m^*) = b_i^*$, $Y(m^*) = y^*$, and $X_i(m^*) = x_i^*$, for all $i \in N$. The remaining arguments are the same as those in the proof of Proposition 2 and thus omitted. Q.E.D.

Thus we have proved that $N_{M,h}(e) = LCSE(e)$ for all $e \in E$ and the proof of Theorem 2 is completed. Q.E.D.

Remark 3. From the proof of Proposition 3, one can see that at Nash equilibria we must have $w_i^* = \mathring{w}_i$, which is different from the case of withholding of endowments. Thus, the above mechanism, like the mechanisms of Hurwicz *et al* [6], forces agents to state correctly their endowments in equilibrium.

5. Concluding Remarks

In this paper, we have presented simple mechanisms which fully implement the Linear Cost Share equilibrium allocations when preferences and endowments are both private information and unknown to the designer for both cases of withholding and destruction of endowments. The mechanism dealing with the case of withholding has the advantage that consumers are not required to report their true endowments even at equilibrium, and thus the incentive compatibility problem for endowments is well taken. In addition, both mechanisms require that each consumer announce only his own endowment, but not others' endowments. The mechanisms are also well behaved in the sense that they are individually feasible, balanced (weakly balanced for the destruction mechanism), and continuous. Furthermore, we allow preferences of consumers to be nontotal–non-transitive and discontinuous. Though this paper only considers Nash-implementation of the Linear Cost Share equilibrium allocations for public goods economies with one private good, the mechanism presented here can be modified to implement the Linear Cost Share equilibrium allocations for public goods economies with any number of goods provided the notion of Linear Cost Share equilibria are extended to the case of many private goods. Also, if one reinterprets the commodity space, our mechanisms result in Pareto efficient allocations to economies with purely private goods or with externalities at Nash equilibria, without considering the profit-maximization principle. Thus, our mechanisms are sufficiently general to cover both Walrasian equilibria of pure exchange economies and public goods economies using either Lindahl equilibria or cost share equilibria. In other words, the mechanisms in the paper appear to represent "generic" mechanisms to implement competitive-type allocations in private and public goods economies.

Finally, some of the techniques developed in the paper may be applied to implement the Linear Cost Share equilibrium allocations using other solutions consepts such as those of subgame perfect equilibrium, undominated Nash equilibrium, and Bayesian Nash equilibrium when information on economic environments is incomplete. It is well known that the Nash equilibrium approach may have a problem in the case of multiple equilibria. Some Nash equilibrium approach may have a problem in the case of multiple equilibria. Some Nash equilibria may be more believable than others. Because of this, some equilibrium concepts may need to be used. Moore and Repullo [16] and Abreu and Sen [1] use subgame perfect equilibrium as the solution concept and gave conditions for subgame perfect implementation of social choice corespondences. Palfrey and Srivastava [17] and Jackson [7] use undominated Nash equilibrium as the solution concept. This concept, like subgame perfect equilibrium, is a refinement of Nash equilibrium. However, the mechanisms of Moore and Repullo [16], Abreu and Sen [1], Palfrey and Srivastava [17] are given just to show what is possible for implementation of a general social choice correspondence, but not what is realistic, since these mechanisms have some seriously undesirable properties (for instance, they are discontinuous and have message spaces of infinite dimensions). Thus even to the implementation of the existing social choice correspondence such as the Walrasian correspondence and Lindahl correspondence, by more realistic mechanisms, they are important and unsolved subjects.

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