Furthermore, the sequences $T^n w_1$ and $T^n w_2$ are increasing and decreasing, respectively. Hence by Dini's theorem both converge to $u(x)$ uniformly on bounded intervals. So does $T_n v$ as

$$T^n w_1(x) \leqslant T^n v(x) \leqslant T^n w_2(x).$$

This completes the proof.

Remark. By Theorem 1 we can get an estimate for the nonzero solution $u(x)$, if it exists. Assume that the function $v(x)$ satisfies

$$Tv(x) \leqslant v(x), \qquad \text{for} \quad 0 \leqslant x \leqslant c.$$

Then

$$u(x) \leqslant v(x) \qquad \text{for} \quad 0 \leqslant x \leqslant c.$$

In particular we have the following.

COROLLARY 2. *Let $\{v_n(x)\}_{n=1}^{\infty}$ be a sequence of positive increasing functions such that*

$$\lim_{n \to \infty} v_n(x) = 0, \qquad \text{for} \quad x \geqslant 0,$$

and

$$Tv_n(x) \leqslant v_n(x), \qquad \text{for} \quad x \geqslant 0.$$

Then the equation $Tu(x) = u(x)$ has no positive solutions.

In a forthcoming paper we will use Corollary 2 to prove that if $\phi(x) = \sqrt{x}$ and $a(x, y) = f(x - y)$ is an invariant kernel given by the function

$$f(x) = e^{-e^{1/x}},$$

then Eq. (2) admits no nonzero solutions.

REFERENCES

1. P. J. BUSHELL, On a class of Volterra and Fredholm non-linear integral equations, *Math. Proc. Cambridge Philos. Soc.* **79** (1979), 329–335.
2. P. J. BUSHELL AND W. OKRASIŃSKI, Uniqueness of solutions for a class of non-linear Volterra integral equations with convolution kernel, *Math. Proc. Cambridge Philos. Soc.* **106** (1989), 547–552.
3. W. OKRASIŃSKI, Non-negative solutions of some non-linear integral equations, *Ann. Polon. Math.* **44** (1984), 209–218.
4. W. OKRASIŃSKI, On a nonlinear Volterra equation, *Math. Methods Appl. Sci.* **8** (1986), 345–350.

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **170**, 457–471 (1992)

Generalizations of the FKKM Theorem and the Ky Fan Minimax Inequality, with Applications to Maximal Elements, Price Equilibrium, and Complementarity

GUOQIANG TIAN*

Department of Economics, Texas A & M University, College Station, Texas 77843

Submitted by C. Foias

Received May 21, 1990

This paper generalizes the Fan–Knaster–Kuratowski–Mazurkiewicz (FKKM) theorem of Ky Fan ("Game Theory and Related Topics," pp. 151–156, North-Holland, Amsterdam, 1979; and *Math. Ann.* **266**, 1984, 519–537) and the Ky Fan minimax inequality by introducing a class of the generalized closedness and continuity conditions, which are called the transfer closedness and transfer continuities. We then apply these results to prove the existence of maximal elements of binary relations under very weak assumptions. We also prove the existence of price equilibrium and the complementarity problem without the continuity assumptions. Thus our results generalize many of the existence theorems in the literature. © 1992 Academic Press, Inc.

1. INTRODUCTION

The classical Knaster–Kuratowski-Mazurkiewicz (KKM) theorem is a basic result for combinatorial mathematics, which is equivalent to many basic theorems such as Sperner's lemma, Brouwer's fixed point theorem, and Ky Fan's minimax inequality. Since Knaster, Kuratowski, and Mazurkiewicz [8] gave this theorem, many generalizations of the KKM theorem have been given. The most important generalization is the Fan–Knaster–Kuratowski–Mazurkiewicz (FKKM) theorem which was obtained by Ky Fan [4, 5] and can be used to prove and/or generalize many existence theorems such as fixed point and coincidence theorems for non-compact convex sets and intersection theorems for sets with convex

* I thank A. Mas-Colell, J. Zhou, and an anonymous referee for useful comments and suggestions. Of course, any remaining errors are my own.

0022-247X/92 $5.00

sections (cf. Fan [5]). This paper offers a further generalization of the KKM theorem and the FKKM theorem of Fan [4, 5] by introducing a class of generalized closedness conditions, which are called the transfer closedness and transfer continuities. We then use our FKKM theorem (Theorem 3) to generalize the Ky Fan minimax inequality by relaxing the compactness and convexity of sets, the lower semi-continuity and quasi-concavity of functions. Since the Ky Fan minimax inequality is a fundamental variational inequality, many existence theorems for variational inequalities and convex analysis can also be generalized by our minimax inequality.

As applications of these results to economics and optimization theory, we generalize a class of existence theorems on the maximal elements of binary relations, price equilibrium, and the complementarity problem by relaxing the compactness and convexity of choice sets, the closedness (openness) of upper (lower) contour sets, and the continuity of excess demand functions. The motivation comes from economic applications showing that the feasible sets or the budget constraints are generally not (weakly) compact in an infinite dimensional commodity space and are not convex in the case of the indivisibility of commodities. Thus, relaxation of convexity of choice sets enables us to deal with the existence of maximal elements even though commodities are indivisible. Further, it may be remarked that Theorem 3, the minimax inequality in Theorem 4, and the existence theorems on maximal elements (Theorems 5 and 6) below are equivalent to one another.

The plan of this paper is as follows. Section 2 states some notation and definitions. Section 3 gives generalizations of the FKKM theorem by relaxing the closedness condition. In Section 4, we generalize the Ky Fan minimax inequality. Section 5 gives the existence theorems on maximal elements of strict and weak preferences which may be nontotal-nontransitive. Finally, in Section 6, we use our minimax inequality to prove the existence of price equilibrium and the complementarity problem.

2. Notation and Definitions

Before the formal discussion, we begin with some notation and definitions. Let S be a subset of a topological (vector) space T and let $D \subset S$. Denote the collections of all subsets, convex hull, closure, and interior of the set D by 2^D, $\operatorname{co} D$, $\operatorname{cl} D$, and $\operatorname{int} D$, respectively. Denote by $\operatorname{cl}_S D$ and $\operatorname{int}_S D$ the relative closure and relative interior of D in S. Throughout the paper all topological vector spaces are assumed to be Hausdorff and denoted by E.

Let X be a topological space. A function $f: X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *lower semi-continuous* if for each point x', we have

$$\liminf_{x \to x'} f(x) \geqslant f(x'),$$

or equivalently, if its epigraph $\operatorname{epi} f \equiv \{(x, a) \in X \times \mathbb{R}: f(x) \leqslant a\}$ is a closed subset of $X \times \mathbb{R}$. A function $f: X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *upper semi-continuous* if $-f$ is upper semi-continuous.

Definition 1 (FS-Convexity[1]). Let Y be a convex subset of E and let $\emptyset \neq X \subset Y$. A correspondence $F: X \to 2^Y$ is said to be FS-*convex* on X if for every finite subset $\{x_1, x_2, ..., x_m\}$ of X

$$\operatorname{co}\{x_1, x_2, ..., x_m\} \subset \bigcup_{j=1}^{m} F(x_j).$$

Remark 1. Note that the FS-convexity of F implies that every point $x \in X$ is a fixed point of $F(x)$, i.e., $x \in F(x)$.

Definition 2 (SS-Convexity[2]). Let Y be a convex subset of E. A correspondence $F: Y \to 2^Y$ is said to be SS-*convex* if $x \notin \operatorname{co} F(x)$ for all $x \in Y$.

Definition 3 (γ-Diagonal Quasi-Concavity). Let Y be a convex subset of E and let $\emptyset \neq X \subset Y$. A function $\phi(x, y): X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *γ-diagonally quasi-concave* (γ-DQCV) in x, if for every finite subset $\{x_1, ..., x_m\} \subset X$ and any $x_\lambda \in \operatorname{co}\{x_1, ..., x_m\}$, we have

$$\inf_{1 \leqslant j \leqslant m} \phi(x_j, x_\lambda) \leqslant \gamma.$$

Remark 2. The above definition on γ-DQCV is more general than that of Zhou and Chen [21]. Here we do not require that $X = Y$ and X be convex.

Remark 3. It is easily shown that a function $\phi: X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ is γ-DQCV in x if and only if the correspondence $F: X \to 2^Y$ defined by $F(x) = \{y \in Y: \phi(x_j, y) \leqslant \gamma\}$ for all $x \in X$ is FS-convex on X.

In the literature, there are two approaches to nontransitive-nontotal preference theory: one through "weak" (i.e., reflexive) preferences (see, e.g., Sonnenschein [13] and Shafer [12[3]), and the other through "strict" (i.e.,

[1] The FS is for Fan [4] and Sonnenschein [13].

[2] The SS is for Shafer and Sonnenschein [12].

[3] Both Sonnenschein and Shafer assume that preferences are complete, and work with a weak preference relation as the underlying source of the strict preferences.

irreflexive) preferences (see, e.g., Schmeidler [11], Mas-Colell [9], Gale and Mas-Colell [10], Yannelis and Prabhakar [20], Tian [16]). The distinction becomes important when preferences are not total. Kim and Richter [7] made the connection between the weak preference approach and the strict preference approach. We will generalize the existence theorems on maximal elements for both types of preferences by relaxing the compactness and convexity conditions. Suppose that the (weak or strict) preference relation is defined on Z and is a subset of $Z \times Z$. Here Z may be considered as a consumption space. Let $\succcurlyeq$ be the weak preference relation. An element (x, y) in $\succcurlyeq$ is written as $x \succcurlyeq y$ and read as "x is at least as good as y." Let $\succ$ be the strict preference relation. An elements (x, y) in $\succ$ is written as $x \succ y$ and read as "x is (strictly) preferred to y." For each x, the weakly upper, weakly lower, strictly upper, and strictly lower contour sets (sections) of x are denoted by $U_w(x) = \{y \in Z : y \succcurlyeq x\}$, $L_w(x) = U_w^{-1}(x) = \{y \in Z : x \succcurlyeq y\}$, $U_s(x) = \{y \in Z : y \succ x\}$, and $L_s(x) = U_x^{-1}(x) = \{y \in Z : x \succ y\}$, respectively.

In some cases, not all points in Z can be chosen, so let $B \subset Z$ be a choice set, which may be considered as, say, the budget set or feasible set.

Definition 4 (Greatest Element). A weak binary relation $\succcurlyeq$ is said to have a *greatest element* on the subset B of Z if there exists a point $x^* \in B$ such that $x^* \succcurlyeq x$ for all $x \in B$, or equivalently $\bigcap_{x \in B} U_w(x) \neq \varnothing$ on B.

Definition 5 (Maximal Element). A strict binary relation $\succ$ is said to have a *maximal element* on the subset B of Z if there exists a point $x^* \in B$ such that $\neg\, x \succ x^*$ for all $x \in B$, i.e., $U_s(x^*) = \varnothing$ on B, where $\neg$ stands for "it is not the case that."

Remark 4. In general there is no relationship between the $\succcurlyeq$-greatest elements and $\succ$-maximal elements. However, when $\succ$ is the asymmetric part of the preference $\succcurlyeq$, $\succcurlyeq$-greatest elements are $\succ$-maximal elements and further they coincide if $\succcurlyeq$ is also complete.[4]

3. Generalizations of the FKKM Theorem

We begin by stating the FKKM theorem due to Fan [4, 5] which is a generalization of the KKM theorem by relaxing the compactness and convexity conditions.

[4] The strict preference $\succ$ is said to be the asymmetric part of $\succcurlyeq$ if $x \succcurlyeq y$ but not $y \succcurlyeq x$. The preference $\succcurlyeq$ is said to be complete if, for any $x, y \in X$, either $x \succcurlyeq y$ or $y \succcurlyeq x$.

Theorem 1 (FKKM Theorem). *In a Hausdorff topological vector space, let Y be a convex set and $\varnothing \neq X \subset Y$. Let $F: X \to 2^Y$ be a correspondence such that*

(a) *for each $x \in X$, $F(x)$ is a relatively closed subset of Y;*

(b) *F is FS-convex on X;*

(c) *there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y.*

Then $\bigcap_{x \in X} F(x) \neq \varnothing$.

Here, by relaxing the closedness condition, we extend Theorem 1 by relaxing the closedness and FS-convexity of $F(x)$. Accordingly, we introduce

Definition 6 (Transfer Closedness). Let X and Y be two topological spaces. A correspondence $G: X \to 2^Y$ is said to be *transfer closed-valued* on X if for every $x \in X$, $y \notin G(x)$ implies that there exists a point $x' \in X$ such that $y \notin \operatorname{cl} G(x')$.

Definition 7 (Transfer Openness). Let X and Y be two topological spaces. A correspondence $P: X \to 2^Y$ is said to be *transfer open-valued* on X if for every $x \in X$, $y \in P(x)$ implies that there exists a point $x' \in X$ such that $y \in \operatorname{int} P(x')$.

Remark 5. Observe that a correspondence is transfer closed-valued if it is closed-valued; a correspondence is transfer open-valued if it is open-valued by letting $x' = x$. Also a correspondence $P: X \to 2^Y$ is transfer open-valued on X if and only if the correspondence $G: X \to 2^Y$, defined by, for every $x \in X$, $G(x) = Y \setminus P(x)$, is transfer closed-valued on X.

Theorem 2. *In a Hausdorff topological vector space, let Y be a convex set and $\varnothing \neq X \subset Y$. Let $F: X \to 2^Y$ be a correspondence such that*

(2a) *F is transfer closed-valued on X;*

(2b) *the correspondence $\operatorname{cl}_Y F: X \to 2^Y$ is FS-convex on X;*

(2c) *there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} \operatorname{cl}_Y F(x)$ is compact and X_0 is contained in a compact convex subset of Y.*

Then $\bigcap_{x \in X} F(x) \neq \varnothing$.

Remark 6. Observe that in case $F(x)$ is closed in Y, Condition (2a) in Theorem 2 is satisfied by letting $x' = x$. Theorem 2 then reduces to Theorem 1. Also Condition (2b) does not require that F satisfy the FS-convexity condition. Further, Conditon (2c) is satisfied if Y is compact.

Proof of Theorem 2. We first prove that $\bigcap_{x \in X} \mathrm{cl}_Y F(x) = \bigcap_{x \in X} F(x)$. It is clear that $\bigcap_{x \in X} F(x) \subset \bigcap_{x \in X} \mathrm{cl}_Y F(x)$. So we only need to show $\bigcap_{x \in X} \mathrm{cl}_Y F(x) \subset \bigcap_{x \in X} F(x)$. Suppose, by way of contradiction, that there is some y in $\bigcap_{x \in X} \mathrm{cl}_Y F(x)$ but not in $\bigcap_{x \in X} F(x)$. Then $y \notin F(x)$ for some $x \in X$. By Condition (2a), there is some $x' \in X$ such that $y \notin \mathrm{cl}_Y F(x')$, a contradiction. For $x \in X$, let $K(x) = \mathrm{cl}_Y F(x)$. Then $K(x)$ satisfies all conditions of Theorem 1 and thus, by Theorem 1, $\bigcap_{x \in X} F(x) = \bigcap_{x \in X} K(x) \neq \varnothing$. ∎

Remark 7. Even though the transfer closedness of F, in general, is not a necessary condition for Theorem 2, it is, however, very weak. In fact, it is a necessary and sufficient condition for $\bigcap_{x \in X} \mathrm{cl}_Y F(x) = \bigcap_{x \in X} F(x)$. The proof of sufficiency is given in the proof of Theorem 2. Here we show it is also a necessary condition. Indeed, suppose $\bigcap_{x \in X} \mathrm{cl}_Y F(x) = \bigcap_{x \in X} F(x)$. For every $x \in X$, if $y \notin F(x)$, then $y \notin \bigcap_{x \in X} F(x) = \bigcap_{x \in X} \mathrm{cl}_Y F(x)$ and thus $y \notin \mathrm{cl}_Y F(x')$ for some $x' \in X$. So F is transfer closed-valued. Thus it is the weakest condition which enables us to use the finite intersection property to show the nonemptyness of $\bigcap_{x \in X} F(x)$. On the other hand, surprisingly, Tian and Zhou [18] recently proved that the transfer closedness of upper sections (contour sets) of a function f, defined by

$$F(x) = \{y \in X: f(y) \geqslant f(x)\},$$

is a necessary and sufficient condition for the set of maximum points of the function f to be nonempty and compact when X is a compact set.[5] Thus this result generalizes the classical Weierstrass theorem by giving a necessary and sufficient condition. In Section 5, we will generalize this result to the existence of greatest elements of ordering relations.

Theorem 2 does weaken the closedness and FS-convexity conditions of F in Theorem 1. The following simple examples show that Theorem 2 is not included in Theorem 1.

EXAMPLE 1. Let $Y = [0, 1]$ and let X consist of all rational points in Y and thus X is non-compact and non-convex on Y. Let $F: X \to 2^Y$ be defined by, for all $x \in X$, $F(x)$ consisting of all rational points in the interval $[x, 1]$. Note that $F(x)$ is non-closed except for $x = 1$ and the FS-convexity condition is not satisfied, so we cannot apply Theorem 1. But Hypotheses (2a), (2b), and (2c) of Theorem 2 are satisfied. Indeed, for every $x \in X$, if $y \notin F(x)$, we can find a point $x' \in X$, say $x' = 1$, such that $y \notin \mathrm{cl}_Y F(1)$ and $\mathrm{cl}_Y F$ is FS-convex since $\mathrm{cl}_Y F(x) = [x, 1]$ for all $x \in X$. If we let X_0 consist of all rational points in $[1/2, 1]$, then $\bigcap_{x \in X_0} \mathrm{cl}_Y F(x) = \{1\}$ is compact and X_0 is contained in a compact convex subset $[1/2, 1]$ of Y. Hence, by Theorem 2, $\bigcap_{x \in X} F(x) \neq \varnothing$.

EXAMPLE 2. Let $Y = [0, 2] \subset \mathbb{R}$ and let $X = (0, 1/4) \cup (1/3, 1)$ which is non-compact and non-convex. For each $x \in X$, let $F(x) = (x, 2]$. Then, for any $x \in X$, $F(x)$ is not closed in Y and the FS-convexity condition is not satisfied (since $x \notin F(x)$) so we cannot apply Theorem 1. But all the hypotheses of Theorem 2 are satisfied. In fact, for any $x \in X$, if $y \notin F(x)$, then $y \leqslant x$. If we take any x' with $x < x'$ and $x' \in X$, we have $y \notin [x', 2]$, i.e., $y \notin \mathrm{cl}_Y F(x')$. So Condition (2a) is satisfied. Let $X_0 = (1/2, 1)$. Then $\bigcap_{x \in X_0} \mathrm{cl}_Y F(x) = [1, 2]$ is compact and X_0 is contained in the compact convex subset $[0, 1]$. Thus Condition (2c) is satisfied. Condition (2b) is clearly satisfied. Hence $\bigcap_{x \in X} F(x) \neq \varnothing$ by Theorem 2. (It can be easily verified that $\bigcap_{x \in X} F(x) = \bigcap_{x \in X} \mathrm{cl}_Y F(x) = [1, 2] \neq \varnothing$.)

It may be remarked that Example 2 also shows that Theorem 2 cannot guarantee $y^* \in X$ even if $y^* \in \bigcap_{x \in X} F(x)$.[6] If we require that y^* be in X, we need to strengthen Condition (2c) and have the following theorem which is the key mathematical tool in this paper:

THEOREM 3. *Suppose all the conditions in Theorem* 2 *hold except that Hypothesis* (2c) *is replaced by*

(3c) *there exists a nonempty set* $X_0 \subset X$ *such that for each* $y \in Y \setminus X_0$ *there exists a point* $x \in X_0$ *with* $y \notin \mathrm{cl}_Y F(x)$ *and* X_0 *is contained in a compact convex subset of* Y.

Then $X \cap (\bigcap_{x \in X} F(x)) \neq \varnothing$.

Proof of Theorem 3. By Condition (3c), we know that $D \equiv \bigcap_{x \in X_0} \mathrm{cl}_Y F(x) \subset X_0$. Since X_0 is contained in a compact convex subset of Y, D is compact. Hence, by Theorem 2, $\bigcap_{x \in X} F(x) \neq \varnothing$. Now for any $y \in \bigcap_{x \in X} F(x)$, we must have $y \in D \subset X_0$, for otherwise $y \notin \mathrm{cl}_Y F(x)$ for some $x \in X_0$. Hence $y \in X$. ∎

4. GENERALIZATIONS OF THE KY FAN MINIMAX INEQUALITY

Tian [14, 15] generalized the minimax inequalities of Fan [3], Allen [1], and Zhou and Chen [21] by relaxing the convexity of sets and showed that they are equivalent to the FKKM theorems. We now give further generalizations of the Ky Fan minimax inequality by relaxing the

[5] This condition can be equivalently stated as follows: If for points $x, y \in X$, $f(y) < f(x)$ implies that there exists a point $x' \in X$ and a neighborhood $\mathcal{N}(y)$ of y such that $f(z) < f(x')$ for all $z \in \mathcal{N}(y)$. A function f which satisfies this condition is said to be transfer upper continuous.

[6] This is also true for Theorem 1.

lower semi-continuity and γ-diagonal quasi-convexity conditions. We first introduce

DEFINITION 8 (γ-Transfer Lower Semi-Continuity). Let X and Y be two topological spaces. A function $\phi: X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ is said to be γ-transfer lower semi-continuous in y if for all $x \in X$ and $y \in Y$, $\phi(x, y) > \gamma$ implies that there exists some point $x' \in X$ and some neighborhood $\mathcal{N}(y)$ of y such that $\phi(x', z) > \gamma$ for all $z \in \mathcal{N}(y)$.

THEOREM 4. *Let Y be a nonempty convex subset of a Hausdorff topological vector space E, let $\varnothing \neq X \subset Y$, let $\gamma \in \mathbb{R}$, and let $\phi: X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be a function such that*

(4i) *it is γ-transfer lower semi-continuous in y;*

(4ii) *for every finite subset $\{x_1, x_2, ..., x_m\}$ of X,* $\mathrm{co}\{x_1, x_2, ..., x_m\} \subset \bigcup_{j=1}^m \mathrm{cl}_Y\{y \in Y: \phi(x_j, y) \leqslant \gamma\}$;

(4iii) *there exists a nonempty subset $C \subset X$ such that for each $y \in Y \backslash C$ there exists a point $x \in C$ with $y \in \mathrm{int}_Y\{z \in Y: \phi(x, z) > \gamma\}$ and C is contained in a compact convex subset of Y.*

Then there exists a point $y^* \in X$ such that $\phi(x, y^*) \leqslant \gamma$ for all $x \in X$.

Proof of Theorem 4. For $x \in X$, let $F(x) = \{y \in Y: \phi(x, y) \leqslant \gamma\}$. Then, by Conditions (4i) and (4ii), $F(x)$ satisfies Conditions (2a) and (2b) of Theorem 3. By Condition (4iii) and the definition of F, we know that for each $y \in Y \backslash C$ there exists a point $x \in C$ with $y \notin \mathrm{cl}_Y F(x)$. Hence, by Theorem 3, $X \cap (\bigcap_{x \in X} F(x)) \neq \varnothing$. Thus there is a point $y^* \in X$ such that $\phi(x, y^*) \leqslant \gamma$ for all $x \in X$. ∎

Remark 8. Note that Condition (4i) is satisfied if $\phi(x, y)$ is lower semi-continuous in y, Condition (4ii) is satisfied if ϕ is γ-diagonally quasi-concave in $x \in X$, and Condition (4iii) is satisfied if $X = Y$ and Y is compact.

Remark 9. Theorem 4 generalizes the minimax inequality of: Fan [3] by relaxing the quasi-concavity and lower semi-continuity of ϕ and the convexity and compactness of X; Allen [1] by relaxing the quasi-concavity and lower semi-continuity of ϕ and the convexity of X; Zhou and Chen [21] by relaxing the γ-diagonal quasi-concavity and lower semi-continuity of ϕ and the convexity of X; Tian [14] by relaxing the γ-diagonal quasi-concavity and lower semi-continuity of ϕ; and Tian [15] by relaxing the γ-diagonal quasi-concavity of ϕ.

Remark 10. Similar to those in Zhou and Chen [21] and Tian [14], Theorem 4 can also be proved independently of Theorem 3 by using the Brouwer's fixed point theorem.

Remark 11. Theorem 3 and Theorem 4 are in fact equivalent. This is because Theorem 3 can also be derived from Theorem 4 by defining $\phi: X \times Y \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi(x, y) = \begin{cases} \gamma & \text{if } (x, y) \in G \\ +\infty & \text{otherwise,} \end{cases}$$

where $\gamma \in \mathbb{R}$ and $G = \{(x, y) \in X \times Y: y \in F(x)\}$, and then applying Theorem 4.

5. THE EXISTENCE OF MAXIMAL BINARY RELATIONS

In this section, we use Theorem 3 to prove the following theorems (Theorems 5, 6) which give sufficient conditions for the existence of a greatest element for weak preferences and a maximal element for strict preferences on non-compact and non-convex infinite dimensional choice sets. It will be noted that the weakly upper (strictly lower) contour sets may not be closed (open). The preference relations may be nontransitive-nontotal. Thus we can obtain the demand correspondence even though the budget set is non-compact (for instance, with zero prices for some commodities) and non-convex (e.g., with indivisible commodities), preferences are nontransitive-nontotal, and further the weakly upper (strictly lower) contour sets may not be closed (open).

DEFINITION 9 (Transfer Upper Continuity). Let Z be a topological space and let $\varnothing \neq B \subset Z$. A preference relation $\succcurlyeq$ defined on Z is said to be *transfer weakly upper continuous* on B if for all $x \in B$ and $y \in Z$, $x \succ y$ implies that there exists a point $x' \in B$ and a neighborhood $\mathcal{N}(y)$ of y such that $x' \succ z$ for all $z \in \mathcal{N}(y)$.

DEFINITION 10 (Transfer Weakly Upper Continuity). Let Z be a topological space and let $\varnothing \neq B \subset Z$. A preference relation $\succcurlyeq$ defined on Z is said to be *transfer weakly upper continuous* on B if for all $x \in B$ and $y \in Z$, $x \succ y$ implies that there exists a point $x' \in B$ and a neighborhood $\mathcal{N}(y)$ of y such that $x' \succcurlyeq z$ for all $z \in \mathcal{N}(y)$.

Remark 12. It is clear that a preference relation $\succcurlyeq$ is transfer upper continuous on B if and only if $U_w: B \to 2^Z$ is transfer closed-valued on B.

DEFINITION 11 (Generalized SS-Convexity). Let Y be a convex subset of E and let $\varnothing \neq X \subset Y$. A correspondence $F: X \to 2^Y$ is said to be *generalized* SS-*convex* on X if for every finite subset $\{x_1, x_2, ..., x_m\}$ of X and $x_0 \in \mathrm{co}\{x_1, x_2, ..., x_m\}$, $x_j \notin F(x_0)$ for some $1 \leqslant j \leqslant m$.

Remark 13. Note that the SS-convexity of F implies the generalized SS-convexity. The converse statement may not be true unless $X = Y$.

The following theorem shows the existence of greatest elements on a non-compact and non-convex set for weak preferences.

THEOREM 5. *Let Z be a convex subset of a Hausdorff topological vector space E, let $\emptyset \neq B \subset Z$, and let $\succcurlyeq$ defined on Z be a weak binary relation such that*

(5i) *U_w is transfer closed-valued on B;*

(5ii) *$\mathrm{cl}_Z\, U_w$ is FS-convex on B;*

(5iii) *there exists a nonempty set $C \subset B$ such that for each $y \in Z \backslash C$ there exists a point $x \in C$ with $y \notin \mathrm{cl}_Z\, U_w(x)$ and C is contained in a compact convex subset of Z.*

Then $\succcurlyeq$ has a greatest element on B.

Proof of Theorem 5. The proof is a consequence of Theorem 3. For $x \in B$, let $F(x) = U_w(x)$. Then $F(x)$ satisfies all the assumptions of Theorem 3. Hence, by Theorem 3, $B \cap (\bigcap_{x \in B} U_w(x)) \neq \emptyset$. Thus there is a point y^* in $B \cap (\bigcap_{x \in B} U_w(x))$ which means $y^* \succcurlyeq x$ for all $x \in B$. ∎

Remark 14. Theorem 5 is also equivalent to Theorem 3 since we can get Theorem 3 from Theorem 5 by defining a relation $\succcurlyeq$ on Z by, for each $x \in X$, $y \succcurlyeq x$ if and only if $y \in F(x)$, and then applying Theorem 5.

When preference relations become orderings (i.e., relations are reflexive, transitive, and total), the transfer closedness of U_w completely characterizes the existence of greatest elements of $\succcurlyeq$. The following propositions generalize the well-known Weierstrass theorem by giving necessary and sufficient conditions and are slight extensions of the results obtained in Tian and Zhou [18] to preference relations. For the completeness, we give the proofs of these propositions even though they are essentially the same as those in Tian and Zhou [18].

PROPOSITION 1. *Let B be a nonempty compact subset in a topological space and let $\succcurlyeq$ be an ordering. Then $\succcurlyeq$ attains its maximum if and only if $\succcurlyeq$ is transfer weakly upper continuous on B.*

Proof. Sufficiency. By way of contradiction, suppose $\succcurlyeq$ does not attain its maximum on B. Then for each $y \in B$, there exists a point $x \in B$ such that $x \succ y$. By the transfer weakly upper continuity of $\succcurlyeq$, there exists a point $x' \in B$ and a neighborhood $\mathcal{N}(y)$ such that $x' \succcurlyeq y'$ for all $y' \in \mathcal{N}(y)$. It follows that $B \subset \bigcup_{y \in B} \mathcal{N}(y)$. Since B is compact, there exist finite points $\{y_1, y_2, ..., y_n\}$ such that $B \subset \bigcup_{i=1}^n \mathcal{N}(y_i)$. Let x_i' be the associated point such that $x_i' \succcurlyeq y'$ for all $y' \in \mathcal{N}(y_i)$. For the finite subset $\{x_1', x_2', ..., x_n'\}$, $\succcurlyeq$ has a greatest point, say, x_1', i.e., $x_1' \succcurlyeq x_i'$ for $i = 1, 2, ..., n$. Since $\succcurlyeq$ has no maximum point on B by the hypothesis, x_1' is not a maximum point of $\succcurlyeq$ on B, thus there is $x \in B$, such that $x \succ x_1'$. But $x \in \mathcal{N}(y_j')$ for some $1 \leqslant j \leqslant n$, therefore $x \succ x_1' \succcurlyeq x_j' \succcurlyeq x$, a contradiction. Hence $\succcurlyeq$ attains its maximum on B.

Necessity. This is trivial. Just let x' be any maximal element of $\succcurlyeq$. Then $x' \succcurlyeq y'$ for all $y' \in B$. ∎

Sometimes, we want the set of greatest elements of $\succcurlyeq$ to be not only nonempty but also compact, say, we want the demand correspondence to be nonempty and compact in order to use fixed point theorems to show the existence of competitive equilibrium or equilibrium for abstract economies (cf. Tian [17]). Then we have the following proposition.

PROPOSITION 2. *Let B be a nonempty compact subset in a Hausdorff topological space and let $\succcurlyeq$ be an ordering. Then the set of greatest elements of $\succcurlyeq$ is nonempty and compact if and only if U_w is transfer closed-valued on B, i.e., if and only if $\succcurlyeq$ is transfer upper continuous on B.*

Proof. Necessity ($\Rightarrow$). Suppose that the set of greatest elements of $\succcurlyeq$ is nonempty compact. Since $\succcurlyeq$ is an ordering, then for every $x \in B$, if $y \notin U_w(x)$ for some $y \in B$, $y \notin U_w(x')$, where $x' \in B$ is a greatest element of $\succcurlyeq$ on B. Since the set of greatest elements is compact, there exists a neighborhood $\mathcal{N}(y)$ of y such that $y' \notin U_w(x')$ for all $y' \in \mathcal{N}(y)$. Thus, $y \notin \mathrm{cl}_Z\, U_w(x')$. Hence U_w is transfer closed-valued on B.

Sufficiency ($\Leftarrow$). Since U_w is transfer closed-valued, $\bigcap_{x \in B} \mathrm{cl}\, U_w(x) = \bigcap_{x \in B} U_w(x)$. Now for any finite subset $\{x_1, x_2, ..., x_m\} \subset B$, $\succcurlyeq$ has a greatest element, say, x_1, on the finite set, i.e., $x_1 \succcurlyeq x_i$ for $i = 1, ..., m$. Then $x_1 \in U_w(x_i)$ for $i = 1, ..., m$. Therefore, $\emptyset \neq \bigcap_{i=1}^m U_w(x_i) \subset \bigcap_{i=1}^m \mathrm{cl}_Z\, U_w(x_i)$. Hence the family of sets $\{\mathrm{cl}_Z\, U_w(x)\colon x \in B\}$ has the finite intersection property on B. Also, since $\{\mathrm{cl}_Z\, U_w(x)\colon x \in B\}$ is a family of closed subsets in the compact set B, $\emptyset \neq \bigcap_{x \in B} \mathrm{cl}_Z\, U_w(x) = \bigcap_{x \in B} U_w(x)$ which means that there exists a point $x^* \in B$ such that $x^* \succcurlyeq x$ for all $x \in B$. Since the set of greatest elements $\bigcap_{x \in B} \mathrm{cl}_Z\, U_w(x)$ is a closed subset of B, it is compact. ∎

Theorem 3 can also be used to prove the existence of maximal elements on a non-compact and non-convex set for strict preferences.

THEOREM 6. *Let Z be a convex subset of a Hausdorff topological vector space E, let $\emptyset \neq B \subset Z$, and let $\succ$ be a strict binary relation on Z such that*

(6a) *L_s is transfer open-valued on B;*

(6b) *$\mathrm{int}_Z\, U_s$ is generalized* SS-*convex on B;*

(6c) *there exists a nonempty set $C \subset Z$ such that for each $y \in Z \backslash C$ there exists a point $x \in C$ with $y \in \mathrm{int}_Z\, L_s(x)$ and C is contained in a compact convex subset of Z.*

Then $\succ$ has a maximal element on B.

Proof of Theorem 6. Let $F(x) = Z \backslash L_s(x)$. Then $\{x \in X: U_s(x) = \varnothing\} = \bigcap_{x \in X} F(x)$, F is transfer closed-valued, and Condition (6c) implies Condition (3c). So we only need to show that U_w is FS-convex. Suppose, by way of contradiction, that there exists a point $x_0 \in \mathrm{co}\{x_1, x_2, ..., x_m\}$ of Z which is not in $\mathrm{cl}_Z\, F(x_j) = Z \backslash \mathrm{int}_Z\, L_s(x)$ for all j. Then $x_0 \in \mathrm{int}_Z\, L_s(x_j)$, so $x_j \in \mathrm{int}_Z\, U_s(x_0)$ for all j, which contradicts the generalized SS-convexity of $\mathrm{int}_Z\, U_s(x_0)$. Hence, by Theorem 3, $\bigcap_{x \in X} F(x) \neq \varnothing$. So there exists some point $x^* \in X$ such that $U_s(x^*) = \varnothing$. ∎

Remark 15. Theorem 6 generalizes the results of Sonnenschein [13] and Yannelis and Prabhakar [20] by relaxing the openness of the strictly lower contour sets and the compactness and convexity of the choice sets. Note that Theorem 5 can be also derived from Theorem 6 if we define a strict binary relation $\succ$ on Z by $x \succ y$ if and only if $\neg\, y \succcurlyeq x$ and then apply Theorem 6. Thus our Theorem 3, Theorem 4, Theorem 5, and Theorem 6 are equivalent to one another.

It may be interesting to know the relationships among the various convexities for preference relations. For example, many economists (say Shafer and Sonnenschein [12], Border [2], Yannelis [20], etc.) use the SS-convexity hypothesis to prove existence theorems in economics when preferences are nontotal-nontransitive. Is this hypothesis weaker than the convexity of $U_s(\cdot)$ or the weak-convexity[7] of relation $\succcurlyeq$ when $\succcurlyeq$ is an ordering? The following proposition proves that these convexity conditions are equivalent to one another when a preference relation becomes an ordering.

PROPOSITION 3. *Let preference relation $\succcurlyeq$ on Z be an ordering, and let $\succ$ be the asymmetric part of $\succcurlyeq$. Then the following statements are equivalent to one another.*

(1) *The relation $\succcurlyeq$ is weakly convex on Z.*

(2) *$U_w(x)$ is convex for every $x \in Z$.*

(3) *$U_s(x)$ is convex for every $x \in Z$.*

[7] A relation $\succcurlyeq$ is weakly convex if $y \succcurlyeq x$ implies $\lambda y + (1-\lambda)x \succcurlyeq x$ for all $0 \leqslant \lambda \leqslant 1$.

(4) *U_s is* SS-*convex on Z.*

(5) *U_w is* FS-*convex on Z.*

Proof. Implications from (1) to (2), (2) to (3), (3) to (4), and (5) to (1) are obvious. We only need to show that (4) implies (5). Indeed, suppose, by way of contradiction, that there exists a point x_λ in the convex hull of some finite subset $\{x_1, x_2, ..., x_m\}$ of Z which is not in $U_w(x_j)$ for any j. Then $x_\lambda \in L_s(x_j)$, so $x_j \in U_s(x_\lambda)$ for all j. But then $x_\lambda \in \mathrm{co}\, U_s(x_\lambda)$, a contradiction. ∎

6. Price Equilibrium and the Complementarity Problem

In this section we will study the existence of price equilibrium and complementarity by using Theorem 4. The equilibrium price problem is to find a price vector p which clears the markets for all commodities (i.e., the excess demands $f(p) \leqslant 0$ for the free disposal equilibrium price or $f(p) = 0$) under the assumption of Walras' law. Here we give an existence theorem on price equilibrium by relaxing the lower semi-continuity of the excess demand functions. For simplicity, we only work with the Euclidean space $\mathbb{R}^{n+1}$.

THEOREM 7. *Let Δ_n be the closed standard n-simplex and let $f: \Delta_n \to \mathbb{R}^{n+1}$ be an excess demand function such that*

(i) *the function $\phi: \Delta_n \times \Delta_n \to \mathbb{R}$ defined by $\phi(p, q) = p \cdot f(q)$ is 0-transfer lower semi-continuous in q;*

(ii) *for all $p \in \Delta_n$, $p \cdot f(p) \leqslant 0$ (Walras' Law).*

Then there exists a price vector $q^* \in \Delta_n$ such that $f(q^*) \leqslant 0$.

Proof of Theorem 7. Let $\phi(p, q) = p \cdot f(q)$. Then ϕ satisfies all the conditions of Theorem 4 with $\gamma = 0$ (by noting Δ_n is compact so that Condition (4iii) is satisfied) and thus there exists some $q^* \in \Delta_n$ such that $\phi(p, q^*) = p \cdot f(q^*) \leqslant 0$ for all $p \in \Delta_n$. Hence $f(q^*) \leqslant 0$. ∎

Remark 16. The 0-transfer lower semi-continuity of $\phi(p, \cdot)$ means that, if the excess demand $f(q)$ at price vector q is not affordable at price vector p, then there exists a price vector p' such that the excess demand $f(z)$ at any price vector which is sufficiently close to q is also not affordable at the price vector p'. Note that, since $p \geqslant 0$, this condition is satisfied if every component of f is lower semi-continuous by letting $p' = p$.

We now consider a mathematically more general problem which is known as the nonlinear complementarity problem.

Let X be a convex cone of a vector space, let $f: X \to E^*$ (dual of E). The problem is to find a p such that $f(p) \in X^* \subset E^*$ (which is the polar cone) and $\langle p, f(p) \rangle = 0$. In particular, if $C = \mathbb{R}_+^{n+1}$, then the condition that $f(p) \in X^*$ becomes $f(p) \leqslant 0$. In the following, we give an existence theorem on the complementarity problem which generalizes the results of Karamardian [6, Theorem 3.1] and Allen [1, Corollary 2].

THEOREM 8. *Let X be a cone in a Hausdorff topological vector space E and let $f: X \to E^*$ be a mapping such that the function $\phi: X \times X \to \mathbb{R} \cup \{\pm\infty\}$ defined by, for every $(p, q) \in X \times X$, $\phi(p, q) = \langle p - q, f(q) \rangle$, satisfies the following conditions:*

(i) *$\phi(p, q)$ is 0-transfer lower semi-continuous $q \in X$;*

(ii) *there exists a nonempty set $D \subset X$ such that for each $q \in X \setminus D$ there exists some $p \in D$ and some neighborhood $\mathcal{N}(q)$ of q with $\phi(p, z) > 0$ for all $z \in \mathcal{N}(q)$ and D is contained in a compact convex subset of X.*

Then there exists a price vector $q^* \in X$ such that $f(q^*) \in X^*$ and $\langle q^*, f(q^*) \rangle = 0$.

Proof of Theorem 8. Since ϕ is linear in p, it satisfies Condition (4ii) of Theorem 4. Conditions (4i) and (4iii) are clearly satisfied. Then by applying Theorem 4 to $\phi(p, q)$ with $\gamma = 0$, we have the existence of some $q^* \in X$ such that $\phi(p, q^*) \leqslant 0$ for all $p \in X$. That is, $\langle p - q^*, f(q^*) \rangle \leqslant 0$ for all $p \in X$. From Lemma 1 of Allen [1], we have $\langle q^*, f(q^*) \rangle = 0$. So $f(q^*) \in X^*$. ∎

REFERENCES

1. G. ALLEN, Variational inequalities, complementarity problems, and duality theorems, *J. Math. Anal. Appl.* **58** (1977), 1–10.
2. K. C. BORDER, "Fixed Point Theorems with Application to Economics and Game Theory," Cambridge Univ. Press, London/New York, 1985.
3. K. FAN, Minimax theorem, *Proc. Nat. Acad. Sci. U.S.A.* **39** (1953), 42–47.
4. K. FAN, Fixed-point and related theorems for non-compact sets, *in* "Game Theory and Related Topics" (O. Moeschlin and D. Pallaschke, Eds.), pp. 151–156, North-Holland, Amsterdam, 1979.
5. K. FAN, Some properties of convex sets related to fixed points theorems, *Math. Ann.* **266** (1984), 519–537.
6. S. KARAMARDIAN, Generalized complementarity problem, *J. Optim. Theory Appl.* **8** (1971), 416–427.
7. T. KIM AND M. RICHTER, Nontransitve-nontotal consumer theory, *J. Econ. Theory* **38** (1986), 324–363.
8. B. KNASTER, C. KURATOWSKI, AND S. MAZURKIEWICZ, Ein Beweis des Fixpunktsatze *n*-dimensionale Simpliexe, *Fund. Math.* **14** (1929), 132–137.

9. A. MAS-COLELL, An equilibrium existence theorem without complete or transitive preferences, *J. Math. Econom.* **1** (1974), 237–246.
10. D. GALE AND A. MAS-COLELL, An equilibrium existence theorem for a general model without ordered preferences, *J. Math. Econom.* **2** (1975), 9–15.
11. D. SCHMEIDLER, Competitive equilibrium in markets with a continuum of traders and incomplete preferences, *Econometrica* **37** (1969), 578–585.
12. W. SHAFER AND H. SONNENSCHEIN, Equilibrium in abstract economies without ordered preferences, *J. Math. Econom.* **2** (1975), 345–348.
13. H. SONNENSCHEIN, Demand theory without transitive preferences, with application to the theory of competitive equilibrium, *in* "Preferences, Utility, and Demand" (J. S. Chipman, L. Hurwicz, M. K. Richter, and H. Sonnenschein, Eds.), Harcourt Brace Jovanovich, New York, 1971.
14. G. TIAN, Minimax inequalities equivalent to the Fan–Knaster–Kuratowski–Mazurkiewicz theorems, mimeo, 1989.
15. G. TIAN, Generalizations of the Fan–Knaster–Kuratowski–Mazurkiewicz theorem and the Ky-Fan minimax inequality, mimeo, 1989.
16. G. TIAN, Equilibrium in abstract economies with a non-compact infinite dimensional strategy space, an infinite number of agents and without ordered preferences, *Econom. Lett.* **33** (1990), 203–206.
17. G. TIAN, Fixed points theorems for mappings with non-compact and non-convex domains, *J. Math. Anal. Appl.* **158** (1991), 161–167.
18. G. TIAN AND J. ZHOU, Transfer continuities, generalizations of the Weierstrass theorem and maximum theorems—A full characterization, Texas A&M University, mimeo, 1990.
19. M. WALKER, On the existence of maximal elements, *J. Econom. Theory* **16** (1977), 470–474.
20. N. C. YANNELIS AND N. D. PRABHAKAR, Equilibria in abstract economies with an infinite number of agents, an infinite number of commodities and without ordered preferences, *J. Math. Econom.* **12** (1983), 223–245.
21. J. ZHOU AND G. CHEN, Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities, *J. Math. Anal. Appl.* **132** (1988), 213–225.