Journal of Mathematical Economics 45 (2009) 113–123

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Implementation of Pareto efficient allocations

Guoqiang Tian[a,b,*]

[a] Department of Economics, Texas A&M University, College Station, TX 77843, USA
[b] School of Economics and Institute for Advanced Research, Shanghai University of Finance and Economics, Shanghai 200433, China.

ARTICLE INFO

Article history:
Received 10 October 2005
Received in revised form 17 July 2008
Accepted 22 July 2008
Available online 5 August 2008

JEL classification:
C72
D61
D71
D82

Keywords:
Incentive mechanism design
Implementation
Pareto efficiency
Price equilibrium with transfer

ABSTRACT

This paper considers Nash implementation and double implementation of Pareto efficient allocations for production economies. We allow production sets and preferences are unknown to the planner. We present a well-behaved mechanism that fully implements Pareto efficient allocations in Nash equilibrium. The mechanism then is modified to fully doubly implement Pareto efficient allocations in Nash and strong Nash equilibria. The mechanisms constructed in the paper have many nice properties such as feasibility and continuity. In addition, they use finite-dimensional message spaces. Furthermore, the mechanism works not only for three or more agents, but also for two-agent economies.

1. Introduction

1.1. Motivation

This paper considers implementation of Pareto efficient allocations for production economies by presenting well-behaved and simple mechanisms that are continuous, feasible, and use finite-dimensional spaces. Pareto optimality is a highly desirable property in designing incentive compatible mechanisms. The importance of this property is attributed to what may be regarded as minimal welfare property. Pareto optimality requires that resources be allocated efficiently. If an allocation is not efficient, there is a waste in allocating resources and therefore there is room for improvement so that at least one agent is better off without making the others worse off under given resources.

However, due to technical difficulties, study on implementation of Pareto efficient allocations in Nash equilibrium so far has been largely devoted to implementing a subset, but not the whole domain, of Pareto efficient allocations. The most commonly used equilibrium principles that result in Pareto optimal allocations are the Walrasian equilibrium, proportional equilibrium and Lindahl distributive equilibrium solutions for private goods economies, Lindahl equilibrium, ratio equilibrium and cost share equilibrium solutions for public goods economies. Many specific mechanisms have been provided in the literature that implement theseequilibrium principles such as those in Hurwicz (1979); Schmeidler (1980); Hurwicz et

* Corresponding author at: School of Economics and Institute for Advanced Research, Shanghai University of Finance and Economics, Shanghai 200433, China.

E-mail address: gtian@tamu.edu.

0304-4068/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.jmateco.2008.07.011

al. (1995); Postlewaite and Wettstein (1989); Tian (1989, 1994, 1996, 1999, 2003); Hong (1995); Peleg (1996a, b); Suh (1995, 1997); Yoshihara (1999); Duggan (2003) among others. The implementation literature rarely discusses implementation of the whole set of Pareto efficient allocations under private ownership or more generally under other ownership structures.

From a positive point of view, Pareto outcomes are often taken as a benchmark-outcomes that expect to arise even though information is incomplete or private. A fundamental question studied in the mechanism design literature is whether or not a social choice correspondence is (fully) implementable on a rich domain of economic environments, and in particular a social choice correspondence is the Pareto correspondence. Although many mechanisms, such as the Groves mechanism (that does not take care of individual rationality or equity) and the market type mechanisms, have been proposed to implement social/Pareto optimal allocations, an open question is if the social choice rule that is given by the Pareto efficient correspondence can be (fully) implemented by an incentive compatible mechanism.

From a normative perspective, an often considered price equilibrium allocation (such as Walrasian or Lindahl equilibrium allocation) that may result in Pareto efficiency is only concerned with efficiency of allocations and has nothing to say about distribution of welfare and equity of allocations. Even if we agree with Pareto optimality, we still do not know whether or not a Pareto allocation that satisfies a desirable property such as equity can be implemented by an incentive compatible mechanism. However, by the well-known Second Theorem of Welfare Economics, such an equitable allocation can be obtained by a price equilibrium allocation with transfers. Indeed, under some regularity conditions such as continuity, convexity, and monotonicity, the Second Theorem of Welfare Economics shows that every Pareto efficient allocation can be supported as a price equilibrium with transfers. As such, one may want to know if every Pareto efficient allocation is implementable. If so, one may further desire that all Pareto efficient allocations can be implemented by a single incentive mechanism. In other words, one may desire to have a mechanism that implements the full set of Pareto efficient allocations.

1.2. Related literature

Groves and Ledyard (1977) were the first to consider Nash implementation of Parato efficient allocations for public goods economies. However, their mechanism only (weakly) implements, but does not fully implement, the Pareto correspondence, that is, it only implements a subset of Pareto efficient allocations. Furthermore, the mechanism is not individually feasible. Osann (1997) discussed Nash implementation of weak Pareto efficient allocations for decomposable (externality-free) economic environments. Again, his mechanism only implements a subset of weak Pareto efficient allocations although the set is much bigger than the set of Lindahl allocations. Besides, the mechanism is not individually feasible and continuous.

Anderlini and Siconolei (2004) consider implementation of Pareto efficient and individually rational allocations for public goods economies. However, their mechanisms are not continuous on the boundary of the feasible set either, and further they consider public goods economies with only one private and one public good. It seems that their equilibrium notion and approach to implementation of Pareto efficient allocations may be hard to extend to a more general case of public goods economies with arbitrary numbers of private and public goods.

Thus, an important unanswered question is whether one can design an incentive compatible mechanism that implements the whole domain, but not just a subset, of Pareto efficient allocations. If so, the mechanism should be well-behaved in the sense that it is continuous, feasible, realistic, has a finite-dimensional dimension message space, and works for any number of agents, etc. Some of these properties have been studied systematically in Dutta et al. (1995); Saijo et al. (1996). We will answer this question by using the approach of the Second Theorem of Welfare Economics, which tells us, under some conditions, including essential condition of convexity of preferences and production sets, that any desired Pareto optimal allocation can be achieved as a market-based equilibrium with transfers.

1.3. Results of the paper

In this paper, we deal with the problem of designing an incentive mechanism that implements Pareto efficient allocations for general neoclassical production economies. We allow production sets and preferences to be unknown to the planner, and present a mechanism that fully implements Pareto efficient allocations in Nash equilibrium in the sense that the set of Nash equilibrium allocations of a mechanism coincides with the set of Pareto optimal allocations over the class of general convex production economies.

To do so, we first introduce the notion of constrained price equilibrium with transfers which is equivalent to the conventional price equilibrium with transfer when each agent's consumption bundle at equilibrium is non-zero. We then reduce the implementation problem of Pareto efficient allocations to the problem of implementing constrained price equilibrium allocations with transfers. This approach is based on the Second Theorem of Welfare Economics. It is a converse of the First Theorem of Welfare Economics. It is this general notion of price equilibrium with transfers that permits an arbitrary distribution of wealth among consumers.

The mechanism is then slightly modified to doubly implement Pareto efficient allocations in Nash and strong Nash equilibria. That is, by the mechanism, not only Nash equilibrium allocations, but also strong Nash equilibrium allocations coincide with Pareto optimal allocations. By double implementation, the solution can cover the situation where agents in coalitions may cooperate and in some other coalitions may not. Thus, the designer does not need to know which coalitions are permissible and, consequently, it allows the possibility for agents to manipulate coalition patterns.

The mechanisms presented in the paper are well-behaved and elementary mechanisms that have many desired properties such as feasibility and continuity. In addition, they are price-quantity market type mechanisms, and use finite-dimensional message spaces. Furthermore, our mechanisms work not only for three or more agents, but also for two-agent economies, and thus they are unified mechanisms that are irrespective of the number of agents.

It may be remarked that, because of transfer payments, the implementation of all Pareto efficient allocations is harder than those for the conventional solutions mentioned above and the mechanisms we will construct in the paper are quite different from the existing ones for Walrasian or Lindahl solutions. For instance, we need to give the managers of firms special incentives to maximize profits at equilibrium, but at the same time the profit cannot affect individuals' wealth. Thus, some of the techniques used in implementing other market-like-equilibrium solutions such as Walrasian equilibrium for private goods economies or Lindahl equilibrium for public goods economies may not be applicable, and some new techniques must be developed. Tian (in press) used similar techniques to consider the problem of incentive mechanism design in non-convex production economies when production sets and preferences both are unknown to the designer.

The remainder of this paper is as follows. Section 2 sets up a general model, introduces the notion of constrained price equilibrium with transfers, and gives the definition of implementation by a mechanism. Section 3 presents a well-behaved mechanism that fully implements constrained price equilibrium allocations with transfers, and consequently fully implements Pareto efficient allocations in Nash equilibrium. Section 4 modifies the mechanism so that the modified mechanism doubly implements Pareto efficient allocations in Nash and strong Nash equilibria. These mechanisms have the desired properties mentioned above. Concluding remarks are presented in Section 5.

2. The setup

2.1. Economic environments

We consider production economies with L commodities, $n \geqq 2$ consumers and J firms.[1] Let $N = \{1, 2, \ldots, n\}$ denote the set of consumers. Each agent's characteristic is denoted by $e_i = (C_i, R_i)$, where $C_i = \mathbb{R}^L_+$ is the consumption set, and R_i is the preference ordering defined on $\mathbb{R}^L_+$. Let P_i denote the asymmetric part of R_i (i.e., $a\,P_i\,b$ if and only if $a\,R_i\,b$, but not $b\,R_i\,a$). We assume that R_i is continuous, convex, and strictly monotonically increasing on $\mathbb{R}^L_+$.[2] Let $\hat{w}$ be the total initial endowment vector of commodities.

Production technologies of firms are denoted by $\mathcal{Y}_1, \ldots, \mathcal{Y}_j, \ldots, \mathcal{Y}_J$. We assume that, for $j = 1, \ldots, J$, $\mathcal{Y}_j$ is closed, convex, contains 0 (possibility of inaction), and $-\mathbb{R}^L_+ \subseteq \mathcal{Y}_j$ (free-disposal).

We assume that there are no externalities or public goods. An economy is the full vector $e = (e_1, \ldots, e_n, \hat{w}, \mathcal{Y}_1, \ldots, \mathcal{Y}_J)$ and the set of all such economies is denoted by E which is assumed to be endowed with the product topology.

2.2. Pareto efficiency and constrained pricing equilibrium with transfers

An allocation of the economy e is a vector $(x_1, \ldots, x_n, y_1, \ldots, y_J) \in \mathbb{R}^{L(n+J)}$ such that: (1) $x := (x_1, \ldots, x_n) \in \mathbb{R}^{nL}_+$, and (2) $y_j \in \mathcal{Y}_j$ for $j = 1, \ldots, J$. Denote by $y = (y_1, \ldots, y_J)$ the profile of production plans of firms.

An allocation (x, y) is *feasible* if

$$\sum_{i=1}^{n} x_i \leqq \sum_{i=1}^{n} \hat{w} + \sum_{j=1}^{J} y_j. \tag{1}$$

Denote the aggregate consumption and production by $\hat{x} = \sum_{i=1}^{n} x_i$ and $\hat{y} = \sum_{j=1}^{J} y_j$, respectively.[3] Then the feasibility condition can be written as

$$\hat{x} \leqq \hat{w} + \hat{y}.$$

An allocation (x, y) is *Pareto-optimal* with respect to the preference profile $R = (R_1, \ldots, R_n)$ if it is feasible and there is no other feasible allocation (x', y') such that $x'_i R_i x_i$ for all $i \in N$ and $x'_i P_i x_i$ for some $i \in N$. Denote by $P(e)$ the set of all such Pareto optimal allocations.

From Debreu (1952); Mas-Colell et al. (1995), one knows that Pareto efficient allocations can be characterized by price equilibrium with transfers under the conditions imposed in the paper.[4] This equivalence result is referred as the well-known Second Theorem of Welfare Economics. We will use this characterization result to study implementation of Pareto efficient allocations. The notion of price equilibrium with transfers is the generalization of Walrasian (competitive) equilibrium

---

1 As usual, vector inequalities, $\geqq$, $\geq$, and ¿, are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \ldots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \ldots, m$.

2 R_i is convex if, for bundles a and b, $a\,P_i\,b$ implies $\lambda a + (1 - \lambda)b\,P_i\,b$ for all $0 < \lambda \leqq 1$. Note that the term "convex" is defined as in Debreu (1959), not as in some recent textbooks.

3 For notational convenience, $\hat{a}$ will be used throughout the paper to denote the sum of vectors a_i, i.e., $\hat{a} := \sum a_l$.

4 A price equilibrium with transfers is also called an equilibrium relative to a price system in Debreu (1952).

allocations. While the Walrasian equilibrium concept allies to the case of private ownership economy, the more general notion of a price equilibrium with transfers allows instead for an arbitrary distribution of wealth among consumers.

A price equilibrium with transfers for an economy e is a list of consumption plans (x_i^*), a list of production plans (y_j^*), and a price vector p^* such that (a) every consumer maximizes his preferences subject to his budget constraint $\{x_i \in \mathbb{R}_+^L : p^* \cdot x_i \leqq p^* \cdot x_i^*\}$, (b) every firm maximizes its profit on $\mathcal{Y}_j$ for all $j = 1, \ldots, J$, and (c) the excess demand over supply is zero. Formerly, we have the following definition:

An allocation $z^* = (x^*, y^*) = (x_1^*, x_2^*, \ldots, x_n^*, y_1^*, y_2^*, \ldots, y_J^*) \in \mathbb{R}_+^{nL} \times \mathcal{Y}$ is a *price equilibrium allocation with transfers* for an economy e if there is a price vector $p^* \in \mathbb{R}_+^L$ such that

(1) for every $i \in N$, $x_i P_i x_i^*$ implies $p^* \cdot x_i > p^* \cdot x_i^*$, i.e., x_i^* is a greatest element for R_i in the budget set $\{x_i \in \mathbb{R}_+^L : p^* \cdot x_I \leqq p^* \cdot x_i^*\}$,
(2) for $j = 1, \ldots, J$, $p^* \cdot y_j^* \geqq p^* \cdot y_j$ for all $y_j \in \partial\mathcal{Y}_j$,
(3) $\sum_{i=1}^n x_i^* = \hat{w} + \sum_{j=1}^J y_j^*$.

Denote by PE(e) the set of all such price equilibrium allocations.

Debreu (1952) proved that, under the conditions imposed in the paper, an allocation (x^*, y^*) is Pareto efficient if and only if it is a price equilibrium with transfers (also see Propositions 16.C.1, 16.D.1 and 16.D.3. in Mas-Colell et al., 1995).[5] Thus, if we can construct a mechanism that implements price equilibrium allocations with transfers, the mechanism then implements Pareto efficient allocations with transfers. Therefore, the problem of implementing Pareto efficient allocations can bereduced to the problem of implementing price equilibrium allocations with transfers.

However, due to the difficulty of constructing a feasible mechanism that implements price equilibrium allocations with transfers, we would introduce the notion of constrained price equilibrium with transfers which resembles the one where the constrained Walrasian principle is used instead of the Walrasian principle when one considers implementation of competitive equilibrium allocations (cf. Hurwicz et al., 1995). As we show below, while the set of constrained Walrasian allocations in general is bigger than the set of Walrasian allocations, the set of constrained price equilibrium allocations with transfers coincides with price equilibrium allocations with transfers for non-zero consumption bundles. Thus, we can reduce the problem of implementing Pareto efficient allocations to the problem of implementing constrained price equilibrium allocations with transfers.

An allocation $z^* = (x^*, y^*) = (x_1^*, x_2^*, \ldots, x_n^*, y_1^*, y_2^*, \ldots, y_J^*) \in \mathbb{R}_+^{nL} \times \mathcal{Y}$ is a constrained price equilibrium allocation with transfers for an economy e if there is a price vector $p^* \in \mathbb{R}_+^L$ such that

(1) for every $i \in N$, $x_i P_i x_i^*$ and $x_i^* \leqq \hat{w} + \sum_{j=1}^J y_j^*$ implies that $p^* \cdot x_I > p^* \cdot x_i^*$,
(2) for $j = 1, \ldots, J$, $p^* \cdot y_j^* \geqq p^* \cdot y_j$ for all $y_j \in \partial\mathcal{Y}_j$,
(3) $\sum_{i=1}^n x_i^* = \hat{w} + \sum_{j=1}^J y_j^*$.

Denote by PE$_c(e)$ of the set of all such price equilibrium allocations.

Remark 1. A constrained price equilibrium allocation with transfers differs from a price equilibrium allocation with transfers, only in the way that each consumer maximizers his preferences not only subject to his budget constraint but also subject to the total supply. Thus, it is clear that every price equilibrium allocation with transfers is a constrained price equilibrium allocation with transfers. What about the converse? We know that a constrained Walrasian allocation may not be a Walrasian allocation on boundary points. Contrary to the conventional Walrasian equilibrium, the set of constrained price equilibrium allocations with transfers actually coincides with the set of price equilibrium allocations with transfers when each consumer's consumption bundle at equilibrium is non-zero.

To show this, we first show that, under local non-satiation of preferences, every constrained price equilibrium allocation with transfers is Pareto efficient, i.e., PE$_c(e) \subseteq P(e)$.

Lemma 1. *Suppose (p, x, y) is a constrained price equilibrium with transfers. If each consumer is locally non-satiated at x, then (x, y) is Pareto efficient.*

Proof. Suppose, by way of contradiction, that (x, y) is not Pareto efficient. Then there is another feasible allocation (x', y') such that $x' R_i x$ for all $i \in N$ and $x' P_i x$ for some $i \in N$. Since x is a constrained price equilibrium with transfers, $x_i' P_i x_i$ and $x_i' \leqq \hat{w} + \sum_{j=1}^J y_j$ imply that $p \cdot x_i' > p \cdot x_i$ for some i, and, by local non-satiation of preferences, $x' R_i x$ and $x_i' \leqq \hat{w} + \sum_{j=1}^J y_j$ imply that $p \cdot x_i' \geqq p \cdot x_i$ for all $i \in N$. Therefore, if (x', y') is Pareto superior to (x, y), we have $\sum_{i=1}^n p \cdot x_i' > \sum_{i=1}^n p \cdot x_i$ by local non-satiation. However, because y_j is a profit maximizing production plan for firm j at price p, we have $\sum_{j=1}^J p \cdot y_j \geqq \sum_{j=1}^J p \cdot y_j'$,

[5] The condition they used are actually weaker. For instance, only local non-satiation is required. Also a consumer's consumption space is not necessarily given by $\mathbb{R}_+^L$.

and thus $\sum_{i=1}^{n} p \cdot x_i' > \sum_{i=1}^{n} p \cdot x_i = \sum_{i=1}^{n} p \cdot \hat{w} + \sum_{j=1}^{J} p \cdot y_j \geqq \sum_{i=1}^{n} p \cdot \hat{w} + \sum_{j=1}^{J} p \cdot y_j'$. This contradicts the fact that (x', y') is a feasible allocation. Q.E.D. □

We now show the equivalence on the two notions of constrained price equilibrium with transfers and price equilibrium with transfers.

Lemma 2. *Suppose for each $i \in N$, preference orderings R_i are continuous on $\mathbb{R}_+^L$ and strictly increasing on $\mathbb{R}_{++}^L$, and production sets $\mathcal{Y}_j$ are non-empty, convex, and closed. Then, an allocation (x, y) with $x_i \neq 0$ is Pareto efficient if and only if it is a constrained price equilibrium with transfers.*

Proof. By Propositions 16.C.1, 16.D.1 and 16.D.2. in Mas-Colell et al. (1995), we know that the set of price equilibrium allocations with transfers coincides with the set of Pareto efficient allocations with non-zero consumptions for all individuals, i.e, $\mathrm{PE}(e) = P(e)$ with $x_i \neq 0$ under the assumptions imposed. Also, by Lemma 1, we have $\mathrm{PE}_c(e) \subseteq P(e)$. Thus, combining these facts, we have $P(e) = \mathrm{PE}(e) \subseteq \mathrm{PE}_c(e) \subseteq P(e)$, and thus $\mathrm{PE}_c(e) = P(e)$ for all $e \in E$. Q.E.D.

Thus, in this paper, we consider implementation of Pareto efficient allocations in which every consumer's consumption x_i is non-zero. The set of such Pareto efficient allocation is still denoted by $P(e)$. We will present a feasible and continuous mechanism which fully implements Pareto efficient allocations and has finite-dimensions of message spaces. To do so, we first give some basic concepts on economic mechanism. □

2.3. Mechanism

In this subsection, we give some basic concepts, notation and definitions used in the mechanism design literature. Let $F : E \to\to \mathbb{R}_+^{L(n+J)}$ be a social correspondence to be implemented. Let M_i denote the i-th agent's message domain. Its elements are written as m_i and are called messages. Let $M = \prod_{i=1}^{n} M_i$ denote the message space which is assumed to be endowed with the product topology. Denote by $h : M \to \mathbb{R}_+^{L(n+J)}$ the outcome function, or more explicitly, $h(m) = (X_1(m), \ldots, X_n(m), Y_1(m), \ldots, Y_J(m))$. Then a mechanism, which is defined on E, consists of a message space M and an outcome function. It is denoted by $\langle M, h\rangle$.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is said to be a *Nash equilibrium* of the mechanism $\langle M, h\rangle$ for an economy e if, for all $i \in N$ and $m_i \in M_i$,

$$X_i(m^*) R_i X_i(m_i, m_{-i}^*), \tag{2}$$

where $(m_i, m_{-i}^*) = (m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. The outcome $h(m^*)$ is then called a *Nash* (*equilibrium*) *allocation* of the mechanism for the economy e. Denote by $V_{M,h}(e)$ the set of all such Nash equilibria and by $N_{M,h}(e)$ the set of all such Nash equilibrium allocations.

A mechanism $\langle M, h\rangle$ is said to *Nash-implement* a social choice correspondence F on E, if, for all $e \in E$, $N_{M,h}(e) \subseteq F(e)$. It is said to *fully Nash-implement* a social choice correspondence F on E, if, for all $e \in E$, $N_{M,h}(e) = F(e)$.

A *coalition* C is a non-empty subset of N.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is said to be a strong Nash equilibrium of the mechanism $\langle M, h\rangle$ for an economy e if there does not exist any coalition C and $m_C \in \prod_{i \in C} M_i$ such that for all $i \in C$,

$$X_i(m_C, m_{-C}^*) P_i X_i(m^*). \tag{3}$$

The outcome $h(m^*)$ is then called a *Nash* (*equilibrium*) *allocation* of the mechanism for the economy e. Denote by $SV_{M,h}(e)$ the set of all such strong Nash equilibria and by $SN_{M,h}(e)$ the set of all such strong Nash equilibrium allocations.

A mechanism $\langle M, h\rangle$ is said to (fully) doubly *implement* a social choice correspondence in Nash and strong Nash equilibria F on E, if, for all $e \in E$, $SN_{M,h}(e) = N_{M,h}(e) \subseteq F(e)$ ($SN_{M,h}(e) = N_{M,h}(e) = F(e)$).

A mechanism $\langle M, h\rangle$ is said to be *feasible*, if, for all $m \in M$, (1) $X(m) \in \mathbb{R}_+^{nL}$, (2) $Y_j(m) \in \mathcal{Y}_j$ for $j = 1, \ldots, J$, and (3) $\sum_{i=1}^{n} X_i(m) \leqq \hat{w} + \sum_{j=1}^{J} Y_j(m)$.

A mechanism $\langle M, h\rangle$ is said to be *continuous*, if the outcome function h is continuous on M.

3. Implementation of pareto efficient allocations

3.1. The mechanism for pareto efficient principle

In the proposed mechanism below, the designer does not need to know firms' true production sets. However, it is well known by now that to have feasible implementation, the designer has to know some information about production sets. That is, we have to require that the mechanism designer can identify one of the consumers as a firm's manager, who is asked to announce a production plan from the production set for each firm. Although this requirement is a strong assumption, it is necessary to guarantee feasibility at all points, including disequilibrium points.[6] Thus, it is assumed that the manager of a firm knows the firm's production possibility set while others may or may not have this information. Although we can reasonably assume that one individual is at most a manager for one firm, to simplify the exposition, without loss of generality, consumer 1 is assumed to be the manager of all firms.

The message space of the mechanism is defined as follows. For each $i \in N$, let the message domain of agent i be of the form

$$M_i = \begin{cases} \Delta_{++}^{L-1} \times \tilde{\mathcal{Y}} \times \mathcal{Y} \times Z_i \times \mathbb{R}_{++} \times (0, 1] & \text{if } i = 1 \\ \Delta_{++}^{L-1} \times \mathbb{R}^{JL} \times Z_i \times \mathbb{R}_{++} \times (0, 1] & \text{if } i = 2 \\ Z_i \times \mathbb{R}_{++} \times (0, 1] & \text{if } i \geqq 3 \end{cases}$$

where

$$\Delta_{++}^{L-1} = \left\{ p \in \mathbb{R}_{++}^L : \sum_{l=1}^{L} p^l = 1 \right\}, \tag{4}$$

$$\tilde{\mathcal{Y}} = \left\{ y \in \mathcal{Y} : \hat{w} + \sum_{j=1}^{J} y_j \geq 0 \right\}, \tag{5}$$

and

$$Z_i = \{(z_{i1}, \ldots, z_{in}) \in \mathbb{R}_{+}^{nL} : z_{i,i+1} \neq 0\} \tag{6}$$

for $i = 1, \ldots, n$, where $n + 1$ is read to be 1.

Note that, unlike the existing mechanisms, we do not require all consumers, except for first two consumers, to announce prices and production profiles of production plans. Thus, the dimension of the message space is lower than that of some of the existing mechanisms that implement Walrasian allocations for convex production economies with more than two agents.

A generic element of M_i is $m_1 = (p_1, y_1, t_1, z_1, \gamma_1, \eta_1)$, $m_2 = (p_2, y_2, z_2, \gamma_2, \eta_2)$, and $m_i = (z_i, \gamma_i, \eta_i)$ for $i = 3, \ldots, n$, whose components have the following interpretations. The component p_i is the price vector proposed by agent i and is used as a price vector by the other agents $k \neq i$. The components $y_1 = (y_{11},\ldots,y_{1J})$ and $t_1 = (t_{11}, \ldots, t_{1J})$ are production profiles announced by agent 1 who works as the manager of firms, and will be used to induce the feasible production plans and profit maximizing rule, respectively. The component $y_2 = (y_{21}, \ldots, y_{2J})$ is a production profile announced by agent 2 who may be interpreted as the owner or a group of investors of firms, and will determine the profit maximizing production profile. While y_1 and t_1 are required to be production profiles in $\mathcal{Y}$,[7] the production profile y_2 is not necessarily in $\mathcal{Y}$ since agent 2 is not assumed to know production sets.

Note that in order for a mechanism to have a finite-dimensional message space, unlike the mechanism constructed in Hurwicz et al. (1995), which requires agents to report production sets that result in an infinite-dimensional message space, we only require agents to report production plans for firms, but not the production sets. The component $z_i = (z_{i1} \ldots, z_{in})$ is an allocation proposed by agent i, where z_{ik} is the consumption bundle of agent k proposed by agent i. The component γ_i is a shrinking index of agent i used to shrink the consumption of other agents in order to have a feasible outcome. The component η_i is the penalty index imposed to agent i if she proposes consumption bundles for the others which are different from those announced by agent $i + 1$.

Before we formally define the outcome function of the mechanism, we give a brief description and explain why the mechanism works. For each announced message $m \in M$, the price vector facing consumer 1 is determined by p_2 announced by agent 2, and the other's price vector is determined by p_1 announced by agent 1. Thus each individual takes prices as given and cannot change them by changing his own messages. The feasible production outcome $Y(m) \in \tilde{\mathcal{Y}}$ is then determined by the feasible production profile y_1 proposed by agent 1, who is the manger of the firms. Also to give the manager of firms an incentive to produce at a profit maximizing outcome at equilibrium, a special compensation τ_1 is provided to the manager of firm j in the way that she will receive a positive amount of compensation if she proposes a more profitable production plan t_1 than one announced by agent 2. Her preliminary consumption outcome is determined by the product of her consumption in $B_1(m)$ that is closest to the consumption bundle z_1 announced by the consumer 1 and the penalty discount factor $[1/(1 + \eta_1(\|z_{12} - x_2'(m)\| + \|z_{1,-1} - z_{2,-1}\| + \tau_1(m))) + \tau_1(m)]$ if she announced a different consumption profile $z_{1,-1}$ from $z_{2,-1}$ and a different production profile t_1 from y_2, where $z_{j,-i} = (z_{j,1}, \ldots, z_{j,i-1}, z_{j,i+1}, \ldots, z_{j,n})$ which denotes consumption bundles proposed by agent j for all agents except for agent i.

To give incentives for agent 2, regarded as the owner of firms, to match the price vector and production profile announced by the manager of firms and match the consumption bundles announced by agent 3, agent 2's preliminary consumption

[6] If one does not like this assumption and is willing to give up the requirement of feasibility of a mechanism like most mechanisms that implement Walrasian or Lindahl allocations in the economics literature, it is much easier to construct a mechanism that implements Pareto efficient allocations.

[7] Actually, we require y_1 be a feasible production profile so that non-negative aggregate consumptions for consumers.

outcome is determined by the product of the consumption in $B_2(m)$ that is closest to the consumption bundle z_2 announced by the consumer 2 and the penalty discount factor, $1/(1+\|p_1-p_1\|+\|y_1-y_2\|+\eta_2(\|z_{23}-x'_3(m)\|+\|z_{2,-2}-z_{3,-2}\|))$. For agent $i \geqq 3$, the preliminary consumption outcome $x_i(m)$ is determined by the product of her consumption in $B_i(m)$ that is closest to the consumption bundle z_i announced by the consumer 3 and the penalty discount factor $1/(1+\eta_i(\|z_{i,i+1}-x'_{i+1}(m)\|+\|z_{i,-i}-z_{i+1,-i}\|))$ if he announces a different consumption profile $z_{i,-i}$ from $z_{i+1,-i}$. To obtain the feasible outcome consumption $X(m)$, we need to shrink the preliminary outcome consumption $x_i(m)$ in the way specified below. We will show that the mechanism constructed in such a way have the properties we desire, and fully implements Pareto efficient allocations over the class of production economies under consideration.

Now we formally present the outcome function of the mechanism.

In order for each agent to take prices as given, define agent i's proposed price vector $p_i : M \to \Delta^{L-1}_{++}$ by

$$p_i(m) = \begin{cases} p_2 & \text{if } i=1 \\ p_1 & \text{otherwise} \end{cases}$$

Define the feasible production outcome function $Y : M \to \tilde{\mathcal{Y}}$ by

$$Y(m) = y_1. \tag{7}$$

To give the manager incentives to follow the profit maximizing rule at Nash equilibrium, some special compensation τ_1 will be provided to her according to the following formula. Define $\tau_1 : M \to \mathbb{R}_+$ by

$$\tau_1(m) = \sum_{j=1}^{J} \tau_{1j}(m) \tag{8}$$

where

$$\tau_{1j}(m) = \max\{0, p_1(m)\cdot(t_{1j}-y_{2j})\}. \tag{9}$$

The compensation formula τ_1 means that agent 1 will receive a positive amount of compensation $\tau_{1j}(m)$ if the manager can announce a more profitable production plan y_{1j} than the one proposed by agent 2, or she will receive a zero amount of compensation. In other words, $\tau_{1j}(m) > 0$ if and only if $p_1(m)\cdot t_{1j} > p_1(m)\cdot y_{2j}$.

Remark 2. The reason the rewarding system works is the following. In determining the profit maximizing production outcome $Y(m)$, if the production plan y_{2j}, announced by agent 2, does not maximize $p_1(m)\cdot y_j$ over production set $\mathcal{Y}_j$, then there is a 'test' production plan $y_j \in \mathcal{Y}_j$ such that $p_1(m)\cdot y_j > p_1(m)\cdot y_{2j}$, and thus agent 1 can gain from using this test production plan.

The feasible consumption set $B_i(m)$ is then defined by

$$B_i(m) = \begin{cases} \{x_1 \in \mathbb{R}^L_+ : p_2\cdot x_1 = p_2\cdot z_{n,1} \,\&\, x_1 \le \hat{w}+\hat{y}_1\} & \text{if } i=1 \\ \{x_i \in \mathbb{R}^L_+ : p_1\cdot x_i \leqq p_1\cdot z_{i-1,i} \,\&\, x_i \le \hat{w}+\hat{y}_1\} & \text{otherwise} \end{cases},$$

which are continuous correspondences with non-empty, compact, and convex values.

Define x'_i: $M \to B_i$ by

$$x'_i(m) = \{x_i : \min_{x_i \in B_i(m)} \|x_i - z_{ii}\|\}, \tag{10}$$

which is the closest to z_{ii}.

Define agent i's preliminary consumption outcome function x_i: $M \to \mathbb{R}^L_+$ by

$$x_i(m) = \begin{cases} \left[\dfrac{1}{1+\eta_1(\|z_{12}-x'_2(m)\|+\|z_{1,-1}-z_{2,-1}\|+\tau_1(m))} + \tau_1(m)\right]x'_1(m) & \text{if } i=1 \\ \dfrac{1}{1+\|p_1-p_2\|+\|y_1-y_2\|+\eta_2(\|z_{23}-x'_3(m)\|+\|z_{2,-2}-z_{3,-2}\|)}x'_2(m) & \text{if } i=2 \\ \dfrac{1}{1+\eta_i(\|z_{i,i+1}-x'_{i+1}(m)\|+\|z_{i,-i}-z_{i+1,-i}\|)}x'_i(m) & \text{if } i \geqq 3 \end{cases},$$

where $z_{i,-i} = (z_{i,1}, \ldots, z_{i,i-1}, z_{i,i+1}, z_{i,n})$.

Define the γ-correspondence A: $M \to 2^{\mathbb{R}_+}$ by

$$A(m) = \{\gamma \in \mathbb{R}_+ : \gamma\gamma_i \le 1 \forall i \in N \,\&\, \gamma \sum_{i=1}^{n} \gamma_i x_i(m) \leqq \hat{w} + \sum_{j=1}^{J} Y_j(m)\} \tag{11}$$

Let $\bar{\gamma}(m)$ be the largest element of $A(m)$, i.e., $\bar{\gamma}(m) \in A(m)$ and $\bar{\gamma}(m) \ge \gamma$ for all $\gamma \in A(m)$.

Finally, define agent i's outcome function for consumption goods X_i: $M \to \mathbb{R}^L_+$ by

$$X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m), \tag{12}$$

which is agent i's consumption resulting from the strategic configuration m.

Thus the outcome function $(X(m), Y(m))$ is continuous and feasible on M since, by the construction of the mechanism, $(X(m), Y(m)) \in \mathbb{R}^{nL}_+ \times \mathcal{Y}$, and

$$\hat{X}(m) \leqq \hat{w} + \hat{Y}(m) \tag{13}$$

for all $m \in M$.

Remark 3. The above mechanism works not only for three or more agents, but also for a two-agent world.

3.2. The result

The remainder of this section is devoted to proving the following theorem.

Theorem 1. *For the class of production economic environments E specified in* Section 2, *if the following assumptions are satisfied*:

(1) $\hat{w} > 0$;
(2) *For each $i \in N$, preference orderings, R_i, are continuous, and strictly increasing on $\mathbb{R}^L_+$*;
(3) *The production sets $\mathcal{Y}_j$ are non-empty, convex, closed, $0 \in \mathcal{Y}_j$, and $-\mathbb{R}^L_+ \in \mathcal{Y}_j$.*

then the mechanism defined in the above subsection, which is continuous, feasible, and uses a finite-dimensional message space, fully implements constrained price equilibrium allocations with transfers and consequently fully implements Pareto efficient allocations with non-zero consumption bundles in Nash equilibrium on E.

Proof. The proof of Theorem 1 consists of the following two propositions which show the equivalence between Nash equilibrium allocations and constrained price equilibrium allocations with transfers. Proposition 1 below proves that every Nash equilibrium allocation is a constrained price equilibrium allocation with transfers. Proposition 2 below proves that every constrained price equilibrium allocation with transfers is a Nash equilibrium allocation. Therefore, we show that the mechanism constructed in the previous section fully implements constrained price equilibrium allocations. Then, by Lemma 2, we show that the mechanism fully implements Pareto efficient allocations. To show these propositions, we first prove the following lemmas. □

Lemma 3. *Suppose $x_i(m)P_i x_i$. Then agent i can choose a very large γ_i such that $X_i(m)P_i x_i$.*

Proof. If agent i declares a large enough γ_i, then $\bar{\gamma}(m)$ becomes very small (since $\bar{\gamma}(m)\gamma_i \leqq 1$) and thus almost nullifies the effect of other agents in $\bar{\gamma}(m)\sum_{i=1}^{n}\gamma_i x_i(m) \leqq \hat{w} + \sum_{j=1}^{J} Y_j(m)$. Thus, $X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m)$ can arbitrarily approach as close to $x_i(m)$ as agent i wishes. From $x_i(m)P_i x_i$ and continuity of preferences, we have $X_i(m)P_i x_i$ if agent i chooses a very large γ_i. Q.E.D. □

Lemma 4. *If $m^* \in V_{M,h}(e)$, then $X_i(m^*) \neq 0$, $x_i(m^*) \neq 0$ and $x'_{ii}(m^*) \neq 0$ for $i \in N$.*

Proof. We argue by contradiction. Suppose $X_i(m^*) = 0$ for some $i \in N$. Since $\hat{w} + \sum_{j=1}^{J} Y_j(m^*) > 0$, $p_i(m^*) > 0$, and $z^*_{i-1,i} \neq 0$ by construction, we have $p_i(m^*) \cdot z^*_{i-1,i} > 0$. Thus there is some $x_i \in \mathbb{R}^L_{++}$ such that $p_i(m^*) \cdot x_i \leqq p_i(m^*) \cdot z^*_{i-1,i}$, $x_i \leqq \hat{w} + \sum_{j=1}^{J} Y_j(m^*)$, and $x_i P_i X_i(m^*)$ by monotonicity of preferences. Now suppose that agent i chooses $z_{ii} = x_i$, $\gamma_i > \gamma_i^*$, and keeps the other components of the message unchanged. Then, $z_{ii} \in B_i(m_i, m^*_{-i})$, and thus $x'_i(m_i, m^*_{-i}) = z_{ii}$. Since $x_i(m_i, m^*_{-i})$ is proportional to $x'_i(m_i, m^*_{-i})$, $x_i(m_i, m^*_{-i}) > 0$, we have $x_i(m_i, m^*_{-i})P_i X_i(m^*)$ by monotonicity of preferences. Therefore, by Lemma 3, agent i can choose a very large γ_i such that $X_i(m_i, m^*_{-i})P_i X_i(m^*)$. This contradicts $m^* \in V_{M,h}(e)$ and thus we must have $X_i(m^*) \neq 0$ for all $i \in N$. Since $X_i(m^*)$ is proportional to $x_i(m^*)$ and $x'_{ii}(m^*)$, $x_i(m^*) \neq 0$ and $x'_{ii}(m^*) \neq 0$ for $i \in N$.Q.E.D. □

Lemma 5. If *m^* is a Nash equilibrium, then $p_1^* = p_2^*$, $y_1^* = y_2^*$, $\tau_1(m^*) = 0$, $z^*_{i,i+1} = x_{i+1}(m^*)$, and $z^*_{i,-i} = z^*_{i+1,-i}$ for all $i \in N$. Consequently, $p(m^*) \equiv p_i(m^*) = p_1^* = p_2^*$* for *$i \in N$*, *$Y(m^*) = y_1^* = y_2^*$*, *$x_i(m^*) = x'_i(m^*)$ for all $i \in N$, $z_1^* = z_2^* = \ldots = z_n^* = x'(m^*) = x(m^*)$.*

Proof. We first show that $p_1^* = p_2^*$ and $y_1^* = y_2^*$. Suppose, by way of contradiction, that $p_1^* \neq p_2^*$, or $y_1^* \neq y_2^*$. Since $x'_i(m^*) \neq 0$ for all agent i by Lemma 4, agent 2 can choose $p_2 = p_1^*$, or $y_2 = y_1^*$ so that her consumption becomes larger and she would be better off by monotonicity of preferences. Hence, m^* is not a Nash equilibrium strategy if $p_1^* \neq p_2^*$, or $y_1^* \neq y_2^*$. Thus, we must have $p_1^* = p_2^*$ and $y_1^* = y_2^*$ at Nash equilibrium.

We now show that $\tau_1(m^*) = 0$, $z^*_{i,i+1} = x_{i+1}(m^*)$, and $z^*_{i,-i} = z^*_{i+1,-i}$ for all $i \in N$. Suppose not. Then agent i can choose a smaller $\eta_i < \eta_i^*$ in $(0, 1]$ so that her consumption becomes larger and she would be better off by monotonicity of preferences. Hence, no choice of η_i could constitute part of the Nash equilibrium strategy when $\tau_1(m^*) \neq 0$, $z^*_{i,i+1} = x_{i+1}(m^*)$, or $z^*_{i,-i} \neq z^*_{i+1,-i}$. Thus, we must have $\tau_1(m^*) = 0$, $z^*_{i,i+1} = x_{i+1}(m^*)$, and $z^*_{i,-i} = z^*_{i+1,-i}$ for all $i \in N$.

Consequently, by the construction of the mechanism, $p(m^*) \equiv p_i(m^*) = p_1^* = p_2^*$ for all $i \in N$, $Y(m^*) = y_1^* = y_2^*$, $x(m^*) = x'(m^*) = z_1^* = z_2^* = \ldots = z_n^*$. Q.E.D. □

Lemma 6. *If $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, then $\bar{\gamma}(m^*)\gamma_i^* = 1$ for all $i \in N$ and thus $X(m^*) = x(m^*) = x'(m^*)$.*

Proof. Suppose, by way of contradiction, that $\bar{\gamma}(m^*)\gamma_i^* < 1$ for some $i \in N$. Then $X_i(m^*) = \bar{\gamma}(m^*)\gamma_i^* x_i(m^*) < x_i(m^*)$, and thus $x_i(m^*)P_iX_i(m^*)$ by monotonicity of preferences. Therefore, by Lemma 3, agent i can choose a very large γ_i such that $X_i(m_i, m_{-i}^*)P_iX_i(m^*)$. This contradicts $m^* \in V_{M,h}(e)$. Thus we must have $\bar{\gamma}(m^*)\gamma_i^* = 1$, and therefore $X(m^*) = x(m^*) = x'(m^*)$. Q.E.D. □

Lemma 7. *If m^* is a Nash equilibrium, then $Y(m^*)$ maximizes profits on $\mathcal{Y}_j$ for $j = 1, \ldots, J$, i.e., $p(m^*) \cdot Y_j(m^*) \geqq p(m^*) \cdot y_j$ for all $y_j \in \mathcal{Y}_j$. Consequently, $\pi_j(m_i, m_{-i}^*) = \pi_j(m^*) = 0$ for all $m_i \in G_i(M)$.*

Proof. Suppose, to the contrary, that for some firm j, there exists a production plan $y_j \in \mathcal{Y}_j$ such that $p(m^*) \cdot Y_j(m^*) < p(m^*) \cdot y_j$. Then, if agent 1 chooses $t_{1j} = y_j$, $\eta_i = 1$, and $\gamma_1 > \gamma_1^*$, and keeps other components of the message unchanged, we have $\tau_1(m_i, m_{-1}^*) = p(m^*) \cdot y_j - p(m^*) \cdot Y_j(m^*) > 0$, and thus $x_1(m_1, m_{-1}^*) = [\tau_1(m_i, m_{-1}^*) + 1/(1 + \eta_1^*\tau_1(m_i, m_{-1}^*))]x_1(m^*) > x_1(m^*) = X_1(m^*)$ by noting $\tau_1(m^*) = 0$ and the function $f(\tau) = \tau_1 + 1/(1 + \eta_1^*\tau_1)$ is strictly increasing in τ_1. Then, $x_1(m_1, m_{-1}^*)P_1X_1(m^*)$ by monotonicity of preferences, and thus, by Lemma 3, agent 1 can choose a very large γ_1 such that $X_1(m_1, m_{-1}^*)P_1X_1(m^*)$, which contradicts the fact m^* is a Nash equilibrium. Hence $Y_j(m^*)$ must a profit maximizing production plan for firm j. Finally, since there is no production plan $y_j \in \mathcal{Y}_j$ such that $p(m^*) \cdot y_j > p(m^*) \cdot Y_j(m^*)$, we must have $\tau_{1j}(m_i, m_{-i}^*) = \tau_{1j}(m^*) = 0$ for all $m_I \in G_i(M)$. Q.E.D. □

Proposition 1. *If the mechanism defined above has a Nash equilibrium m^*, then the Nash equilibrium allocation $(X(m^*), Y(m^*), p(m^*))$ is a constrained price equilibrium with transfers, i.e., $N_{M,h}(e) \subseteq \mathrm{PE}(e)$.*

Proof. Let m^* be a Nash equilibrium. We need to prove that $(X(m^*), Y(m^*), p(m^*))$ is a constrained price equilibrium allocation with transfers. Note that the feasibility condition is also satisfied by the construction of the mechanism, all individuals take $p(m^*)$ as given by the construction of the mechanism, and $Y(m^*)$ maximizes the profits of firms over $\mathcal{Y}_j$ for all j by Lemma 7. So we only need to show that $X_i(m^*)$ is the greatest element for R_i in the budget set $\{x_i \in \mathbb{R}_+^L : p(m^*) \cdot x_i \leqq p(m^*) \cdot X_i(m^*)\}$ for all $i \in N$.

Suppose, by way of contradiction, that there is some $x_i \in \mathbb{R}_+^L$ and $x_i \leqq \hat{w} + \sum_{j=1}^J Y_j(m^*)$ such that $x_iP_iX_i(m^*)$ and $p(m^*) \cdot x_i \leqq p(m^*) \cdot X_i(m^*)$. Then, if agent i chooses $z_{ii} = x_i$, $\gamma_i > \gamma_i^*$, and keeps other components of the message unchanged. Then, $z_{ii} \in B_i(m_i, m_{-i}^*)$, and thus $x_i'(m_i, m_{-i}^*) = z_{ii}P_iX_i(m^*)$. Therefore, by Lemma 3, agent i can choose a very large γ_i such that $X_i(m_i, m_{-i}^*)P_iX_i(m^*)$. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Thus, $(X(m^*), Y(m^*))$ is a constrained price equilibrium allocation with transfers. □

Proposition 2. *If (x^*, y^*, p^*) is a constrained price equilibrium with transfers and $x_i^* \neq 0$ for all $i \in N$, then there is a Nash equilibrium m^* of the above mechanism such that $Y(m^*) = y^*$, $p(m^*) = p^*$, and $X_i(m^*) = x_i^*$ for all $i \in N$, i.e., $\mathrm{PE}(e) \subseteq N_{M,h}(e)$.*

Proof. We first note that by the strict monotonicity of preference orderings, the normalized price vector p^* must be in Δ_{++}^{L-1}. We need to show that there is a message m^* such that (x^*, y^*) is a Nash equilibrium allocation. For each $i \in N$, define m_i^* by $p_i^* = p^*$, $y_i^* = t_1^* = y^*$ (for $i = 1, 2$), $z_i^* = x^*$, $\gamma_i^* = 1$, and $\eta_i^* = 1$. Then, it can be easily verified that $Y(m^*) = y_1^* = y_2^*$, $p_i(m^*) = p^*$, and $X_i(m^*) = x_i^*$ for all $i \in N$. Then, $p_i(m_i, m_{-i}^*) = p_i(m^*)$, $p(m^*) \cdot Y_j(m^*) \geqq p(m^*) \cdot y_j$ for all $y_j \in \mathcal{Y}_j$, and thus $\tau_1(m_i, m_{-i}^*) = \tau_1(m^*) = 0$ for all $m_i \in M_i$. Hence

$$p(m^*) \cdot X_i(m_i, m_{-i}^*) \leqq p(m^*) \cdot X_i(m^*) \tag{14}$$

for all $m_i \in M_i$. Thus, $X_i(m_i, m_{-i}^*)$ satisfies the budget constraint for all $m_i \in M_i$. Thus, we must have $X_i(m^*)R_iX_i(m_i, m_{-i}^*)$, or else it contradicts the fact that $(X(m^*), Y(m^*))$ is a constrained price equilibrium allocation with transfers. So $(X(m^*), Y(m^*))$ must be a Nash equilibrium allocation. Q.E.D.

Thus, by Propositions 1 and 2, we know that $N_{M,h}(e) = \mathrm{PE}(e)$, which means the mechanism fully implements constrained price equilibrium allocations with transfers in Nash equilibrium, and thus by Lemma 2, the mechanism fully implements Pareto efficient allocations in Nash equilibrium. The proof of Theorem 1 is completed. Q.E.D. □

4. Double implementation of pareto efficient allocations

The above mechanism only fully implements Pareto efficient allocations in Nash equilibrium, but does not fully doubly implement Pareto efficient allocations in Nash and strong Nash equilibria. When agents 1 and 2 form a coalition, they may be better off. However, if one is willing to increase the dimensions of the message by asking every consumer to report a price vector and production profile, the above mechanism can be slightly modified so the modified mechanism fully doubly implementation Pareto efficient allocations. In this section, we briefly discuss such a modified mechanism.

The message domain of agent i is given by

$$M_i = \begin{cases} \Delta_{++}^{L-1} \times \tilde{\mathcal{Y}} \times \mathcal{Y} \times Z_i \times \mathbb{R}_{++} \times (0, 1\,] & \text{if } i = 1 \\ \Delta_{++}^{L-1} \times \mathbb{R}^{JL} \times Z_i \times \mathbb{R}_{++} \times (0, 1\,] & \text{otherwise} \end{cases}$$

where $\tilde{\mathcal{Y}}$ and Z_i are defined by (6). A generic element of M_i is $m_i = (p_i, y_i, t_i, z_i, \gamma_i, \eta_i)$ for $i = 1$ and $m_i = (p_i, y_i, z_i, \gamma_i, \eta_i)$ for $i = 2, \ldots, n$. Note that, while agent 1, the manager of firms, is required to report production profiles $y_1 = (y_{11}, \ldots, y_{1J})$ and $t_1 = (t_{11}, \ldots, t_{1J})$ in the production sets, the others are not required to announce production profiles in the production sets.

Agent i's price vector $p_i : M \to \Delta_+^{L-1}$ is defined by

$$p_i(m) = p_{i+1} \tag{15}$$

and the feasible production outcome function $Y : M \to \tilde{\mathcal{Y}}$ by

$$Y(m) = y_1. \tag{16}$$

Agents 1's compensation function τ_1 for inducing firms to produce at the profit maximizing production plans is specified as the same as before. The feasible consumption set $B_i(m)$ and x_i' for agent i are also defined as the same as before.

Agent i's preliminary consumption outcome function x_i: $M \to \mathbb{R}_+^L$ is modified to

$$x_i(m) = \begin{cases} [\dfrac{1}{1 + \eta_1(\|z_{12} - x_2'(m)\| + \|z_{1,-1} - z_{2,-1}\| + \tau(m))} + \tau_1(m)]x_1'(m) & \text{if } i = 1 \\ \dfrac{1}{1 + \|p_i - p_{i+1}\| + \|y_i - y_{i+1}\| + \eta_i(\|z_{i,i+1} - x_{i+1}'(m)\| + \|z_{i,-i} - z_{i+1,-i}\|)} x_i'(m) & \text{otherwise} \end{cases}$$

Finally, agent i's outcome function for consumption goods X_i: $M \to \mathbb{R}_+^L$ is given by

$$X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m), \tag{17}$$

which is the same as before. Then, the outcome function $(X(m), Y(m))$ is continuous and feasible on M, and the mechanism clearly satisfies the best response property.

Similarly, we have the following theorem.

Theorem 2. *For the class of production economic environments E specified in* Section 2, *if the following assumptions are satisfied*:

(1) $\hat{w} > 0$;
(2) *For each* $i \in N$, *preference orderings*, R_i, *are continuous on* $\mathbb{R}_+^L$, *and strictly increasing on* $\mathbb{R}_+^L$;
(3) *The production sets* $\mathcal{Y}_j$ *are non-empty, convex, closed*, $0 \in \mathcal{Y}_j$, *and* $-\mathbb{R}_+^L \in \mathcal{Y}_j$.

then the mechanism defined in the above subsection, which is continuous, feasible, and uses a finite-dimensional message space, fully doubly implements constrained price equilibrium allocations with transfers and consequently fully doubly implements Pareto efficient allocations with non-zero consumption bundles in Nash and strong Nash equilibria on E.

Proof. The proof of the equivalence between Nash equilibrium allocations and constrained price equilibrium allocations with transfers is the same as the one of Theorem 1. We only need to prove the following proposition that shows that every Nash equilibrium is a strong Nash equilibrium. As a result, we show that the mechanism constructed in the previous section doubly implements constrained price equilibrium allocations with transfers. Consequently, by Lemma 2, the mechanism constructed in the previous section doubly implements Pareto efficient allocations. □

Proposition 3. *Every Nash equilibrium* m^* *of the mechanism defined above is a strong Nash equilibrium, that is*, $N_{M,h}(e) \subseteq SN_{M,h}(e)$.

Proof. Let m^* be a Nash equilibrium. Since the mechanism fully implements constrained price equilibrium allocations with transfers, we know that $(X(m^*), Y(m^*))$ is a constrained price equilibrium allocation with transfers. Then $(X(m^*), Y(m^*))$ is Pareto optimal by Lemma 2, and thus the coalition N cannot be improved upon by any $m \in M$. Now for any coalition C with $\emptyset \neq C \neq N$, choose $i \in C$ such that $i + 1 \notin C$. Then no strategy played by C can change $p(m)$ and $Y(m)$ since they are determined by m_{i+1}. Furthermore, because $(X(m^*), Y(m^*)) \in P(e)$ and

$$p(m^*) \cdot X_i(m_C, m_{-C}^*) \leqq p(m^*) \cdot X_i(m^*), \tag{18}$$

$X_i(m^*)$ is the maximal consumption in the budget set of i, and thus S cannot improve upon $(X(m^*), Y(m^*))$. Q.E.D.

Thus, we have showed that $N_{M,h}(e) = SN_{M,h}(e) = \mathrm{PE}(e)$ for all $e \in E$. The proof of Theorem 2 is completed. Q.E.D. □

5. Concluding remarks

In this paper we considered implementation and double implementation of Pareto optimal allocations for general production economies. We presented a specific mechanism that uses a finite-dimension of message space and fully Nash implements Pareto optimal allocations when preferences and productions sets are unknown to the designer. The mechanism then is slightly modified to doubly implement Pareto correspondence in Nash and strong Nash equilibria. That is, by the mechanism, not only Nash equilibrium allocations, but also strong Nash equilibrium allocations coincide with constrained pricing equilibrium allocations with transfers. By double implementation, the solution can cover the situation where agents in some coalitions may cooperate and in some other coalitions may not. Thus, the designer does not need to know which coalitions are permissible and, consequently, it allows for the possibility of agents manipulating coalition patterns.

The mechanisms constructed in the paper are well-behaved and have several desired properties: (1) they use finite-dimensional message spaces. (2) They are stable in the sense that they are continuous. A slight change of strategies does not result in a drastic change of the outcome. (3) They are credible in the sense that they are feasible. Every participant receive a consumption bundle in her consumption set. Every production plan is in the production set, and aggregate consumption do not excess aggregate supply even at non-equilibrium. A mechanism would not be credible if it was not feasible. (4) They are market type mechanisms. The price and quantity are components of the message spaces. (5) They are more realistic and relatively more informationally efficient. The mechanism that implements Pareto efficient allocations in Nash equilibrium only requires two agents, one as the manager and the other as the owners of firms, to announce price vectors and production plans of firms. Thus, this mechanism not only uses smaller message spaces, but also they are more realistic since the others that are not involved in or related to production are not required to announce production plans. (6) They work not only for three or more agents, but also for two-agent economies. Thus they are unified mechanisms that are irrespective of the number of agents.

Although this paper only considers implementation of Pareto efficient allocations for private goods economies, the techniques developed in the paper can be used and the mechanisms can be modified to consider implementation of Pareto efficient allocations for public goods economies.

Acknowledgements

I wish to thank an anonymous referee for helpful comments and suggestions. Financial support from the National Natural Science Foundation of China (NSFC-70773073) and the Private Enterprise Research Center at Texas A&M University as well as from Cheung Kong Scholars Program at the Ministry of Education of China is gratefully acknowledged.

References

Anderlini, L., Siconolei, P., 2004. Efficient provision of public goods with endogenous redistribution. Review of Economic Design 8, 413–447.
Debreu, G., 1952. A social equilibrium existence theorem. In: Proceedings of the National Academy of Sciences of the USA, 38, pp. 386–393.
Duggan, J., 2003. Nash Implementation with a Private Good. Economic Theory 21, 117–131.
Dutta, B., Sen, A., Vohra, R., 1995. Nash implementation through elementary mechanisms in economic environments. Economic Design 1, 173–204.
Groves, T., Ledyard, J., 1977. Optimal allocation of public goods: a solution to the free rider problem. Econometrica 45 (4), 783–811.
Hong, L., 1995. Nash implementation in production economy. Economic Theory 5, 401–417.
Hurwicz, L., 1979. Outcome function yielding Walrasian and Lindahl allocations at Nash equilibrium point. Review of Economic Studies 46, 217–225.
Hurwicz, L., Maskin, E., Postlewaite, A., 1995. Feasible Nash implementation of social choice rules when the designer does not know endowments or production sets. In: Ledyard, J.O. (Ed.), The Economics of Informational Decentralization: Complexity, Efficiency, and Stability (Essays in Honor of Stanley Reiter). Kluwer Academic Publishers.
Mas-Colell, A., Whinston, M., Green, J., 1995. Microeconomic Theory. Oxford University Press.
Osann, H., 1997/1998. Nash-implementation of the weak Pareto choice rule for indecompposable environments. Review of Economic Design 3, 57–74.
Peleg, B., 1996a. A continuous double implementation of the constrained Walrasian equilibrium. Economic Design 2, 89–97.
Peleg, B., 1996b. Double implementation of the Lindahl equilibrium by a continuous mechanism. Economic Design 2, 311–324.
Postlewaite, A., Wettstein, D., 1989. Continuous and feasible implementation. Review of Economic Studies 56, 603–611.
Schmeidler, D., 1980. Walrasian analysis via strategic outcome functions. Econometrica 48, 1585–1593.
Saijo, T., Tatamitani, Y., Yamato, T., 1996. Toward natural implementation. International Economic Review 37, 949–980.
Suh, S., 1995. A mechanism implementing the proportional solution. Economic Design 1, 301–317.
Suh, S., 1997. Double implementation in Nash and strong Nash equilibria. Social Choice and Welfare 14, 439–447.
Tian, G., 1989. Implementation of the Lindahl correspondence by a single-valued, feasible, and continuous mechanism. Review of Economic Studies 56, 613–621.
Tian, G., 1994. Implementation of linear cost share equilibrium allocations. Journal of Economic Theory 64, 568–584.
Tian, G., 1996. Continuous and feasible implementation of rational expectation Lindahl allocations. Games and Economic Behavior 16, 135–151.
Tian, G., 1999. Double implementation in economies with production technologies unknown to the designer. Economic Theory 13, 689–707.
Tian, G., 2003. A solution to the problem of consumption externalities. Journal of Mathematical Economics 39, 831–847.
Tian, G., Implementation in economies with non-convex production technologies unknown to the designer, Games and Economic Behavior, in press.
Yoshihara, N., 1999. Natural and double implementation of public ownership solutions in differentiable production economies. Review of Economic Design 4, 127–151.