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A solution to the problem of consumption externalities

Guoqiang Tian [a,b,*]

[a] *Department of Economics, Texas A&M University, College Station, TX 77843, USA*
[b] *Department of Economics, School of Economics and Management, Tsinghua University, Beijing 100084, China*

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Abstract

This paper considers the problem of incentive mechanism design that results in efficient allocations for economies with consumption externalities when preferences, individual endowments, and coalition patterns among individuals are unknown to the planner. We do so by introducing the notion of constrained distributive Lindahl equilibrium. We give a mechanism that implements constrained distributive Lindahl allocations in Nash and strong Nash equilibria. Since the Pigouvian mechanism is a special case of the distributive Lindahl mechanism, the mechanism also implements the Pigouvian allocations. The mechanism is feasible and continuous. It works not only for three or more agents, but also for two-agent economies.

JEL classification: C72; D61; D71; D82

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1. Introduction

This paper studies the incentive mechanism design that selects Pareto efficient allocations for economies with consumption externalities in which agents have preferences defined on allocations rather on individual commodity bundles. Allocative efficiency and incentive compatibility of an economic system are two highly desired properties for an economic mechanism to have. Pareto optimality requires resources be allocated efficiently, and it may be regarded as a minimal welfare criterion. The incentive compatibility of an economic mechanism requires the consistency of individual interests and a social goal under the rules

* Tel.: +1-979-845-7393; fax: +1-979-847-8757.
E-mail address: gtian@tamu.edu (G. Tian).

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of the mechanism. Since agents have private information, they may find it advantageous to distort the information they reveal, and thus, they may use such information strategically to advance their own interests. As a result, centralized decision making may be impossible or at least inappropriate and so decentralized decision making is highly preferable. Thus, one needs an economic mechanism (an institution) to provide individuals with appropriate incentives so that individuals' interests are consistent with the goals of an organization. An organization or economic system which has this consistency property is called incentive compatibility.

The incentive compatibility requirements depend upon two basic components: classes of economic environments over which a mechanism is supposed to operate and particular outcomes that a mechanism is required to implement. A mechanism can be viewed as an abstract planning procedure; it consists of a message space in which communication takes place and an outcome function which translates messages into outcomes such as allocations of resources. A mechanism implements a social choice goal if the outcomes given by the outcome function agree with the social choice goal at an equilibrium solution concept which describes individual self-interested behavior. The implementation theory, which regards mechanisms as unknown, deals with precisely this problem by designing a game form such that a prespecified welfare criterion is guaranteed to be achieved by the game across a large domain of possible environments when individuals pursue their personal interests. The implementation problem is fundamental to economics and its related social science disciplines.

For the general equilibrium approach to the efficiency of resource allocation for economic environments with private goods, the most commonly used general equilibrium notion is the Walrasian equilibrium principle. Since the Walrasian mechanism, in general, is not incentive-compatible even for classical economic environments when the number of agents is finite, many incentive-compatible mechanisms have been proposed to implement Walrasian allocations at Nash equilibrium and/or strong Nash equilibrium points such as those in Hurwicz (1979), Schmeidler (1980), Hurwicz et al. (1995), Postlewaite and Wettstein (1989), Tian (1992, 1999, 2000a), Hong (1995), and Peleg (1996a,b) among others.

The Walrasian equilibrium principle, however, has a limited scope. When a consumer's level of preference depends in part on the consumption of others, not every Walrasian equilibrium is necessarily Pareto optimal, and thus one needs to adopt other types of resource allocation principles. The problem of designing incentive-compatible mechanisms over classes of economies displaying externalities has been investigated in the literature mainly in the spacial case wherein the externalities arise out of the provision of public goods such as those in Hurwicz (1979), Hurwicz et al. (1995), Walker (1977), Tian (1989, 1990, 2000b), and Li et al. (1995).

The only exception was Varian (1994) who gave a so-called compensation mechanism whose subgame-perfect equilibria implement efficient allocations in economic environments with certain types of externalities. While Varian's mechanism is simple, it has some limitations. First, it only considers the consumption externalities in the presence of transferable goods. That is, the economies considered by Varian (1994) contain both of goods with externalities and without externalities. Second, he used subgame-perfect equilibrium as a solution concept to describe individuals' self-interested behavior. Subgame-perfect strategy behavior is a weak individual behavior assumption for implementability of a social choice goal. Moore and Repullo (1988) showed that almost all social choice correspondences are implementable in subgame perfect equilibrium so that it makes implementation of a social choice rule much easier. Third, Varian's compensation mechanism requires preferences be continuous. Fourth, the mechanism is not individually feasible: out of equilibria, some outcome allocations may not be in the consumption set; although in equilibrium, they are necessarily in the consumption set.

For social choice functions defined on general economic environments, Boylan (1998) provided a necessary and sufficient condition for a social choice function to be implementable in coalition-proof-Nash equilibrium (CPNE) in environments with complete information. His characterization result on CPNE-implementability, however, requires that individuals' utility functions be bounded with respect to all variables except for money, and that they be either quasi-linear or analytic. Furthermore, due to the general nature of the social choice rules under consideration, the implementing mechanisms turn out to be quite complex. Characterization results show what is possible for the implementation of a social choice rule, but not what is realistic. Thus, like most characterization results in the literature, Boylan's mechanism is not natural in the sense that it is not continuous; small variations in an agent's strategy choice may lead to large jumps in the resulting allocations. Further, it has a message space of infinite dimension.

The purpose of this paper is twofold. First, we introduce the notion of constrained distributive Lindahl equilibrium which yields Pareto efficient allocations for economies with consumption externalities that are characterized by non-malevolent preferences. This equilibrium concept is a slight generalization of distributive Lindahl equilibrium introduced by Bergstrom (1970) who demonstrated that the Lindahlian approach to the analysis of public goods can be also used to analyze a model of wide-spread externalities in which agents have preferences defined on allocations rather on individual commodity bundles. The distributive Lindahl principle accommodates the existence of altruism and income redistribution without resorting to arbitrary equity assessments. Second, we consider double implementation of constrained distributive Lindahl correspondence.[1] That is, we will give a specific mechanism in which not only Nash allocations, but also strong Nash allocations coincide with constrained distributive Lindahl allocations. By double implementation, the solution can cover the situation where agents in some coalitions will cooperate and in some other coalitions will not. Thus, the designer does not need to know which coalitions are permissible and, consequently, it allows the possibility for agents to manipulate coalition patterns.

Note that the mechanisms proposed in the paper also have some nice properties in that they use feasible and continuous outcome functions, and have message spaces of finite-dimension. Furthermore, our mechanism works not only for three or more agents,

[1] It is important to distinguish the constrained distributive Lindahl mechanism from the constrained distributive Lindahl correspondence. The former is an economic mechanism while the latter is a performance correspondence which consists of allocations which can be supported (obtained) by the constrained distributive Lindahl mechanism and, in fact, can be implemented by non constrained distributive Lindahl mechanisms such as the mechanism that will be given in the paper. Just like the (constrained) Walrasian mechanism, the constrained distributive Lindahl mechanism requires that individuals choose consumption bundles taking into account aggregate feasibility constrains, and further it is not compatible with price-taking behavior with a setting with a finite number of individuals so that it may fail to yield Pareto-efficient allocations in the case of a small number of agents. This is one of the main reasons why we want to give some alternative mechanism that implements constrained distributive Lindahl allocations.

but also for two-agent economies, and thus it is a unified mechanism which is irrespective of the number of agents. In addition, the mechanism is a market-type mechanism in the sense that it contains prices and consumption quantities as components of messages.

The double implementation in Nash and strong Nash equilibria was first studied by Maskin (1979a,b). Maskin showed that any social choice correspondence that selects Pareto-efficient and individual allocations are doubly implementable in Nash and strong Nash equilibria. Suh (1997) further provided a necessary and sufficient condition for a social choice correspondence to be doubly implementable in Nash and strong Nash equilibrium for a class of economic environments. Many specific mechanisms with a certain desired properties have been proposed to implement some well-known social choice correspondences such as Walrasian allocation, Lindahl allocations, proportional allocations in the literature such as those in Suh (1995), Peleg (1996a,b), Tian (1999, 2000a,b,c), and Yoshihara (1999). Yamato (1993) also considered double implementation in Nash and un-dominated Nash equilibria.

The remainder of this paper is as follows. Section 2 sets forth the framework of the analysis and gives the notion of constrained distributive Lindahl equilibrium, and notation and definitions used for mechanism design. Section 3 gives a mechanism which has the desirable properties mentioned above when preferences, individual endowments, and coalition patterns are unknown to the designer. In Section 4, we prove that this mechanism doubly implements constrained distributive Lindahl allocations. Concluding remarks are presented in Section 5.

2. Framework and generalized distributive Lindahl equilibrium

2.1. Economic environments with preference externalities

Consider pure exchange economies with L private goods and $n \geqq 2$ consumers who are characterized by their preferences and endowments.[2] Denoted by $N = \{1, \dots, n\}$ the set of consumers. Throughout this paper, subscripts are used to index consumers and superscripts are used to index goods unless otherwise stated. An allocation is a vector $(x_1, x_2, \dots, x_n) \in \mathbb{R}_+^{nL}$ where x_i represents the bundle of goods allocated to consumer i. For the ith consumer, his characteristic is denoted by $e_i = (\mathring{w}_i, R_i)$, where $\mathring{w}_i \in \mathbb{R}_{++}^L$ is his initial endowments of the goods, and, as we are considering economies with preference externalities, R_i is a preference ordering defined on $\mathbb{R}_+^{nL}$. Let P_i be the strict preference (asymmetric part) of R_i. We assume that R_i are convex[3] and strictly monotonically increasing in x_i. We further assume that R_i are non-malevolent. The non-malevolence is needed to ensure the existence of distributive Lindahl equilibrium. Before formalizing the notion of non-malevolence which was introduced by Bergstrom (1970), we give the concept of separability of preferences.

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2 As usual, vector inequalities are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \dots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \dots, m$.

3 R_i is convex if for bundles a, b, c with $0 < \lambda \leqq 1$ and $c = \lambda a + (1 - \lambda)b$, the relation $a \, P_i \, b$ implies $c \, P_i \, b$.

The preference relations R_i of consumer i are said to be *separable between consumers* if for all $j \in N$ and all x and y in $\mathbb{R}_+^{nL}$ such that $x_k = y_k$ for all $k \neq j$, xR_iy implies $x'R_iy'$ for any x' and y' such that $x'_j = x_j$, $y'_j = y_j$, and $x'_k = y'_k$ for all $k \neq j$.

In words, preferences are separable between individuals if each consumer's preference between any two allocations which contain the same commodity bundles for everyone except some consumer j is unaffected by what consumers other than j consume, so long as in each of the two allocations compared the amount consumed by the others is the same. It rules out such effects as the desire to imitate the consumption of others or a desire for a commodity solely because of its scarcity but does allow persons to be concerned about the consumption of others. This is the notion of separability familiar in consumption theory. When R_i are represented by a continuous utility function, then they are separable between consumers if and only if they can be represented by a utility function of the form $u_i(g_1(x_1), \dots, g_n(x_n))$, where $g_j(\cdot)$ is a continuous real-valued function and x_j is the consumption bundle received by consumer j.

When preferences are separable between individuals, one can define the notion of a private preference ordering $\succcurlyeq_i$ for consumer i on his individual consumption set $\mathbb{R}_+^L$ as follows: for x_i and $y_i \in \mathbb{R}_+^L$, we say that $x_i \succcurlyeq_i y_i$ if and only if uR_iv whenever $u_i = x_i$, $v_i = y_i$, and $u_j = v_j$ for all $j \in N$ with $j \neq i$. It is clear that $\succcurlyeq_i$ is a complete preordering on $\mathbb{R}_+^L$ if preferences of i are separable between individuals.

We are now in the position to define the concept of non-malevolence. Consumer i is said to be *non-malevolently* (*benevolently*) related to consumer j if preferences of i and j are separable between individuals and for any x and y in $\mathbb{R}_+^{nL}$ such that $x_k = y_k$ for $k \neq j$ and $x_j \succcurlyeq_j y_j$ implies that xR_iy.

In words, consumer i is non-malevolently related to consumer j if for any two allocations x and y which contain the same bundles for everyone except j and such that j privately prefers his bundle in x to his bundle in y, consumer i respects j's private preference to the extent that he prefers x to y. Non-malevolence rules out the possibility that i disagrees with j about what kinds of goods j should consume. When each consumer is non-malevolently related to every other consumer, we say that the economy is characterized by non-malevolent preference externalities.

When R_i are represented by a continuous utility function, then consumer i is non-malevolently related to all consumers if and only if R_i can be represented by a utility function of the form $u_i(f_1(x_1), \dots, f_n(x_n))$, where $f_j(\cdot)$ is a continuous real-valued function which represents the private preferences of consumer j.

A pure exchange economy with non-malevolent preference externalities (in short, a non-malevolent pure exchange economy) is the full vector $e = (e_1, \dots, e_n)$ and the set of all such pure exchange economies with non-malevolent preference externalities is denoted by E.

An allocation $x \in \mathbb{R}^{nL}$ is said to be *feasible* if $x \in \mathbb{R}_+^{nL}$ and $\sum_{i=1}^n x_i \leqq \sum_{i=1}^n \mathring{w}_i$.

An allocation x is said to be *Pareto efficient* if it is feasible and there does not exist another feasible allocation x' such that $x'R_ix$ for all $i \in N$ and $x'P_ix$ for some $i \in N$. The set of all such allocations is denoted by $\mathcal{P}(e)$.

An allocation x is said to be *individually rational* if $xR_i\mathring{w}$ for all $i \in N$. The set of all such allocations is denoted by $\mathcal{I}(e)$.

2.2. Constrained distributive Lindahl allocations

Let

$$\Delta^{L-1} = \left\{ p^l \in \mathbb{R}^L_{++} : \sum_{l=1}^{L} p^l = 1 \right\}$$

be the normalized commodity price space which is the $n-1$ dimensional unit simplex, and let

$$A = \left\{ \alpha = [\alpha_{ij}] : \alpha_{ij} \geq 0 \text{ for all } i,\, j \in N \quad \text{and} \quad \sum_{i \in N} \alpha_{ij} = 1 \right\}$$

be the admissible share space.

For pure exchange economies with non-malevolent preference externalities, we now introduce an equilibrium principle that is called the constrained distributive Lindahl equilibrium and described as follows. Let p be a normalized price vector in the commodity price space Δ^{L-1}, and let $\alpha = [\alpha_{ij}]$ be an $n \times n$ share matrix system in the admissible share space A where α_{ij} are the shares of the cost of consumer j's consumption to be borne by consumer i which are assigned to each $i \in N$ for each $j \in N$. For a commodity price vector p, a share matrix α, and the initial endowment vector $\mathring{w} = (\mathring{w}_1, \ldots, \mathring{w}_n)$, each consumer has an initial wealth distribution of property rights which is determined by the institutions of the economy. Each consumer $i \in N$ then states the allocation which he likes best among those feasible allocations which he can afford if for each $j \in N$ he must pay the fraction α_{ij} of the cost of any bundle allocated to consumer j. At an equilibrium set of prices and shares, all consumers agree on the same allocation and there are no excess demand or supplies in any commodity market.

Formally, for a price vector $p \in \Delta^{L-1}$ and a share matrix $\alpha \in A$, an initial wealth distribution $\mathring{W}$ is a function $\mathring{W}(p, \alpha) = (\mathring{W}_1(p, \alpha), \ldots, \mathring{W}_n(p, \alpha))$ whose value depends on prices, shares, and the initial distribution of property rights and such that for any price-share system $(p, \alpha) \in \Delta^{L-1} \times A$, $\sum_{i=1}^{n} \mathring{W}_i(p, \alpha) = \sum_{i=1}^{n} p \cdot \mathring{w}_i$. It is assumed that $\mathring{W}_i(\cdot)$ is continuous in (p, α, w) and strictly increasing in $\mathring{w}_i$ for $i = 1, \ldots, n$.

Notice that the initial wealth distribution function $\mathring{W}(p, \alpha) = (\mathring{W}_1(p, \alpha), \ldots, \mathring{W}_I(p, \alpha))$ given in the present paper is more general and includes the initial wealth distribution function $\mathring{W}(p) = (\mathring{W}_1(p), \ldots, \mathring{W}_I(p))$ given by Bergstrom (1970) as a special case since Bergstrom's initial wealth distribution function $\mathring{W}(p)$ depends only on prices, but not on α. It will be seen that the initial wealth distribution function $\mathring{W}(p, \alpha)$ defined in the paper has some advantages that the one given in Bergstrom (1970) does not share.

An allocation $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^{nL}_+$ is a *constrained distributive Lindahl allocation* for an economy e if there is a price vector $p \in \Delta^{L-1}$ and a Lindahl share system $\alpha \in A$ such that

(1) $$\sum_{j=1}^{n} \alpha_{ij}[p \cdot x_j] \leqq \mathring{W}_i(p, \alpha);$$

(2) or all $i = 1, \ldots, n$, there does not exist $x' \, P_i \, x \in \mathbb{R}^{nL}_+$ such that

(2.1) $x' \, P_i \, x$;

(2.2) $\sum_{j=1}^{n} \alpha_{ij}[p \cdot x'_j] \leqq \mathring{W}_i(p, \alpha)$;

(2.3) $\sum_{j=1}^{n} x'_n \leqq \sum_{j=1}^{n} \mathring{w}_j$,

(3) $\sum_{j=1}^{n} x_j \leqq \sum_{j=1}^{n} \mathring{w}_j$.

The price-share-allocation triple (p, α, x) is then called a constrained distributive Lindahl equilibrium. Thus, at a constrained distributive Lindahl equilibrium, there is unanimous agreement about what the consumption bundle of each consumer should be, given total initial endowments and that each i must pay α_{ij} of the cost of the bundle consumed by each j. Denote by $\mathrm{DL_c}(e)$ the set of all such allocations.

Remark 1. Note that a constrained distributive Lindahl allocation differs from a distributive Lindahl allocation, which was introduced by Bergstrom (1972), only in a way that each agent maximizes his preferences not only subject to his budget constraint, but also subject to total endowments available to the economy. Thus, it is clearly that every distributive Lindahl allocation is a constrained distributive Lindahl allocation, but a constrained distributive Lindahl allocation may not be a distributive Lindahl allocation.

Remark 2. The reason we introduce the notion of the constrained distributive Lindahl equilibrium is that, like the Walrasian and Lindahl solutions, one can show that distributive Lindahl allocations violate Maskin's monotonicity condition so that it cannot be Nash implemented by any feasible mechanism. However, as we will do, it is possible to design a feasible mechanism whose Nash allocations coincide with a slightly larger set than distributive Lindahl allocations, namely, constrained distributive Lindahl allocations which are Pareto-efficient as shown in Lemma 1.

Remark 3. Bergstrom (1970) has shown that there exists a distributive Lindahl equilibrium, if the economy is characterized by non-malevolent preference externalities and there exists an initial allocation of resource ownership $w' = (w'_1, \ldots, w'_I)$, such that the initial wealth distribution function has a special form given by $\mathring{W}_i(p, \alpha) = p \cdot w'_i$ and $\sum_{i=1}^{I} w'_i = \sum_{i=1}^{I} \mathring{w}_i$, and if the other usual conditions (such as the convexity, local non-satiation, and continuity of preferences) are satisfied.[4] It will be seen that, although non-malevolence, continuity, and convexity of preferences are needed to ensure the existence of distributive Lindahl equilibrium, they are not needed to ensure that the mechanism presented in the paper

[4] Two examples for the initial wealth distribution function $\mathring{W}_i(p, \alpha)$ having such a special form can be given: (1) $\mathring{W}_i(p, \alpha) = p \cdot \mathring{w}_i$ by letting $w'_i = \mathring{w}_i$. Such an initial wealth distribution function has been used by Asdrubali (1996) to argue coalitional instability of the distributive Lindahl equilibrium. (2) $\mathring{W}_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[p \cdot \mathring{w}_j]$ by letting $w'_i = \sum_{j \in N} \alpha_{ij} \mathring{w}_j$. We guess under such a specification about the initial wealth function, Asdrubali's conclusion may not be true.

to implement constrained distributive Lindahl allocations. In fact, as long as preferences are monotonic, constrained distributive Lindahl allocations can be implemented by the mechanism we present in the paper.

We now show that, under the local non-satiation of preferences, every constrained distributive Lindahl allocation is Pareto efficient.

Lemma 1. *If (p, α, x) is a constrained distributive Lindahl equilibrium and if each consumer is locally nonsatiated at x, then x is Pareto efficient.*

Proof. Suppose, by way of contradiction, that x is not Pareto efficient. Then there is another feasible allocation x such that $x' R_i x$ for all $i \in N$ and $x' P_i x$ for some $i \in N$. Since x' is a constrained distributive Lindahl equilibrium, $x' P_i x$ and $\sum_{j=1}^n x'_j \leqq \sum_{j=1}^n \mathring{w}_j$ imply that $\sum_{j=1}^n \alpha_{ij}[p \cdot x'_j] > \mathring{W}_i(p, \alpha)$, and by local non-satiation of preferences, $x' R_i x$ and $\sum_{j=1}^n x'_j \leqq \sum_{j=1}^n \mathring{w}_j$ imply that $\sum_{j=1}^n \alpha_{ij}[p \cdot x'_j] \geqq \mathring{W}_i(p, \alpha)$. Therefore, if x' is Pareto superior to x, we have $\sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}[p \cdot x'_j] > \sum_{i=1}^n \mathring{W}_i(p, \alpha)$. But, since $\sum_{i=1}^n \alpha_{ij} = 1$ for all $j \in N$, this implies that $\sum_{j=1}^n x'_j > \sum_{j=1}^n \mathring{w}_i$. This contradicts the fact that x' is feasible. □

In general, a distributive Lindahl allocation may not be individually rational, and further there may be no relationship with the so-called Pigouvian equilibrium for a general wealth distribution function $\mathring{W}$. However, when the initial wealth distribution function is given by $\mathring{W}_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[p \cdot \mathring{w}_j]$, every distributive Lindahl allocation with such a spacial form of initial wealth distribution is clearly individually rational so that $\mathrm{DL}(e) \subset \mathcal{I}(e) \cap \mathcal{P}(e)$ for all $e \in E^{\mathrm{DL}}$, and further it is a Pigouvian allocation.[5] To see this, let us first define the Pigouvian equilibrium.

Let $t \in \mathbb{R}^{LI^2}$ be a transfer which satisfy the condition

$$\sum_{i=1}^{I} t_{ij} = 0.$$

For each price-transfer system $(p, t) \in \mathbb{R}^L \times \mathbb{R}^{LI^2}$, the budget set $B_i(p, t)$ of the ith consumer is given by

$$\left\{ x=(x_1, \ldots, x_I) \in \mathbb{R}^{IL} : x_i \in \mathbb{R}_+^L \quad \text{and} \quad \sum_{j=1}^{I} t_{ij} \cdot x_j + p \cdot x_i = \sum_{j=1}^{I} t_{ij} \cdot \mathring{w}_j + p \cdot \mathring{w}_I \right\}.$$

An allocation $x = (x_1, x_2, \ldots, x_I) \in \mathbb{R}^{IL}$ is a *Pigouvian allocation* for an economy e if it is feasible and there is a price vector $p \in R^L$ and a transfer system $t \in \mathbb{R}^{LI^2}$ such that

(1) $x \in B_i(p, t)$ for all $i = 1, \ldots, I$;
(2) for all $i = 1, \ldots, I$, $x' P_i x$ implies $\sum_{j \in N} t_{ij} \cdot x'_j + p \cdot x'_i > \sum_{j \in N} t_{ij} \cdot \mathring{w}_j + p \cdot \mathring{w}_i$.

[5] Because of these desired properties, it motivates us to generalize Bergstrom's initial wealth distribution function to the general one given in the present paper.

The price-transfer-allocation (p, t, x) is then called the *Pigouvian equilibrium*. Nakamura (1988) has shown that there is always a Pigouvian equilibrium under the regular conditions of convexity and continuity of preferences as well as interior endowments. It is clear that every Pigouvian is Pareto efficient and individually rational.

Now, for $i = 1, \dots, I$ and $j = 1, \dots, I$, if we let $t_{ij} = \alpha_{ij} p$ for $i \neq j$ and $t_{ii} = (\alpha_{ii} - 1)p$, then we have $\sum_{i=1}^{I} t_{ij} = 0$ so that it is a transfer system, and thus every distributive Lindahl allocation with respect to the initial wealth distribution function $\mathring{W}_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[p \cdot \mathring{w}_j]$ is a Pigouvian allocation, but the reverse may not be true.

2.3. Allocation mechanism

Let M_i denote the ith message (strategy) domain. Its elements are written as m_i and called messages. Let $M = \prod_{i=1}^{n} M_i$ denote the message (strategy) space. Let $X : M \to \mathbb{R}_+^{nL}$ denote the outcome function, or more explicitly, $X_i(m)$ is the ith consumer's consumption outcome at m. A mechanism consists of $\langle M, X\rangle$, which is defined on E. A message $m^* = (m_1^*, \dots, m_n^*) \in M$ is a *Nash equilibrium* (NE) of the mechanism $\langle M, X\rangle$ for an economy $e \in E$ if for all $i \in N$ and $m_i \in M_i$, it is not true that

$$X(m_i, m_{-i}^*)\; P_i\; X(m^*), \tag{1}$$

where $(m_i, m_{-i}^*) = (m_1^*, \dots, m_{i-1}^*, m_i, m_{i+1}^*, \dots, m_n^*)$. The allocation outcome $X(m^*)$ is then called a *Nash* (*equilibrium*) *allocation*. Denote by $V_{M,X}(e)$ the set of all such Nash equilibria and by $N_{M,X}(e)$ the set of all such Nash (equilibrium) allocations.

A mechanism $\langle M, X\rangle$ *Nash-implements* the constrained distributive Lindahl correspondence $\mathrm{DL_c}$ on E if for all $e \in E$, $N_{M,X}(e) = \mathrm{DL_c}(e)$.

Let C be a *coalition* that is a non-empty subset of N.

A message $m^* = (m_1^*, \dots, m_n^*) \in M$ is said to be a *strong Nash equilibrium* of the mechanism $\langle M, X\rangle$ for an economy $e \in E$ if there does not exist any coalition C and $m_C \in \prod_{i \in C} M_i$ such that for all $i \in C$,

$$X(m_C, m_{-C}^*)\; P_i\; X(m^*). \tag{2}$$

$X(m^*)$ is then called a *strong Nash* (*equilibrium*) *allocation* of the mechanism for the economy e. The set of all such strong Nash equilibria is denoted by $SV_{M,X}(e)$ and the set of all such strong Nash (equilibrium) allocations is denoted by $SN_{M,X}(e)$.

The mechanism $\langle M, h\rangle$ is said to doubly *implement* the constrained distributive Lindahl correspondence $\mathrm{DL_c}$ on E, if, for all $e \in E$, $SN_{M,N}(e) = N_{M,X}(e) = \mathrm{DL_c}(e)$.

A mechanism $\langle M, X\rangle$ is *feasible* if for all $m \in M$, $X(m) \in \mathbb{R}_+^{nL}$ and

$$\sum_{j=1}^{n} X_j(m) \leqq \sum_{j=n}^{n} \mathring{w}_j. \tag{3}$$

3. A feasible and continuous mechanism

In this section, we present a simple feasible and continuous mechanism which doubly implements the constrained distributive Lindahl correspondence on E.

For each $i \in N$, let the message domain of agent i be of the form

$$M_i = (0, \mathring{w}_i] \times \Delta^{L-1} \times A \times \mathbb{R}^{nL}. \tag{4}$$

A generic element of M_i is $m_i = (w_i, p_i, a_i, x_{i1}, \dots, x_{in})$ whose components have the following interpretations. The component w_i denotes a profession of agent i's endowment, the inequality $0 < w_i \leqq \mathring{w}_i$ means that the agent cannot overstate his own endowment; on the other hand, the endowment can be understated, but the claimed endowment w_i must be positive. Note that, although the true endowment is the upper bound of the reported endowment, the designer does not need to know this upper bound. This is because whenever an agent claims an endowment of a certain amount, the designer can ask him to *exhibit* it (one may, for instance, imagine that the rules of the game require that the agent 'put on the table' the reported amount w_i). The component p_i is the price vector proposed by agent i and is used as a price vector of agent $i-1$, where $i-1$ is read to be n when $i = 1$. The component a_i is an $I \times I$ share matrix proposed by agent i and is used as a share matrix of agent $i-1$. The component $(x_{i1}, \dots, x_{in})$ is the proposed allocation by agent i, where x_{ij} may be interpreted as the contribution that agent i is willing to make to agent j (a negative x_{ij} means agent i wants to get $-x_{ij}$ amount of goods from agent j).

Define agent i's price vector $p_i : M \to \Delta^{L-1}$ by

$$p_i(m) = p_{i+1}, \tag{5}$$

where $n+1$ is to be read as 1.

Define agent i's share matrix $\alpha^i : M \to A$ by

$$\alpha^i(m) = a_{i+1}, \tag{6}$$

Thus, by the definitions of $p_i(\cdot)$ and $\alpha^i(\cdot)$, they are independent of choices of consumer i so that each consumer takes them as given. Note that although $p_i(\cdot)$ and $\alpha^i(\cdot)$ are functions of proposed price vector and share matrix by consumer $i+1$, respectively, for simplicity, we have written $p(\cdot)$ and $\alpha^i(\cdot)$ as functions of m without loss of generality. Also it may be remarked that the construction of $p_i(m)$ is much simpler than the one used in Postlewaite and Wettstein (1989) and Tian (1992), in which it is determined by proposed price vector of all individuals, while ours is only involved one person's proposed price.

Define a feasible correspondence $B : M \to \mathbb{R}^{nL}_+$ by

$$B(m) = \left\{ x \in \mathbb{R}^{nL}_+ : \sum_{i=1}^n x_i \leqq \sum_{i=1}^n \frac{w_i}{1 + \|p_i - p_i(m)\| + \|a_i - \alpha^i(m)\|} \quad \text{and} \right.$$
$$\left. \sum_{j=1}^n \alpha^i_{ij}(m)[p_j(m) \cdot x_j] \leqq \frac{W_i(p_i(m), \alpha^i(m))}{1 + \|p_i - p_i(m)\| + \|a_i - \alpha^i(m)\|} \ \forall i \in N \right\}, \tag{7}$$

which is clearly nonempty compact convex (by the total resource constraints) for all $m \in M$. Note that, by definition, $\alpha^i_{ij}(m)$ is an element in the ith row and jth column of a_{i+1}. We will show the following lemma in the Appendix A.

Lemma 2. *$B(\cdot)$ is continuous on M.*

Let $\tilde{x}_j = \sum_{i=1}^n x_{ij}$ which is the sum of contributions that agents are willing to make to agent j and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$.

The outcome function X: $M \to \mathbb{R}_+^{nL}$ is given by

$$X(m) = \{y \in \mathbb{R}_+^{nL} : \min_{y \in B(m)} \|y - \tilde{x}\|\}, \tag{8}$$

which is the closest to $\tilde{x}$. Then X is single-valued and continuous on M.[6] Also, since $X(m) \in \mathbb{R}_+^{nL}$ and

$$\sum_{i=1}^n X_i(m) \leqq \sum_{i=1}^n \mathring{w}_i \tag{9}$$

for all $m \in M$, the mechanism is feasible and continuous.

Remark 4. Note that the above mechanism does not depend on the number of agents. Thus, it is a unified mechanism which works for two-agent economies as well as for economies with three or more agents.

4. Results

The remainder of this paper is devoted to the proof of equivalence among Nash allocations, strong Nash allocations, and constrained distributive Lindahl allocations. Proposition 1 below proves that every Nash allocation is a constrained distributive Lindahl allocation. Proposition 2 below proves that every constrained distributive Lindahl allocation is a Nash allocation. Proposition 3 below proves that every Nash equilibrium is a strong Nash equilibrium. Therefore, we show that the mechanism constructed in the previous section doubly implements constrained distributive Lindahl allocations, which is stated in Theorem 1 below. To show these results, we first prove the following lemmas.

Lemma 3. *If $m^* \in V_{M,X}(e)$, then $p_1^* = p_2^* = \cdots = p_n^*$ and $a_1^* = a_2^* = \cdots = a_n^*$. Consequently, $p_1(m^*) = p_2(m^*) = \cdots = p_n(m^*) = p(m^*) \in \Delta_{++}^{L-1}$ and $\alpha^{*1}(m^*) = \alpha^{*2}(m^*) = \cdots = \alpha^{*n}(m^*) = \alpha(m^*) \in A$.*

Proof. Suppose, by way of contradiction, that $p_i^* \neq p_{i+1}^*$ and/or $a_i^* \neq a_{i+1}^*$ (i.e. $p_i^* \neq p_i(m^*)$ and/or $a_i^* \neq \alpha^i(m^*)$) for some $i \in N$. Then $\sum_{j=1}^n \alpha_{ij}^i(m^*)[p_j(m^*) \cdot X_j(m^*)] \leqq W_i(p_i(m^*), \alpha^i(m^*))/(1 + \|p_i - p_i(m^*)\| + \|a_i - \alpha^i(m^*)\|) < W(p_i(m^*), \alpha^i(m^*))$, and $\sum_{j=1}^n X_j(m^*) \leqq \sum_{j=1}^n w_j/(1 + \|p_j - p_j(m)\| + \|a_j - \alpha^j(m)\|) < \sum_{j=1}^n w_j$. Thus, there is $x \in \mathbb{R}_+^{nL}$ such that $\sum_{j=1} \alpha_{ij}^i(m^*)[p_j(m^*) \cdot x_j] \leqq W(p_i(m^*), \alpha^i(m^*))$, $\sum_{j=1}^n x_j \leqq \sum_{j=1}^n w_j$

[6] This is because X is an upper semi-continuous correspondence by Berge's Maximum Theorem (see Debreu, 1959, p. 19), and single-valued (see Mas-Colell, 1985, p. 28).

and $x\, P_i\, X(m^*)$ by strict monotonicity of preferences. Now if agent i chooses $p_i = p_i(m^*)$, $a_i = \alpha^i(m^*)$, $x_{ij} = x_j - \sum_{t\neq i} x^*_{tj}$ for $j = 1, \ldots, n$, and keeps w^*_i unchanged, then $x \in B(m_i, m^*_{-i})$, and thus $X(m_i, m^*_{-i}) = x$. Therefore, $X(m_i, m^*_{-i})\, P_i\, X(m^*)$. This contradicts $X(m^*) \in N_{M,X}(e)$. Thus, we must have $p^*_1 = p^*_2 = \cdots = p^*_n$, and $a_{*1} = a_{*2} = \cdots = a_{*n}$. Therefore, $p_1(m^*) = p_2(m^*) = \cdots = p_n(m^*) = p(m^*) \in \Delta^{L-1}_{++}$ and $\alpha^{*1}(m^*) = \alpha^{*2}(m^*) = \cdots = \alpha^{*n}(m^*) = \alpha(m^*) \in A$. □

Lemma 4. *If $m^* \in N_{M,X}(e)$, then $w^*_i = \mathring{w}_i$ for all $i \in N$.*

Proof. Suppose, by way of contradiction, that $w^*_i \neq \mathring{w}_i$ for some $i \in N$. Then, by the monotonicity of $W_i(p, \alpha)$, $\sum_{j=1} \alpha^*_{ij}[p(m^*) \cdot X_j(m^*)] \leqq W_i(p(m^*), \alpha(m^*)) < \mathring{W}_i(p(m^*), \alpha(m^*))$, and $\sum^n_{j=1} X_j(m^*) \leqq \sum^n_{j=1} w_j \leq \sum^n_{j=1} \mathring{w}_j$. Thus, there is $x \in \mathbb{R}^{nL}_+$ such that $\sum_{j=1} \alpha^*_{ij}[p(m^*) \cdot x_j] \leqq \mathring{W}_i(p(m^*), \alpha(m^*))$, $\sum^n_{j=1} x_j \leqq \sum^n_{j=1} \mathring{w}_j$, and $x\, P_i\, X(m^*)$ by strict monotonicity of preferences. Now if agent i chooses $w_i = \mathring{w}_i$, $x_{ij} = x_j - \sum_{t\neq i} x^*_{tj}$ for $j = 1, \ldots, n$, and keeps p^*_i and a^*_i unchanged, then $x \in B(m_i, m^*_{-i})$, and thus $X(m_i, m^*_{-i}) = x$. Hence, $X(m_i, m^*_{-i})\, P_i\, X(m^*)$. This contradicts $X(m^*) \in N_{M,X}(e)$ and thus $w^*_i = \mathring{w}_i$ for all $i \in N$. □

Lemma 5. *If $X(m^*) \in N_{M,X}(e)$, then $\sum^n_{j=1} X_j(m^*) = \sum^n_{j=1} \mathring{w}_j$, and $\sum_{j=1} \alpha_{ij}(m^*)[p(m^*) \cdot X_j(m^*)] = \mathring{W}_i(p(m^*), \alpha(m^*))$ for all $i \in N$.*

Proof. Suppose, by way of contradiction, that $\sum^n_{j=1} X_j(m^*) \neq \sum^n_{j=1} \mathring{w}_j$. Then, $\sum^n_{j=1} X_j(m^*) \leq \sum^n_{j=1} \mathring{w}_j$, and thus we must have $\sum_{j=1} \alpha_{ij}(m^*)[p(m^*) \cdot X_i(m^*)] < \mathring{W}_i(p(m^*), \alpha(m^*))$ for some $i \in N$, and thus there is $x \in \mathbb{R}^{nL}_+$ such that $\sum_{j=1} \alpha_{ij}(m^*)[p(m^*) \cdot x_j] \leqq \mathring{W}_i(p(m^*), \alpha(m^*))$, $\sum^n_{j=1} x_j \leqq \sum^n_{j=1} \mathring{w}_j$, and $x\, P_i\, X(m^*)$ by strict monotonicity of preferences. Now if agent i chooses $x_{ij} = x_j - \sum_{t\neq i} x^*_{tj}$ for $j = 1, \ldots, n$, and keeps p^*_i, a^*_i, and w^*_i unchanged, then $x \in B(m_i, m^*_{-i})$, and thus $X(m_i, m^*_{-i}) = x$. Hence, $X(m_i, m^*_{-i})\, P_i\, X(m^*)$. This contradicts $X(m^*) \in N_{M,X}(e)$. We must have $\sum^n_{j=1} X_j(m^*) = \sum^n_{j=1} \mathring{w}_j$, and consequently, $\sum_{j=1} \alpha_{ij}(m^*)[p(m^*) \cdot X_j(m^*)] = \mathring{W}_i(p(m^*), \alpha(m^*))$ for all $i \in N$. □

Proposition 1. *If the mechanism $\langle M, X\rangle$ defined above has a Nash equilibrium m^* for $e \in E$, then $X(m^*)$ is a constrained distributive Lindahl allocation with $p(m^*)$ as an equilibrium price vector and $\alpha(m^*)$ as an equilibrium share matrix, i.e. $N_{M,X}(e) \subset \mathrm{DL_c}(e)$ for all $e \in E$.*

Proof. Let m^* be a Nash equilibrium. Then $X(m^*)$ is a Nash equilibrium allocation. We wish to show that $X(m^*)$ is a constrained distributive Lindahl allocation. By Lemmas 3–5, $p_1(m^*) = \cdots = p_n(m^*) = p(m^*) \in \Delta^{L-1}_+$, $\alpha^1(m^*) = \cdots = \alpha^n(m^*) = \alpha(m^*) \in A$, $w^*_i = \mathring{w}_i$, $\sum_{j=1} \alpha_{ij}(m^*)[p(m^*) \cdot X_j(m^*)] = \mathring{W}_i(p(m^*), \alpha(m^*))$ for all $i \in N$, and $\sum^n_{j=1} X_j(m^*) = \sum^n_{j=1} \mathring{w}_j$. Also, by the construction of the mechanism, we know that $X(m^*) \in \mathbb{R}^{nL}_+$. So we only need to show that each individual is maximizing his/her preferences subject to his budget constraint and resource constraints. Suppose, by way of contradiction, that for some agent i, there exists some $\tilde{x} \in \mathbb{R}_+^{nL}$ such that $\sum_{j=1}^n \tilde{x}_j \leqq \sum_{j=1}^n \mathring{w}_j$, $\sum_{j=1} \alpha_{ij}(m^*)[p(m^*) \cdot \tilde{x}_j] \leqq \mathring{W}_i(p(m^*), \alpha(m^*))$ and $\tilde{x}\ P_i\ X(m^*)$. Let $x_{ij} = \tilde{x}_j - \sum_{t \neq i} x_{tj}^*$, for $j = 1, \ldots, n$, and keep p_i^*, a_{*i}, and w_i^* unchanged, then $\tilde{x} \in B(m_i, m_{-i}^*)$, and thus $X(m_i, m_{-i}^*) = \tilde{x}$. Therefore, we have $X(m_i, m_{-i}^*)\ P_i\ X(m^*)$. This contradicts $X(m^*) \in N_{M,X}(e)$. So $X(m^*)$ is a constrained distributive Lindahl allocation. □

Proposition 2. *If $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is a constrained distributive Lindahl allocation with an equilibrium price vector $p^* \in \Delta_+^{L-1}$ and an equilibrium share matrix $\alpha^* \in A$ for $e \in E$, then there exists a Nash equilibrium m^* of the mechanism $\langle M, X\rangle$ defined above such that $X_i(m^*) = x_i^*$, $p_i(m^*) = p^*$, and $\alpha^i(m^*) = \alpha^*$, for all $i \in N$, i.e.* $\mathrm{DL_c}(e) \subset N_{M,X}(e)$ *for all $e \in E$.*

Proof. Since preferences satisfy the strict monotonicity condition and x^* is a constrained distributive Lindahl allocation, we must have $p^* \in \Delta^{L-1}$, $\alpha^* \in A$, $\sum_{j=1}^n x_j^* = \sum_{j=1}^n \mathring{w}_j$ and $\sum_{j=1} \alpha_{ij}(m^*)[p(m^*) \cdot x_j^*] = \mathring{W}_i(p(m^*), \alpha(m^*))$ for $i \in N$. Now for each $i \in N$, let $m_i^* = (\mathring{w}_i, p*, \alpha^*, x_{i1}^*, \ldots, x_{in}^*)$.

Then x^* is an outcome with p^* as a price vector and α^* as a share matrix, i.e. $X_i(m^*) = x_i^*$ for all $i \in N$, $p_i(m^*) = p^*$, and $\alpha^i(m^*) = \alpha^*$. We show that m^* yields this allocation as a Nash allocation. In fact, agent i cannot change $p_i(m^*)$ and $\alpha(m^*)$ by changing his proposed price and proposed share matrix (i.e. $p_i(m_i, m_{-i}^*) = p_i(m^*)$ and $\alpha^i(m_i, m_{-i}^*) = \alpha^i(m^*)$ for all $m_i \in M_i$). Announcing a different message m_i by agent i may yield an allocation $X(m_i, m_{-i}^*)$ such that $X(m_i, m_{-i}^*) \in \mathbb{R}_+^{nL}$,

$$p(m^*) \cdot \sum_{j=1} \alpha_{ij}(m^*)[X_i(m_i, m_{-i}^*)] \leqq \mathring{W}_i(p(m^*), \alpha(m^*)), \tag{10}$$

and

$$\sum_{j=1} X_j(m_i, m_{-i}^*) \leqq \sum_{i=1}^n \mathring{w}_i. \tag{11}$$

Now suppose, by way of contradiction, that m^* is not a Nash equilibrium. Then there are $i \in N$ and m_i such that $X(m_i, m_{-i}^*)\ P_i\ X(m^*)$. Since $\sum_{j=1}^n X_j(m_i, m_{-i}^*) \leqq \sum_{i=1}^n \mathring{w}_i$, we must have, by the definition of the constrained distributive Lindahl allocation, $\sum_{j=1} \alpha_{ij}(m^*)$ $[p(m^*) \cdot X_i(m_i, m_{-i}^*)] > \sum_{j=1} \alpha_{ij}(m^*) \mathring{W}_i(p(m^*), \alpha(m^*))$. But this contradicts the budget constraint ((10)). Thus we have shown that agent i cannot improve his/her utility by changing his/her own message while the others' messages remain fixed for all $i \in N$. Hence x^* is a Nash allocation. □

Proposition 3. *Every Nash equilibrium m^* of the mechanism defined above is a strong Nash equilibrium, that is, $N_{M,X}(e) \subseteq SN_{M,X}(e)$.*

Proof. Let m^* be a Nash equilibrium. By Proposition 1, we know that $X(m^*)$ is a constrained distributive Lindahl allocation with $p(m^*)$ as a price vector. Then $X(m^*)$ is Pareto optimal and thus the coalition N cannot be improved upon by any $m \in M$. Now for any coalition C with $\emptyset \neq C \neq N$, choose $i \in C$ such that $i+1 \notin C$. Then no strategy played by C can change the budget set of i since $p_i(m)$ is determined by p_{i+1} and $\alpha_i(m)$ is determined by a_{i+1}. Furthermore, because $X(m^*) \in \mathrm{DL_c}(e)$, it is the preference maximizing consumption with respect to the budget set of i and the total endowments, and thus C cannot improve upon $X(m^*)$. □

Since every strong Nash equilibrium is clearly a Nash equilibrium, then by combining Propositions 1–3, we have the following theorem.

Theorem 1. *For the class of exchange economies E, there exists a feasible and continuous mechanism which doubly implements the constrained distributive Lindahl correspondence. That is, $N_{M,X}(e) = SN_{M,X}(e) = \mathrm{DL_c}(e)$ for all $e \in E$.*

5. Conclusion

In this paper, we have considered the incentive mechanism design which results in efficient allocations for exchange economies with consumption externalities. We do so by introducing the notion of constrained distributive Lindahl solution for economies with consumption externalities. We then presented a market-type mechanism which doubly implements constrained distributive Lindahl allocations when coalition patterns, preferences, and endowments are unknown to the designer. In particular, since the Pigouvian mechanism is a special case of the distributive Lindahl mechanism, the mechanism also implements the Pigouvian allocations. The double implementation covers the case where agents in some coalitions may cooperate and in other coalitions may not, when such information is unknown to the designer. This combining solution concept, which characterizes agents' strategic behavior, may bring about a state which takes advantage of both Nash (social) equilibrium and strong Nash (social) equilibrium, so that it may be easy to reach and hard to leave. The mechanisms constructed in the paper are well-behaved in the sense that they are feasible and continuous. In addition, the mechanism works also for two agents.

For the simplicity of exposition, attention has only been confined to pure exchange economies and consequently, to purely consumption externalities in the present paper. However, the results may likely be extended to economies with convex production technologies by extending the notion of the (constrained)distributive Lindahl equilibrium to convex production economies.

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Appendix A

Proof of Lemma 2. It is clear that $B(\cdot)$ has closed graph by the continuity of $p_i(\cdot)$, $\alpha^i(\cdot)$, and $W_i(\cdot)$. Since the range space of the correspondence $B(\cdot)$ is bounded by the reported total endowments $\sum_{i=1}^n w_i$, it is compact. Thus, $B(\cdot)$ is upper hemi-continuous on M. So we only need to show that $B(m)$ is also lower hemi-continuous at every $m \in M$. Let $m \in M$, $x = (x_1, \ldots, x_n) \in B(m)$, and let $\{m_k\}$ be a sequence such that $m_k \to m$, where $m_k = (m_1^k, \ldots, m_n^k)$ and $m_i^k = (w_i^k, p_i^k, a_i^k, z_{i1}^k, \ldots, z_{in}^k)$. We want to prove that there is a sequence $\{x_k\}$ such that $x_k \to x$, and, for all k, $x_k \in B(m_k)$, i.e. $x_k = (x_{1k}, \ldots, x_{nk}) \in \mathbb{R}_+^{nL}$, $\sum_{j=1}^n \alpha_{ij}^i(m_k)[p_j(m_k) \cdot x_{jk}] \leqq W_i(p_i(m_k), \alpha^i(m_k))/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$ for all $i \in N$, and $\sum_{i\in N} x_{ik} \leqq \sum_{i\in N} w_i^k/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$. We first prove that there is a sequence $\{\hat{x}_k\}$ such that $\hat{x}_k \to x$, and, for all k, $\hat{x}_k \in \mathbb{R}_+^{nL}$ and $\sum_{j=1}^n \alpha_{ij}^i(m_k)[p_j(m_k) \cdot \hat{x}_{jk}] \leqq W_i(p_i(m_k), \alpha^i(m_k))/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$ for all $i \in N$.

Let $N' = \{i \in N : \sum_{j=1}^n \alpha_{ij}^i[p_j(m) \cdot x_j] = W_i(p_i(m), \alpha^i(m))/(1 + \|p_i - p_i(m)\| + \|a_i - \alpha^i(m)\|)\}$. Two cases will be considered.

Case 1. $N' = \emptyset$, i.e. $\sum_{j=1}^n \alpha_{ij}^i(m)[p_j(m) \cdot x_j] < W_i(p_i(m), \alpha^i(m))/(1 + \|p_i - p_i(m)\| + \|a_i - \alpha^i(m)\|)$ for all $i \in N$. Then, by the continuity of $p_i(\cdot)$, $\alpha^i(\cdot)$, and $W_i(\cdot)$, for all k larger than a certain integer k', we have $\sum_{j=1}^n \alpha_{ij}^i(m_k)[p_j(m_k) \cdot x_{jk}] < W_i(p_i(m_k), \alpha^i(m_k))/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$. Let $\hat{x}_k = x$ for all $k > k'$ and $\hat{x}_k = 0$ for $k \leqq k'$. Then, $\hat{x}_k \to x$, and, for all k, $\sum_{j=1}^n \alpha_{ij}^i(m_k)[p_j(m_k) \cdot \hat{x}_{jk}] < W_i(p_i(m_k), \alpha^i(m_k))/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$ for all $i \in N$.

Case 2. $N' \neq \emptyset$. Then $\sum_{j=1}^n \alpha_{ij}^i(m)[p_j(m) \cdot x_j] = W_i(p_i(m), \alpha^i(m))/(1 + \|p_i - p_i(m)\| + \|a_i - \alpha^i(m)\|)$ for all $i \in N'$. Note that, since $W_i(\cdot) > 0$, we must have $\sum_{j=1}^n \alpha_{ij}^i(m)[p_j(m) \cdot x_j] > 0$, and thus, by the continuity of $p_i(m) > 0$ and $\alpha^i(m) > 0$, $f_{ik}(x) \equiv \sum_{j=1}^n \alpha_{ij}^i(m_k)[p_j(m_k) \cdot x_j] > 0$ for all k larger than a certain integer k'. For each $k \geq k'$ and $i \in N'$, let $\omega_i = W_i(p_i(m), \alpha^i(m))/(1+\|p_i - p_i(m)\| + \|a_i - \alpha^i(m)\|)$, $\omega_{ik} = W_i(p_i(m_k), \alpha^i(m_k))/(1+\|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$ let $\lambda_{ik} = \omega_{ik}/f_{ik}(x)$,

$$\hat{x}_k^i = \begin{cases} \lambda_{ik}x, & \text{if } \dfrac{\omega_{ik}}{f_k(x)} \leqq 1; \\ x, & \text{otherwise.} \end{cases}$$

and let $\hat{x}_k = \min_{i\in N'}\{\hat{x}_k^i\} = x \min_{i\in N'}\{1, \lambda_{ik}\}$. Then $\hat{x}_k \leqq \hat{x}_k^i \leqq x$. Also, since $\lambda_{ik} = \omega_{ik}/f_{ik}(\hat{x}) \to \omega_i/\sum_{j=1}^n \alpha_{ij}^i(m)[p_j(m) \cdot x_j] = 1$ for all $i \in N'$, we have $\hat{x}_k \to x$ and $f_{ik}(\hat{x}_k) > 0$ for all k larger than a certain integer k''. Now we claim that $\hat{x}_k$ also satisfies $\sum_{j=1}^n \alpha_{ij}(m_k)[p_j(m_k) \cdot \hat{x}_{jk}] \leqq W_i(p_i(m_k), \alpha^i(m_k))/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$ for all $i \in N$ and $k \geqq \max\{k', k''\}$. Indeed, for each $i \in N'$, we have $\sum_{j=1}^n \alpha_{ij}(m_k)[p_j(m_k) \cdot \hat{x}_{jk}] \leqq W_i(p_i(m_k), \alpha^i(m_k))/(1+\|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$ by the definition of $\hat{x}_k$. For all $i \in N \backslash N'$, since $\sum_{j=1}^n \alpha_{ij}^i(m)[p_j(m) \cdot x_j] < W_i(p_i(m), \alpha^i(m))/(1 + \|p_i - p_i(m)\| +$

$\|a_i - \alpha^i(m)\|)$, we have $\sum_{j=1}^n \alpha_{ij}(m_k)[p_j(m_k) \cdot \hat{x}_{jk}] \leqq W_i(p_i(m_k), \alpha^i(m_k))/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$ for all k larger than a certain integer k''' by the continuity of $p_i(m) > 0$, $\alpha^i(m) > 0$, and $W_i(\cdot)$. Thus, for all $k \geq \max\{k', k'', k'''\}$ and $I \in N$, we have $\sum_{j=1}^n \alpha_{ij}^i(m_k)[p_j(m_k) \cdot \hat{x}_{jk}] \leqq W_i(p_i(m_k), \alpha^i(m_k))/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$ for all $i \in N$.

Thus, in both cases, there is a sequence $\{\hat{x}_k\}$ such that $\hat{x}_k \to x$, and, for all k, $\hat{x}_k \in \mathbb{R}_+^{nL}$ and $\sum_{j=1}^n \alpha_{ij}^i(m_k)[p_j(m_k) \cdot \hat{x}_{jk}] \leqq W_i(p_i(m_k), \alpha^i(m_k))/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$ for all $i \in N$.

We now show that there is a sequence $\{\bar{x}_k\}$ such that $\bar{x}_k \to x$, and, for all k, $\bar{x}_k \in \mathbb{R}_+^{nL}$ and $\sum_{i\in N} \bar{x}_{ik} \leqq \sum_{i\in N} w_i^k$. For each $l = 1, \dots, L$, two cases will be considered.

Case 1. $\sum_{i\in N} x_i^l < \sum_{i\in N} w_i^l/(1+\|p_i - p_i(m)\| + \|a_i - \alpha^i(m)\|)$. Hence, for all k larger than a certain integer k', we have $\sum_{i\in N} x_i^l < \sum_{i\in N} w_i^{lk}/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$. For each $i \in N$, let $\bar{x}_{ik}^l = x_i^l$ for all $k > k'$ and $\hat{x}_{ik}^l = 0$ for $k \leqq k'$. Then, we have $\sum_{i\in N} x_{ik}^l < \sum_{i\in N} w_i^{lk}/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$.

Case 2. $\sum_{i\in N} x_i^l = \sum_{i\in N} w_i^l/(1 + \|p_i - p_i(m)\| + \|a_i - \alpha^i(m)\|)$. Note that, since $w_i > 0$ for all i, we must have $\sum_{i\in N} x_i^l > 0$. For each $i \in N$, define $\bar{x}_{ik}^l$ as follows:

$$\bar{x}_{ik}^l = \begin{cases} \dfrac{\sum_{i\in N} w_i^{lk}/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)}{\sum_{i\in N} x_i^l} x_i^l, & \text{if } \dfrac{\sum_{i\in N} w_i^{lk}(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)}{\sum_{i\in N} x_i^l} \leqq 1; \\ x_i^l, & \text{otherwise}. \end{cases}$$

Then $\bar{x}_{ik}^l \leqq x_i^l$, and $\sum_{i\in N} \bar{x}_{ik}^l \leqq \sum_{i\in N} w_i^{lk}/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$.

Also, since $\sum_{i\in N}[w_i^{lk}/(1+\|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)]/\sum_{i\in N} x_i^l \to \sum_{i\in N}[w_i^l/(1+\|p_i - p_i(m)\| + \|a_i - \alpha^i(m)\|)]/\sum_{i\in N} x_i^l = 1$, we have $\bar{x}_{ik}^l \to x_i^l$. Thus, in both cases, there is a sequence $\{\bar{x}_k\}$ such that $\bar{x}_k \to x$, and, for all k, $\bar{x}_k \in \mathbb{R}_+^{nL}$ and $\sum_{i\in N} x_i^k \leqq \sum_{i\in N} w_i^k/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$. Here, $\bar{x}_k = (\bar{x}_k^1, \dots, \bar{x}_k^L)$.

Finally, let $x'_k = \min(\bar{x}_k, \hat{x}_k)$ with $x'_{ik} = \min(\bar{x}_{ik}, \hat{x}_{ik})$ for $i = 1, \dots, n$. Then $x'_k \to x$ since $\bar{x}_k \to x$ and $\hat{x}_k \to x$. Also, for every k larger than a certain integer $\bar{k}$, we have $x'_k \geq 0$, $\sum_{i\in N} x'_{ik} \leqq \sum_{i\in N} w_i^k/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$ because $x'_k \leqq \bar{x}_k$ and $\sum_{i\in N} \bar{x}_{ik} \leqq \sum_{i\in N} w_i^k/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$, and $\sum_{j=1}^n \alpha_{ij}^i(m_k)[p_j(m_k) \cdot x'_{jk}] \leqq W_i(p_i(m_k), \alpha^i(m_k))/(1 + \|p_i^k - p_i(m_k)\| + \|a_i^k - \alpha^i(m_k)\|)$ for all $i \in N$ by noting that $x'_{ik} \leqq \hat{x}_{ik}$. Let $x_k = x'_k$ for all $k > \bar{k}$ and $x_k = 0$ for $k \leqq \bar{k}$. Then, $x_k \to x$, and $x_k \in B(m_k)$ for all k. Therefore, the sequence $\{x_k\}$ has all the desired properties. So $B_x(m)$ is lower hemi-continuous at every $m \in M$.

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