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Existence of equilibrium in abstract economies with discontinuous payoffs and non-compact choice spaces*

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This paper proves the equilibrium existence for abstract economies with non-compact infinite-dimensional strategy spaces, infinitely many agents, and discontinuous payoff (utility) functions by using the quasi-variational inequality approach. The motivations come from economic applications showing that payoff functions are discontinuous in many cases and the set of feasible allocations generally is not compact in a given topology of the commodity space, a typical situation in infinite dimensional vector space. It will be noted that our results also extend a foundational quasi-variational inequality by relaxing the compactness and concavity conditions. Thus many existence theorems in the quasi-variational inequalities literature can also be generalized by our results.

1. Introduction

In this paper we deal mainly with the problem of the existence of the social equilibrium in abstract economies with discontinuous payoff functions, non-compact infinite-dimensional strategy space, and infinitely many agents.

The abstract economy defined by Debreu (1952) generalizes an N-person Nash noncooperative game in that a player's strategy set depends on the strategy choices of all the other players and can be used to prove the existence of competitive equilibrium since the market mechanism can be regarded as an abstract economy [cf. e.g., Arrow and Debreu (1954)]. Debreu (1952) proved the existence of equilibrium in abstract economies with finitely many agents and finite-dimensional strategy spaces by assuming the existence of continuous utility functions. Since Debreu's seminal work on abstract economies, many existence results have been given in the literature. Shafer and Sonnenschein (1975) extended Debreu's results to abstract economies

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without ordered preferences.[1] For the infinite dimensional space and finite or infinitely many agents cases, the existence results were given by Yannelis and Prabhakar (1983), Khan and Vohra (1984), Khan (1986), Yannelis (1987), and Kim, Prikry, and Yannelis (1989).

To the best of my knowledge, however, all the existence theorems, including those mentioned above, on abstract economies and competitive equilibrium in the literature, are obtained by assuming that the choice sets are compact in some topological vector space and/or payoff functions (or preference relations) are continuous.[2] But, in many cases, payoff functions are discontinuous. Also, in the finite-dimensional setting, the usual (closed) boundedness assumptions imply compactness of the feasible consumption and production sets and thus the compactness hypotheses may be plausible. Since the set of feasible allocations is bounded by total endowments and thus is compact, any equilibrium allocation is feasible and therefore is in the compact set. Thus the consumption and production sets can be compactified by truncating these sets and it can be shown that the equilibrium of the truncated subeconomy is an equilibrium of the original economy as long as the truncated subeconomy is close enough to the original economy. However, in the infinite-dimensional setting, the usual (closed) boundedness assumptions do not imply compactness of the feasible consumption and production sets and thus the feasible sets will not generally be compact in a given topology of the commodity space and they are not even weakly compact for any topology in a vector space.[3] Thus one cannot use the truncated economy approach to prove the existence of competitive equilibrium. To avoid this difficulty in the literature, economists [say, e.g., Zame (1987)] explicitly assume that the feasible sets are (weakly) compact in some topology. However, this is not a satisfactory assumption since, as we mentioned above, in the infinite-dimensional setting, the feasible sets will not generally be compact in any topology of the commodity vector space. Thus the compactness assumption does not solve the problem in this general abstract framework.

An appropriate question then is whether or not we can extend Debreu's theorem to the case of non-compact choice spaces and/or discontinuous payoff functions. The purpose of this paper is to answer this question in the affirmative. Of course, in order to guarantee the existence of equilibria on a

[1]Shafer and Sonnenschein (1975) did not strictly generalize Debreu's results since convexity rather than contractibility assumptions are made.

[2]In the case where an abstract economy reduces to the conventional game, Dasgupta and Maskin (1986) and Tian and Zhou (1992) extended the existence theorems of Nash (1950, 1951). Debreu (1951), Nikaido and Isoda (1955) by relaxing the continuity of payoff functions. But they still assume the strategy spaces are compact.

[3]For some infinite-dimensional spaces such as L_1, l_1, L_2, l_2, L_∞, and l_∞ spaces, the feasible sets with finite number of agents, however, may be (weakly) compact in some weaker topologies if some additional conditions are imposed [cf. Zame (1987, pp. 1084–1085)].

non-compact set, some kind of 'boundary' assumptions (i.e., assumptions on utility functions outside of some compact set) is absolutely necessary. The 'boundary' assumption used in this paper [Assumption (iii.3) in Theorem 2 below] is similar to those used by Border (1985) and Allen (1977) to prove the existence of maximal elements of binary relations and of equilibrium of variational inequality on non-compact sets. This assumption is a natural generalization of the compactness assumption since it is satisfied if the choice space itself is compact.

Even though this paper only considers the existence of equilibrium in abstract economies, we think the results given in this paper can be used, by applying standard techniques, to extend the existence theorems on general equilibrium in the literature such as those in Bewley (1972), Bojan (1974), El'Barkuki (1977), Mas-Colell (1986), and Zame (1987) to economies with non-compact feasible sets, and discontinuous, interdependent, and price-dependent preferences so that we do not need to assume the existence of a weaker topology to guarantee the compactness of the feasible sets. Also the results may be used to extend the equilibrium existence theorems on N-person social coalitional games [Ichiishi (1981)] in the literature. Further, our results extend the results of Aubin (1979), Zhou and Chen (1988), and Tian and Zhou (1991) on a foundational quasi-variational inequality by relaxing the compactness and/or concavity conditions. Thus many existence theorems in quasi-variational inequalities literature can be generalized by our results.

The remainder of this paper is organized as follows. Section 2 states some notation, definitions, and mathematical preliminaries. The main existence theorems and their proofs are given in section 3. Finally, some concluding remarks are presented in section 4.

2. Notation, definitions, and framework

2.1. Mathematical preliminaries

Let X and Y be two topological spaces, and let 2^Y be the collection of all subsets of Y. A correspondence $F: X \to 2^Y$ is said to be *upper semi-continuous* (in short, u.s.c.) if the set $\{x \in X: F(x) \subset V\}$ is open in X for every open subset V of Y. A correspondence $F: X \to 2^Y$ is said to be *lower semi-continuous* (in short, l.s.c.) if the set $\{x \in X: F(x) \cap V \neq \varnothing\}$ is open in X for every open subset V of Y. A correspondence $F: X \to 2^Y$ is said to be *continuous* if it is both u.s.c. and l.s.c. A correspondence $F: X \to 2^Y$ is said to have *open lower sections* if the set $F^{-1}(y) = \{x \in X: y \in F(x)\}$ is open in X for every $y \in Y$. A correspondence $F: X \to 2^Y$ is said to have *open upper sections* if, for every $x \in X$, $F(x)$ is open in Y. A correspondence $F: X \to 2^Y$ is said to be *closed* if the correspondence has a closed graph, i.e., the set $\{(x, y) \in X \times Y: y \in F(x)\}$ is closed in $X \times Y$. A correspondence $F: X \to 2^Y$ is said to have an *open graph* if the set $\{(x, y) \in X \times Y: y \in F(x)\}$ is open in $X \times Y$. Denoted by $\operatorname{co} B$ and $\bar{B}$ denotes the convex hull and the closure of the set B.

Let X be a topological space. A function $f: X \to \mathbb{R} \cup \{-\infty\}$ is said to be *upper semi-continuous* (in short, u.s.c.) on X if for each point $x' \in X$, we have

$$\limsup_{x \to x'} f(x) \leqq f(x'),$$

or equivalently, its epigraph $\operatorname{epi} u \equiv \{(x, a) \in X \times \mathbb{R}: f(x) \geqq a\}$ is a closed subset of $X \times \mathbb{R}$. A function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semi-continuous* (in short, l.s.c.) on X if $-f(x)$ is upper semi-continuous on X.

A functional $\phi(x, y): X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *0-diagonally quasi-concave* (in short, 0-DQCV) in y [cf. Zhou and Chen (1988)], if for any finite subset $\{y_1, \ldots, y_m\} \subset X$ and any $y_\lambda \in \operatorname{co}\{y_1, \ldots, y_m\}$ (i.e., $y_\lambda = \sum_{j=1}^m \lambda_j y_j$ for $\lambda_j \geqq 0$ with $\sum_{j=1}^m \lambda_j = 1$),

$$\min_j \phi(y_\lambda, y_j) \leqq 0.$$

A functional $\phi(x, y): X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is to be *0-diagonally quasi-convex* (in short, 0-DQCX) in y if $-\phi(x, y)$ is 0-diagonally quasi-concave.

Remarks 1. Zhou and Chen (1988) gave a class of diagonal (quasi-) concavity (convexity) conditions which are weaker than the usual (quasi-) concavity (convexity) conditions and from which many theorems in convex analysis and (quasi-) variational inequalities can be generalized.

We first state some technical lemmas which are needed in the discussions of the existence of equilibria for abstract economies below.

Lemma 1. Let X and Y be two topological spaces, and let $F: X \to 2^Y$ be an upper semi-continuous correspondence with closed values. Then F is closed.

Lemma 2. Let X and Y be two topological spaces, and $F: X \to 2^Y$ be a closed correspondence. Then F is upper semi-continuous if Y is compact.

Lemma 3. Let X and Y be two topological spaces, and $\phi: X \to 2^Y$ and $\psi: X \to 2^Y$ be correspondences having open lower sections. Then the correspondence $\theta: X \to 2^Y$ defined by, for all $x \in X$, $\theta(x) = \phi(x) \cap \psi(x)$, has open lower sections.

Lemma 4. Let X be a topological space and let Y be a convex subset of a *topological vector space. Suppose a correspondence $\phi: X \to 2^Y$ has open lower sections. Then the correspondence $\psi: X \to 2^Y$ defined by $\psi(x) = \operatorname{co} \phi(x)$ for all $x \in X$ has open lower sections.*

Lemma 5. *Let X be a paracompact Hausdorff space and Y be a topological vector space. Suppose $F: X \to 2^Y$ is a correspondence with nonempty convex values and has open lower sections. Then there exists a continuous function $f: X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.*

The proofs of Lemmas 1 and 2 can be found, e.g., in Aubin and Ekeland (1984, ch. 3). The proofs of Lemmas of 3–5 can be found in Yannelis and Prabhakar (1983).

We now extend a fundamental quasi-variational inequality [see, e.g., Aubin (1979, Theorem 9.3.1) or Aubin and Ekeland (1984, Theorem 6.4.21)] by relaxing the concavity condition. This also extends a result of Zhou and Chen (1988, Theorem 3.1) by relaxing the 0-DCV condition.

Proposition 1. *Let Z be a nonempty, compact, convex, and metrizable subset in a locally convex Hausdorff topological vector space. Suppose that*

(*a*) *$K: 2^Z$ is a correspondence with nonempty convex values and has open lower sections such that $\bar{K}: Z - 2^Z$ is u.s.c.,*
(*b*) *$\phi: Z \times Z \to \mathbb{R} \cup \{\pm\infty\}$ is lower semi-continuous in x for any $y \in Z$ and is 0-diagonally quasi-concave in y for any $x \in Z$.*

Then there exists $x^* \in \bar{K}(x^*)$ such that $\sup_{y \in K(x^*)} \phi(x^*, y) \leqq 0$.

Proof. Define a correspondence $P: Z \to 2^Z$ by, for each $x \in Z$, $P(x) = \{y \in Z: \phi(x, y) > 0\}$. Thus, to show the conclusions of the proposition, it is equivalent to show that there exists $x^* \in \bar{K}(x^*)$ such that $K(x^*) \cap P(x^*) = \varnothing$.

Since ϕ is l.s.c. in x, then for each $x \in Z$, $P^{-1}(y) = \{x \in Z: \phi(x, y) > 0\}$ is open. Thus P has open lower sections. Also, $x \notin \operatorname{co} P(x)$ for all $x \in Z$. To see this, suppose, by way of contradiction, that there exists some point $x_\lambda \in Z$ such that $x_\lambda \in \operatorname{co} P(x_\lambda)$. Then there exist finite points, $x_1, \ldots, x_m$ in Z, and $\lambda_j \geqq 0$ with $\sum_{j=1}^m \lambda_j = 1$ such that $x_\lambda = \sum_{j=1}^m \lambda_j x_j$ and $x_i \in P(x_\lambda)$ for all $i = 1, \ldots, m$. That is, $\phi(x_\lambda, x_i) > 0$ for all i, which contradicts the hypothesis that ϕ is 0-DQCV in y.

Define another correspondence $F: Z \to 2^Z$ by $F(x) = K(x) \cap \operatorname{co} P(x)$. Let $U = \{x \in Z: F(x) \neq \varnothing\}$. Since K and P have open lower sections in Z, they are in U. Then, by Lemma 4, $\operatorname{co} P$ has open lower sections in U. Hence, by Lemma 3, the corresdpondence $F|U: U \to 2^Z$ has lower open sections in U and for all $x \in U$, $F(x)$ is nonempty and convex. Also, since X is a metrizable space, it is paracompact [cf. Michael (1956, p. 831)]. Hence, by Lemma 5, there exists a continuous function $f: U \to Z$ such that $f(x) \in F(x)$ for all $x \in U$.

Note that, since F has open lower sections and thus is l.s.c. [cf. Yannelis and Prabhakar (1983, p. 237)], U is open. Define the correspondence $G: Z \to 2^Z$ by

$$G(x) = \begin{cases} \{f(x)\} & \text{if } x \in U \\ \bar{K}(x) & \text{otherwise.} \end{cases} \tag{1}$$

Then G is an upper semi-continuous correspondence with nonempty closed convex values. Hence, by Kakutani's fixed point theorem, there exists a point $x^* \in Z$ such that $x^* \in G(x^*)$. Note that, if $x^* \in U$, then $x^* = f(x^*) \in F(x^*) \subset \operatorname{co} P(x^*)$, a contradiction. Hence, $x^* \notin U$ and thus $x^* \in \bar{K}(x^*)$ and $K(x^*) \cap \operatorname{co} P(x^*) = \varnothing$ which implies $K(x^*) \cap P(x^*) = \varnothing$. □

2.2. *Abstract economies*

Let I be the set of agents which is any (finite or infinite) countable set. Each agent has a choice set X_i, a constraint correspondence[4] $A_i: X_i \to 2^{X_i}$, and a payoff (utility) function $u_i: \prod_{j \in I} X_j \to \mathbb{R} \cup \{\pm\infty\}$, where $X = \prod_{i \in I} X_i$. Denote by X_{-i} and A the product $\prod_{j \in I \setminus \{i\}} X_j$ and the product $\prod_{i \in I} A_j$. Denote by x and x_{-i} an element of X and an element of X_{-i}, respectively.

An *abstract economy* (*or a generalized game*) $\Gamma = (X_i, A_i, u_i)_{i \in I}$ is defined as a family of ordered triples (X_i, A_i, u_i). Following Borglin and Keiding (1976), or Yannelis and Prabhakar (1983), an *equilibrium* for Γ is an $x^* \in X$ such that $x^* \in \bar{A}(x^*)$ and $u_i(x^*) \geqq u(x^*_{-i}, x_i)$ for all $x_i \in A(x^*_{-i})$ and all $i \in I$.

If $A_i(x_{-i}) \equiv X_i, \forall i \in I$, the abstract economy reduces to the conventional game $\Gamma = (X_i, u_i)$ and the equilibrium is called a *Nash equilibrium*.

Now we introduce the functional $U: X \times X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$U(x, y) = \sum_{i \in I} \frac{1}{2^i} [u_i(x) - u_i(x_{-i}, y_i)]. \tag{2}$$

The equivalence of the variational inequality (the quasi-variational inequality) and finding a Nash equilibrium point of the game (a generalized N person game) is well-known and due to Nikaido and Isoda (1955). The proof, however, is only given for the case of the finite number of agents. The following lemma is a version for abstract economies with an infinite number of agents.

Lemma 6. A point $x^* \in X$ is an equilibrium for an abstract economy Γ if and only if $x^* \in \bar{A}(x^*)$ and it satisfies

[4]Note that this equilibrium concept only makes sense from an economic point of view if A_i is independent of player i's choice. If this were not the case, we might have an equilibrium, but changing the player i's choice changes A_i so as to admit a preferable strategy.

$$\inf_{y \in A(x^*)} U(x^*, y) \geqq 0. \tag{3}$$

Proof. Let $x^* \in \bar{A}(x^*)$ be a solution of (3) and let $y = (x^*_{-i}, y_i)$. Then we have

$$\frac{1}{2^i} [u_i(x^*) - u_i(x^*_{-i}, y_i)] \geqq 0 \tag{4}$$

for any $y_i \in A_i(x^*_{-i})$. So x^* is an equilibrium of the abstract economy. The converse is obvious by summing up (4) for all agents. □

3. Main results

The main results of this paper are the following theorems:

Theorem 1. Let Z be a nonempty compact convex metrizable subset in a locally convex Hausdorff topological vector space E. Suppose that

(i) *the correspondence $A: Z \to 2^Z$ has nonempty convex values and open lower sections as well as $\bar{A}: Z \to 2^Z$ is u.s.c.,*
(ii) *$U: Z \times Z \to \mathbb{R} \cup \{\pm\infty\}$ is upper semi-continuous in x and is 0-diagonally quasi-convex in y.*

Then Γ has an equilibrium.

Proof. Let $\phi(x, y) = -U(x, y)$. Then $\phi: Z \times Z \to \mathbb{R} \cup \{\pm\infty\}$ is lower semi-continuous in x and is 0-diagonally quasi-concave in y by Assumption (ii). Therefore, by Proposition 1, there exists $x^* \in \bar{A}(x^*)$ such that $\sup_{y \in K(x^*)} \phi(x^*, y) \leqq 0$ and thus $\inf_{y \in A(x^*)} U(z^*, y) \geqq 0$. Finally, from Lemma 6, we know that $x^* \in Z$ is an equilibrium of the abstract economy. □

Remark 2. A sufficient condition for $U(x, y)$ to be upper semi-continuous in x is that u_i is upper semi-continuous in x_i and continuous in x_{-i}.

Remark 3. The assumption that U is 0-diagonally quasi-convex in y is equivalent to

$$\sum_{i \in I} \frac{1}{2^i} u_i(x_\lambda) \geqq \min_j \sum_{i \in I} \frac{1}{2^i} u_i(x_{-i\lambda}, y_{ij}) \tag{5}$$

for any finite subset $\{x_1, \ldots, x_m\} \subset X$ and any $x_\lambda \in \text{co}\{x_1, \ldots, x_m\}$ which implies that each u_i is quasi-concave in x_i.

When $A \equiv Z$, the abstract economy reduces to the conventional game, as a corollary of the above theorem, we have an existence theorem on Nash equilibrium for games with discontinuous payoffs, which generalize the results of Nash (1950, 1951) and Nikaido and Isoda (1955).

Corollary 1. Let Z be a nonempty compact convex metrizable subset in a locally convex Hausdorff topological vector space E. Suppose that $U: Z\times Z\to\mathbb{R}\cup\{\pm\infty\}$ is upper sem-continuous in x and is 0-diagonally quasi-convex in y. Then Γ has an equilibrium.

The Theorem 1 can be generalized by relaxing the compactness of the strategy space.

Theorem 2. Let X be a convex metrizable subset in a locally convex Hausdorff topological vector space E. Suppose that

(i) the correspondence $A: X\to 2^X$ has nonempty convex values and open lower sections as well as $\bar{A}: Z\to 2^Z$ is u.s.c.,
(ii) $U: X\times X\to\mathbb{R}\cup\{\pm\infty\}$ is upper semi-continuous in x and is 0-diagonally quasi-convex in y,
(iii) there exists a nonempty compact convex set $C\subset X$ such that
  (iii.1) $A_i(C)$ is contained in a compact convex set $D_i\subset X_i$, where $C=\prod_{i\in I} C_i$;
  (iii.2) $A_i(x)\cap Z_i\neq\varnothing$ for all $x\in X_{-i}\times Z_i$, where $Z_i=\mathrm{co}\{D_i\cup C_i\}$;
  (iii.3) for each $x_i\in Z_i\backslash C_i$ and $x_{-i}\in X_{-i}$ there exists $y_i\in A_i(x)\cap Z_i$ such that $u_i(x_{-i}, y_i)>u_i(x_{-i}, x_i)$.

Then Γ has an equilibrium.

Proof. Since C_i and D_i are compact and convex subsets of X_i, then Z_i is compact convex. Let $X=\prod_{i\in I} Z_i$. For each $i\in I$, define a correspondence $K_i: Z\to 2^{Z_i}$ by, for each $x\in Z$,

$$K_i(x)=A_i(x)\cap Z_i. \tag{6}$$

Then, by condition (i) and properties of Z_i, $K_i(x)$ is nonempty and convex for all $x\in Z$. Since Z_i is compact and $\bar{A}_i$ is closed by Lemma 1, $\bar{K}_i$ is closed and therefore is upper semi-continuous on Z by Lemma 2. Also, note that

$$K(x)=\begin{cases} A(x) & \text{if } x\in C\\ A(x)\cap Z & \text{otherwise.}\end{cases} \tag{7}$$

Hence, by Theorem 1, there is $x^*\in\bar{K}(x^*)$ such that $\sup_{y\in K(x^*)}\phi(x^*, y)\leqq 0$. Now $x^*\in C$, for otherwise Hypothesis (iii.3) would be violated, and thus

$x^* \in \bar{K}(x^*) = \bar{A}(x^*)$. Then $\sup_{y \in A(x^*)} \phi(x^*, y) \leqq 0$ and thus $\inf_{y \in A(x^*)} U(x^*, y) \geqq 0$. □

Remark 4. From Lemma 6, we know that Theorem 2 is a generalization of Proposition 1 by relaxing the compactness condition of the choice set.

Remark 5. Observe that in case of a compact convex X, Assumptions (iii.1)–(iii.3) in Theorem 2 are satisfied by $C = X$. Assumption (iii.2) is needed to guarantee the existence of a fixed point of $A(x)$.[5] Assumption (iii.3) guarantees the fixed point $x^* \in C$.

4. Concluding remarks

In this paper, we have considered equilibrium existence for abstract economies with non-compact infinite-dimensional strategy spaces, infinitely many agents, and discontinuous and non-monotonic utility functions. Thus our results can be easily used to prove the existence of a competitive equilibrium without the continuity, compactness, and interiority assumptions and some 'bounded' restrictions on the marginal rate of (technical) substitution on consumption (production) sets imposed by Mas-Colell (1986), Zame (1987) and others. Further, our results also extend a foundational quasi-variational inequality by relaxing the compactness and concavity conditions. Thus many existence theorems in the quasi-variational inequalities literature can be generalized by our results.

[5]Tian (1991) proved that a necessary and sufficient condition for the existence of fixed points of an upper semi-continuous correspondence with nonempty closed values F defined on any subset (which may be non-compact and non-convex) of a locally convex Hausdorff topological vector space is that there exists a compact convex subset $B \subset X$ such that $F(x) \cap B \neq \varnothing$ for all $x \in B$.

References

Allen, G., 1977, Variational inequalities, complementarity problems, and duality theorems, Journal of Mathematical Analysis and Applications 58, 1–10.

Arrow, K. and G. Debreu, 1954, Existence of equilibrium for a competitive economy, Econometrica 22, 265–290.

Aubin, J.P., 1979, Mathematical methods of game and economic theory (North-Holland, Amsterdam).

Aubin, J.P. and I. Ekeland, 1984, Applied nonlinear analysis (Wiley, New York).

Bewley, T., 1972, Existence of equilibria with infinitely many commodities, Journal of Mathematical Economics 4, 514–540.

Borglin, A. and H. Keiding, 1976, Existence of equilibrium actions and of equilibrium: A note on the 'new' existence theorems, Journal of Mathematical Economics 3, 313–316.

Border, K.C., 1985, Fixed point theorems with applications to economics and game theory (Cambridge University Press, Cambridge).

Bojan, P., 1974, A generalization of theorems on the existence of competitive economic equilibrium to the case of infinitely many commodities, Mathematical Nalkanika 4, 490–494.

Debreu, G., 1952, A social equilibrium existence theorem, Proceedings of the National Academy of Sciences of the U.S.A. 38, 386–393.

El'Barkuki, R.A., 1977, The existence of an equilibrium in economic structure with a banach space of commodities, Akad. Nauk. Azerbaidjan, SSR Dokl. 33, no. 5, 8–12.

Dasgupta, P. and E. Maskin, 1986, The existence of equilibrium in discontinuous economic games, I: Theory, Review of Economic Studies 53, 1–26.

Ichiishi, T., 1981, A social coalitional equilibrium existence lemma, Econometrica 49, 369–377.

Khan, M. Ali, 1986, Equilibrium points of nonatomic games over a banach space, Transactions of the American Mathematical Society 29, 737–749.

Khan, M. Ali and R. Vohra, 1984, Equilibrium in abstract economies without ordered preferences and with a measure space of agents, Journal of Mathematical Economics 13, no. 2, 133–142.

Kim, T., K. Prikry, and N.C. Yannelis, 1989, Equilibrium in abstract economies with a measure space and with an infinite dimensional strategy space, Journal of Approximation Theory 56, 256–266.

Mas-Colell, A., 1986, The price equilibrium existence problem in topological vector lattices, Econometrica 54, 1039–1054.

Nash, J., 1950, Equilibrium points in N-person games, Proceedings of the National Academy of Sciences 36, 48–49.

Nash, J., 1951, Non-cooperative games, Annals of Mathematics 54, 286–295.

Nikaido, A. and K. Isoda, 1955, Note on noncooperative convex games, Pacific Journal of Mathematics 5, 807–815.

Shafer, W. and H. Sonnenschein, 1975, Equilibrium in abstract economies without ordered preferences, Journal of Mathematical Economics 2, 345–348.

Tian, G., 1991, Fixed points theorems for mappings with non-compact and non-convex domains, Journal of Mathematical Analysis and Applications 158, 161–167.

Tian, G. and J. Zhou, 1991, Quasi-variational inequalities with non-compact sets, Journal of Mathematical Analysis and Applications 160, 583–595.

Tian, G. and J. Zhou, 1992, The maximum theorem and the existence of Nash equilibrium of (generalized) games without lower semicontinuities, Journal of Mathematical Analysis and Applications, forthcoming.

Yannelis, N.C., 1987, Equilibria in noncooperative models of competition, Journal of Economic Theory 41, 96–111.

Yannelis, N.C. and N.D. Prabhakar, 1983, Existence of maximal elements and equilibria in linear topological spaces, Journal of Mathematical Economics 12, 233–245.

Zame, W.R., 1987, Competitive equilibria in production with an infinite-dimensional commodity space, Econometrica 55, 1075–1108.

Zhou, J.X. and G. Chen, 1988, Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities, Journal of Mathematical Analysis and Applications 132, 213–225.