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On informational efficiency and incentive aspects of generalized ratio equilibria

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In this paper, we investigate the informational efficiency and incentive aspects of the Generalized Ratio Equilibrium (GRE) rule in public goods economies with general variable returns and by-product technologies. We extend Hurwicz's (1979a) results to the cases of non-constant returns and prove that any implementable social choice correspondence that selects Pareto efficient and autarkically individually rational allocations contains GRE allocations, and conversely, the set of GRE allocations contains the interior points of the correspondence. Moreover, we prove that the GRE process is informationally the most efficient process among informationally decentralized and Pareto efficient resource allocation processes and thus that the minimal message space needed for the GRE correspondence is never larger than that needed for the Lindahl correspondence, and that when by products technologies are present, it is smaller than that needed by the Lindahl correspondence.

Key words: Mechanism design; Incentives; Informational efficiency; Public goods
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1. Introduction

This paper considers informational efficiency and incentive aspects of the Generalized Ratio Equilibrium (GRE) process that selects Pareto efficient and individually rational allocations in public goods economies with general variable returns and by-product technologies when agents have free access to the production technologies. The incentives and informational requirements are two basic aspects that a social system in general, and an economic system in particular, needs to consider. In studying the efficiency, incentives, and informational requirements of a resources allocation process in economies with public goods through the general equilibrium approach, a widely

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used general equilibrium principle is the conventional Lindahl equilibrium process which selects Pareto efficient and individually rational allocations.

The Lindahl equilibrium principle, however, has some drawbacks. For example, production under the Lindahl equilibrium principle must take place at a price-taking, profit-maximizing point, precluding the existence of an equilibrium if increasing returns to scale (IRS) are present. Furthermore, positive profits must be distributed in accordance with some exogenously given profit distributions. And, unlike the Walrasian equilibrium principle, the core equivalence result for the Lindahl equilibrium does not hold even in the case of a continuum of consumers; the core of a public goods economy can be much larger than the set of Lindahl allocations [cf. Muench (1972)]. Thus, Edgeworth's conjecture, as interpreted through the core, cannot be extended to public goods economies with respect to the Lindahl solution. In addition, even though a Lindahl allocation is individually rational, it may not be autarkically individually rational in the sense that some agent can be better off if he/she consumes a commodity bundle that can be achieved solely by his/her endowment and the technology without using other agents' endowments. This is true for the strict convex technologies [cf. Wilkie (1990)].

In this paper, we consider an alternative selection rule, the Generalized Ratio Equilibrium, which extends the Ratio equilibrium introduced by Kaneko (1977) to allow the presence of by-products and general variable returns [decreasing returns to scale (DRS), constant returns to scale (CRS), and IRS] technologies at different scales of production. The GRE selects Pareto-efficient and autarkically individually rational allocations for public goods economies, and does not have the above mentioned drawbacks of the Lindahl equilibrium principle. Furthermore, as Tian and Li (1994b) did, it can be regarded as a general equilibrium solution concept for the state ownership system for public goods economies. And the GRE process is designed in a decentralized way, which yields efficient outcomes and guarantees that individual contributions are in line with individual benefits. We prove the existence of a GRE even in the presence of IRS by providing a so-called 'Transformation-Equivalence Principle' which shows that an allocation is a GRE allocation for some economy e if and only if the allocation for private goods and input demand for producing public goods is a Lindahl allocation for some other CRS economy e'.

We then investigate the incentive aspects and informational efficiency of the GRE principle. We generalize Hurwicz's (1979a, Theorems 3 and 4) characterization results to the cases of non-constant returns economies (including IRS) and autarkic individual rationality, and prove that the set of Nash allocations of any mechanism which implements Pareto efficient and autarkically individually rational allocations must contain the set of GRE allocations, and conversely, the set of GRE allocations contains the set of

interior Nash allocations. Moreover, we consider an additional virtue of the GRE process over the Lindahl correspondence that arises from consideration of the informational requirements (in the sense of message pace dimension) for 'realization' and 'implementation' of these correspondences, and prove that the GRE process is informationally the most efficient process among privacy preserving (informationally decentralized) and non-wasteful (Pareto efficient) resource allocation processes even in the presence of by-product technologies. Consequently, this implies that the Lindahl allocation process is not uniquely informationally efficient, and further, in the presence of by-products, is not informationally efficient. Thus the paper in this way shows that some results analogous to theorems of Hurwicz, Jordan, Mount and Reiter, Walker, etc. that establish the informational superiority of the competitive mechanism in pure private goods exchange economies do not hold for Lindahl allocations in economies with public goods.[1]

The plan of this paper is as follows. In section 2 we give a formal description of the public goods model and the definition of GRE. We then consider its existence and other properties of the GRE. Section 3 gives some general results on the incentive and information aspects of the GRE principle. Concluding remarks are presented in section 4.

2. Model and generalized ratio equilibria

2.1. Economic environments

In an economy with public goods, there are n agents who consume one private good and K public goods, x being private (as a numeraire) and y public.[2] Denote by $N=\{1,2,\ldots,n\}$ the set of agents. Each agent's characteristic is denoted by $e_i=(w_i,R_i)$, where $w_i\in\mathbb{R}_{++}$ is the initial endowment of the private good and R_i is the preference ordering (P_i denotes the asymmetric part of the preference R_i) defined on $\mathbb{R}_+^{1+K}$ which is strictly monotone increasing. We assume that there are no initial endowments of the public goods, but that the public goods can be produced from the private good by T state-owned firms that may have CRS, DRS, or IRS technologies. Each firm t is given an exclusive franchise to produce K_t public goods, denoted by $y_t\in\mathbb{R}_+^{K_t}$. Then $\sum_{t=1}^{T}K_t=K$. Further we assume that the technology of

[1]Hurwicz (1972), Mount and Reiter (1974), Walker (1977), Hurwicz (1986b) among others showed that, for private goods economies, the competitive (Walrasian) allocation process has a message space of minimal dimension among a certain class of resource allocation processes that are privacy preserving and non-wasteful. Jordan (1982) proved that the competitive allocation process for exchange economies is uniquely informationally efficient. For a class of public goods economies without the presence of by-products, Sato (1981) obtained a similar result showing that the Lindahl allocation process has a message space of minimal dimension among a certain class of resource allocation processes that are privacy preserving and non-wasteful.

[2]As usual, vector inequalities are defined as follows: Let $a,b\in\mathbb{R}^m$. Then $a\geqq b$ means $a_s\geqq b_s$ for all $s=1,\ldots,m$; $a\geqq b$ means $a\geqq b$ but $a\neq b$; $a>b$ means $a_s>b_s$ for all $s=1,\ldots,m$.

production of each firm t is represented by a production function $f_t: \mathbb{R}_+ \to \mathbb{R}_+^{K_t}$ and thus $y_t = f_t(v_t)$, where v_t is the input used by firm t. Thus we allow the presence of by-products when $K_t > 1$.[3] Let $f(v) = (f_1(v_1), \ldots, f_T(v_T))$. The cost function (the input demand function) of firm t is denoted by $C_t(y_t)$. We assume throughout that $C_t(0) = 0$, and that C_t is increasing[4] and continuous. Note that the cost function $C_t(y_t)$ is not differentiable in the presence of by-product technologies. In this case, the cost function is given by $C_t(y_t) = \max\{(f_t^1)^{-1}(y_t^1), \ldots, (f_t^{K_t})^{-1}(y_t^{K_t})\}$, where $(f_t^k)^{-1}(y_t^k)$ is the inverse function of $f_t^k(v_t)$. Let $C(y) = (C_1(y_1), \ldots, C_T(y_T))$. An economy is the full vector $e = (e_1, \ldots, e_n, C^1(\cdot), \ldots, C^T(\cdot))$ and the set of all such economies is denoted by E.

2.2. Generalised ratio equilibria

An allocation $(x, y) = (x_1, \ldots, x_n, y)$ is *feasible* for an economy e if $(x, y) \in \mathbb{R}_+^{n+K}$ and

$$\sum_{i=1}^{n} x_i + \iota \cdot C(y) \leqq \sum_{i=1}^{n} w_i, \tag{1}$$

where ι is a vector of ones of dimension T. An allocation (x, y) is *Pareto efficient* for an economy e if it is feasible and there is no other feasible allocation (x', y') such that $(x_i', y')R_i(x_i, y)$ for all $i \in N$ and $(x_j', y')P_j(x_j, y)$ for some $j \in N$. Denote by $\mathscr{P}(e)$ the set of Pareto efficient allocations for economy e. An allocation (x, y) is *individually rational* for an economy e if $(x_i, y)R_i(w_i, 0)$ for all $i \in N$. An allocation (x, y) is *autarkically individually rational*, or *free access individually rational* if for each agent i, $(\bar{x}_i, \bar{y})R_i(x_i, y_i)$. Here $(\bar{x}_i, \bar{y}_i)$ is a preference maximizer subject to $x_i + C(y_i) \leqq w_i$. The notion of autarkic individual rationality was given by Moulin (1989) and Saijo (1991), which may be a more reasonable definition than individual rationality for the case of strict convex technologies. It is clear that every autarkically individually rational allocation is individually rational, but the reverse statement may not be true unless there is no production. Denote by $AIR(e)$ the set of autarkically individually rational allocations and define $S(e) = \mathscr{P}(e) \cap AIR(e)$ for the economy e. A *coalition* B is a nonempty subset of N. An allocation (x, y) can be *improved by* B if there exists another allocation $(x_i', y')_{i \in B}$ such that (1) $\sum_{i \in B} x_i + \iota \cdot C(y) \leqq \sum_{i \in B} w_i$, and (2) $(x_i', y')R_i(x_i, y)$ for all $i \in B$ with strict preference for some $j \in B$. An allocation (x, y) is *a core allocation* if it cannot be improved by any coalition B. Note that the definition of the core

[3]That is, producing one public good automatically produces another, e.g., building a dam for electricity produces a lake for boating.

[4]I.e., $C(y') > C(y)$ if $y' > y$.

includes Pareto efficiency and autarkic individual rationality as special cases, which correspond to $B=n$ and $B=1$, respectively.

Definition 1. A Generalized Ratio Equilibrium (GRE) for an economy e is a pair formed by a feasible allocation $(x^*, y^*) \in \mathbb{R}_+^{n+K}$ and ratio vectors $r_i^* \in \mathbb{R}_+^T$, one for each i, such that

(a) $x_i^* + r_i^* \cdot C(y^*) \leqq w_i$ for all $i \in N$;
(b) for all $i \in N$, there does not exist $(x_i, y) \in \mathbb{R}_+^{1+K}$ such that $(x_i, y) P_i (x_i^*, y^*)$ and $x_i + r_i^* \cdot C(y) \leqq w_i$;
(c) $\sum_{i=1}^n r_i^* = \iota$.

The allocation (x^*, y^*) is then called GRE allocation. Denote by $GRE(e)$ the set of all such allocations. Condition (a) is the budget constraint of agent i, which means that the sum of the person's consumption on the private good and the cost shares of producing the public goods (in terms of the private good) should not exceed the person's endowment of the private good. Condition (b) is just the utility maximization subject to his budget constraint. Condition (c) means that the sum of all individual's percentages (ratios) of sharing the production cost is equal to one. Observe that the GRE reduces to the Ratio Equilibrium (RE) of Kaneko (1977) when $K_t = 1$ for all t's.

Let $\Delta^{n-1} = \{\theta \in \mathbb{R}_+^n : \sum_{i=1}^n \theta_i = 1\}$ be the $(n-1)$-dimensional unit simplex.

Definition 2. Given a profit share vector $\theta \in \Delta^{n-1}$, a *θ-Lindahl equilibrium allocation* for an economy e is a pair formed by a feasible allocation (x^*, y^*) and a system of personalized price vectors, $q_i^* \in \mathbb{R}^K$, one for each, such that

(1) y^* maximizes profits $q^* \cdot y - C(y)$;
(2) $x_i^* + q_i^* \cdot y^* \leqq w_i + \theta_i [q^* \cdot y^* - C(y^*)]$ *for all* $i \in N$;
(3) for all $i \in N$, there does not exist (x_i, y) such that $(x_i, y) P_i (x_i^*, y^*)$ and $x_i + q_i^* \cdot y \leqq w_i + \theta_i [q^* \cdot y^* - C(y^*)]$;
(4) $\sum_{i=1}^n q_i^* = q^*$.

Denote by $L(e; \theta)$ the set of all such allocations. It is well known that every Lindahl allocation is Pareto efficient.

Proposition 1. If (x, y) is a GRE allocation, then it is in the core and thus it is Pareto efficient and AIR.

Proof. The proof is essentially the same as the one for Ratio allocations which was given by Kaneko (1977) and thus omitted here. Q.E.D.

The remainder of this section considers the existence of a GRE. It may be remarked that we cannot follow the approaches used by Kaneko (1977) or Mas-Colell and Silvestre (1989) to prove the existence of a GRE since these

approaches require that preference orderings be convex and production technologies of state-owned firms display no IRS. Here we give another approach which shows that proving the existence of a GRE for the original economy is equivalent to proving the existence of a Lindahl equilibrium for the transformed CRS economy. We call this approach the 'Transformation Equivalence Principle' (TE-Principle) which plays an important role in this paper.

For an economy e with $e_i=(w_i, R_i)$ and production functions $f_t: \mathbb{R}_+ \to \mathbb{R}_+^{K_t}$ which displays CRS, DRS, or IRS, we can define a CRS economy e' with $e_i'=(R_i', w_i)$, x being a private good, $v \in \mathbb{R}_+^T$ being public goods, and preferences R_i' defined by $(x_i, v)R_i'(\bar{x}_i, \bar{v})$ if and only if $(x_i, f(v))R_i(\bar{x}_i, f(\bar{v}))$.[5]

Proposition 2 (TE-Principle). For any economy e, if (x^*, y^*) is a GRE allocation with a ratio system $(r_1^*, \ldots, \tau_N^*) \in R_+^{NT}$, then $(x^*, v^*) \equiv (x^*, C(y^*))$ is a Lindahl allocation with $(r_1^*, \ldots, r_N^*)$ as the personalized price system for the transformed CRS economy e'; and if (x^*, v^*) is a Lindahl allocation with a personalized price system $(q_1^*, \ldots, q_N^*) \in R_+^{NT}$ for the transformed CRS economy e', then $(x^*, y^*) \equiv (x^*, f(v^*))$ is a GRE allocation with $(q_1^*, \ldots, q_N^*)$ as the ratio system for the original economy e.

Proof. We first note that if (x^*, y^*) is a GRE allocation with a ratio system $(r_1^*, \ldots, r_N^*) \in R_+^{NT}$ for e, then, by monotonicity of preferences, $f(C(y^*))=y^*$ (i.e., y^* is produced in an efficient way). We then want to show that if (x^*, y^*) is a GRE allocation with a ratio system $(r_1^*, \ldots, r_N^*) \in R_+^{NT}$ for e, then $(x^*, v^*)=(x^*, C(y^*))$ is a Lindahl allocation with $(r_1^*, \ldots, r_N^*)$ as a personalized price system for e'. Suppose, by way of contradiction, that there is some $i \in N$ and (x_i, v) such that $x_i + r_i^* \cdot v \leqq w_i$ and $(x_i, f(v))P_i(x_i^*, f(v^*))$. Let $y=f(v)$. Then $(x_i, y)P_i(x_i^*, f(v^*))$ and yet $x_i + r_i^* \cdot C(y) \leqq x_i + r_i^* \cdot v \leqq w_i$. But this contradicts the hypothesis that (x^*, y^*) is a GRE allocation.

We now show that $(x^*, y^*)=(x^*, f(v^*))$ is a GRE allocation with $(q_1^*, \ldots, q_N^*) \in R_+^{NT}$ as a ratio system if (x^*, v^*) is a Lindahl allocation with a personalized price system $(q_1^*, \ldots, q_N^*) \in R_+^{NT}$. Suppose, by way of contradiction, that there are some $i \in N$ and (x_i, y) such that $x_i + q_i^* \cdot C(y) \leqq w_i$ and $(x_i, y)P_i(x_i^*, y^*)$. Since $f(C(y)) \geqq y$, then $(x_i, f(C(y)))P_i(x_i^*, f(v^*))$. Let $v=C(y)$. Then we have $(x_i, f(v))P_i(x_i^*, f(v^*))$ and $x_i + q_i^* \cdot v \leqq w_i$. But this contradicts the hypothesis that (x^*, v^*) is a Lindahl allocation. Q.E.D.

Remark 1. Proposition 2, though simple, is useful in proving the existence of a GRE and in discussing incentive and information aspects of the GRE rule by mapping the original economy e to the CRS economy e'. Virtually all

[5]When preferences R_i can be represented by utility functions $u_i(x_i, y)$, R_i' are defined by utility functions $u_i'(x_i, v) \equiv u(x, f(v))$.

of the main results in this paper follow by inverting this mapping to draw 'nice' properties of the Lindahl allocations in the ideal setting of constant returns to scale (such as the existence of a Lindahl equilibrium, the characterization results of Hurwicz (1979a) on the efficiency and individual rationality of a social choice correspondence, the informational efficiency of the Lindahl allocation process without the presence of by-products, etc.) back to the more complex case of IRS. It also gives a simple way to find the GRE by finding the conventional Lindahl equilibrium (cf. Example 1 below). After obtaining the existence and/or solution of Lindahl equilibrium for e' and mechanisms which implement the Lindahl correspondence (there are many such mechanisms in the literature), we transform them back and consequently show the existence of a GRE and implementability of the GRE correspondence. Note that even though the original economy is not smooth in the presence of by-product technologies (since C_t is not differentiable), the transformed economy may be smooth since it has a CRS technology to produce the goods. In the following, we will use this proposition to obtain some interesting results. The first application of the TE-Principle is to prove the existence of a GRE even if the technology displays IRS for some cases.

Theorem 1. For an economy e, if the transformed preference orderings R_i' are continuous and convex,[6] *then there exists a GRE for the economy e. In particular, when preferences R_i can be represented by utility functions $u_i(x_i, y)$, there exists a GRE if the composite function $u(x, f(v))$ is continuous and quasi-concave.*

Proof. Since the transformed economy e' is a CRS economy and transformed preference orderings R_i' are continuous and convex, we know the existence of a Lindahl equilibrium from the result of Foley (1970), Milleron (1972), Roberts (1974). Thus there is a GRE by the TE-Principle. Q.E.D.

Remark 2. Since the continuity and convexity of transformed preference orderings (the continuity and quasi-concavity of the composite utility functions) are general conditions for the existence of a Lindahl equilibrium for CRS economies, Theorem 1 is a general result for the existence of a GRE for public goods economies even with the presence of IRS by the TE-Principle. It can be easily verified that a sufficient condition for transformed preference orderings R_i' to be convex is that preferences R_i are convex and f is concave.

Remark 3. One could easily find examples where, even if production functions exhibit IRS, utility functions of agents after transformation can be

[6]A preference R_i' is convex if, for bundles a, b, c and $0<\lambda\leqq 1$, $c=\lambda a+(1-\lambda)b$, the relation $aP_i'b$ implies $cP_i'b$.

quasi-concave, and thus, by the TE-Principle we know the existence of a GRE. In fact, as long as the transformed utility function u_i' is Cobb–Douglas, it is then quasi-concave [cf. Berge (1963, p. 209)]. Note that this is true if utility functions and production functions are both Cobb–Douglas type functions (with homogeneity of any degree). More generally, we can allow production functions to display different returns at different production scales which result in the standard U-shaped cost functions.

3. Incentives and informational properties of the GRE

In this section we consider the issues of information and incentives of the GRE process by using the TE-principle and the results in the mechanism design literature. To do so, we first give some notation and definitions which are standard in the mechanism design theory literature [say, in Hurwicz (1986a, 1986b), or Sato (1981)].

Let F be a social choice rule, i.e., a correspondence from E to the set Z of resource allocations. Let $\mathcal{M}_i$ denote the ith agent's message (strategy) domain. Its elements are written as m_i and called messages. Let $\mathcal{M}=\prod_{i=1}^{n}\mathcal{M}_i$ denote the message (strategy) space. Let $u_i(e_i)$ be the set of stationary (or equilibrium) messages of agent i. $\mu_i: E_i \to\to \mathcal{M}$ is called a *coordinate and privacy preserving correspondence*. Denote $\mu(e)=\bigcap_{i=1}^{n}\mu_i(e_i)$. We assume that the message correspondence μ is locally threaded, i.e., has locally a continuous, single-valued selection.[7] Let $h: \mathcal{M}\to Z$ denote the outcome function, or more explicitly, $h_i(m)=(X_i(m), Y(m), V(m))$, where $X_i(m)$ is the ith agent's outcome function for the private good, $Y(m)$ is the outcome function for the public goods, and $V(m)$ is the input demand outcome function needed to produce public goods $Y(m)$ and $Y(m)=f(V(m))$.

A privacy preserving resource allocation process (mechanism) $\langle \mathcal{M}, \mu, h\rangle$ defined on E *realizes* the choice rule F if for all $e\in E$, $\mu(e)\neq\emptyset$ and $h(m)\in F(e)$ for all $m\in\mu(e)$. A privacy preserving resource allocation process (mechanism) $\langle \mathcal{M}, \mu, h\rangle$ is said to be *non-wasteful* on E if for all $e\in E$, $\mu(e)\neq\emptyset$ and $h(m)$ is Pareto efficient for all $m\in\mu(e)$. The notion of informational size can be considered a concept which characterizes the relative sizes of topological spaces that are used to convey information in the resource allocation process. This suggests the following definition. Let M and S be two topological spaces. The space M is said to have as much information as the space S by the Fréchet ordering, denoted by $M\geqq_F S$ if S can be embedded homeomorphically in M, i.e., if there is a subspace of M' of M which is homeomorphic to S.

[7]That is, for every $e\in E$, there is a neighborhood $\mathcal{O}(e)\subset E$ and a continuous function $g: E\to M$ such that $g(e')\in\mu(e')$ for all $e'\in\mathcal{O}(e)$.

The literature on incentives has approached the mechanism design problem as one of designing an appropriate noncooperative game which consists of a message space and an outcome function.[8] A message $m^* = (m_1^*, \ldots, m_n^*) \in \mathcal{M}$ is a *Nash equilibrium* (NE) of the game form $\langle \mathcal{M}, h \rangle$ for an economy e if for any $i \in N$ and for all $m_i \in \mathcal{M}_i$,

$$(X_i(m^*), Y_i(m^*)) R_i (X_i(m^*_{-i}, m_i), Y_i(m^*_{-i}, m_i)), \tag{2}$$

where $(m^*_{-i}, m_i) = (m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. The $h(m^*)$ is then called a *Nash (equilibrium) allocation.* Denote by $V_{\mathcal{M},h}(e)$ the set of all such Nash equilibria and by $N_{\mathcal{M},h}(e)$ the set of all such Nash (equilibrium) allocations. A mechanism $\langle \mathcal{M}, h \rangle$ fully *Nash-implements* the social choice rule F on E, if, for all $e \in E$, $N_{\mathcal{M},h}(e) \neq \emptyset$ and $N_{\mathcal{M},h}(e) = F(e)$. Many specific mechanisms have been constructed in the literature to implement various social choice correspondences which select Pareto efficient and individually rational allocations such as those in Hurwicz (1979b), Walker (1981), Hurwicz et al. (1984), Tian (1989, 1990, 1991, 1993) among others. In the following subsection we will give two general results on the incentive properties of GRE.

3.1. Some incentives results on GRE

Let $E_L \subset E$ be the set of economies in which R_i is a preference relation generalized by a utility function of the form: $u_i(x_i, y) = x_i + \beta_i \cdot y$ for some $\beta_i > 0$ for all $i \in N$, and production technologies display CRS. Let $E_C \subset E$ be the set of economies in which R_i is a preference relation generalized by a utility function of the form: $u_i(x_i, y) = x_i + (1-\theta_i)\beta_i \cdot y + \theta_i \sum_k q_k \ln(y_k + d_k)$ with $0 < \theta_i < 1$; $\beta_i > 0$; $q_k > 0$; $d_k > 0$ for $i \in N$ and $k = 1, \ldots, K$ and production technologies display CRS. Let $E_{LC} = E_L \cup E_C$.

The following two theorems extend Hurwicz's (1979a) results to the cases of non-constant returns and autarkic individual rationality.

Theorem 2. Let $\bar{E} \subset E$ be a class of public goods economies which contains E_{LC} as a subset. Let the outcome function h have the following properties:

(i) for each $e \in \bar{E}$, $N_{\mathcal{M},h}(e) \subset S(e)$;
(ii) $N_{\mathcal{M},h}(e) \neq \emptyset$ for all $e \in E_{LC} \subset \bar{E}$;
(iii) if $e_v \in E_{LC}$ and $z_v \in N_{\mathcal{M},h}(e_v)$ for $v = 1, 2, \ldots$; $\lim e_v = e_0 \in E_L$; and $\lim z_v = z_0$; then $z_0 \in N_{\mathcal{M},h}(e_0)$.

Then $GR(e) \subset N_{\mathcal{M},h}(e)$ for all $e \in \bar{E}$.

[8]The game form is also called a mechanism in the incentives literature. Note that messages are chosen in whole space not only in the set of stationary messages.

Theorem 3. Let $\bar{E} \subset E$ be a class of public goods economies which contains E_L as a subset and in which every agent's preference relation R_i is convex and continuous. Let the outcome function h have the following properties:

(i) for each $e \in \bar{E}$, $\mathbb{R}^{n+K}_{++} \cap N_{\mathcal{M},h}(e) \subset S(e)$;
(ii) for each i and m_{-i}, the set $H_i(m_{-i}) = \{x_i, y);\ (x_i, y) \leqq h_i(m_{-i}, m_i)$ for some $m_i \in \mathcal{M}_i\}$ is convex.

Then $\mathbb{R}^{n+K}_{++} \cap N_{\mathcal{M},h}(e) \subset GR(e)$ for all $e \in \bar{E}$.

Proof of Theorems 2 and 3. Suppose that $\langle \mathcal{M}, X(m), V(m), Y(m) \rangle$ is a mechanism defined on E. Then $\langle \mathcal{M}, X(m), V(m) \rangle$ is a mechanism for the transferred economies E' which have CRS technologies and $Y(m) = f(V(m))$. We can easily see that $(X(m^*), V(m^*), Y(m^*)$ is a Nash equilibrium allocation (resp. Pareto efficient and AIR) for $e \in E$ if and only if $(X(m^*), V(m^*))$ is a Nash equilibrium allocation (resp. Pareto efficient and AIR) for the transformed economy $e' \in E'$ by the definition of R'_i. Since production functions are linear, the transformed economies E'_L and E'_C have the same functional forms as E_L and E_C, respectively. Thus all the conditions of Hurwicz's (1979a) Theorems 3 and 4 are satisfied. Then, we know that the set of Nash allocations for the mechanism $\langle \mathcal{M}, X(m), V(m) \rangle$ defined on e' contains the set of Lindahl allocations, and the set of Lindahl allocations contains the set of interior Nash allocations. Therefore, by the TE-Principle, the set of Nash equilibria of the mechanism $\langle \mathcal{M}, X(m), V(m), Y(m) \rangle$ contains GRE allocations and the set of GRE allocations contains the set of interior Nash allocations. Q.E.D.

3.2. Some informational requirement results on GRE

This subsection gives some general results on informational requirements of resource allocation processes which are privacy preserving and non-wasteful. We first show that the lower bound of informational requirements of resources allocation processes is the product of the number of independent public goods and the number of individuals.

Theorem 4. For public goods economies E specified in section 2, let $E_{CD} \subset E$ be a class of public goods economies whose transformed preferences R'_i can be represented by the Cobb–Douglas utility functions of the form $u'_i(x_i, v) = x_i \prod_{t=1}^{T} (v^t)^{\beta_i^t}$ $(\beta_i^t > 0)$. Let $\langle \mathcal{M}, \mu, h \rangle$ be a resource allocation process which is privacy preserving and non-wasteful on E_{CD}. Suppose the production functions and individual endowments are known to the designer, and the process $\langle \mathcal{M}, \mu, h \rangle$ satisfies either of the following three conditions: (1) M is a Hausdorff space; (2) the inverse of the restriction $\mu|_{E_{CF}}$ is upper hemi-continuous; (3) the inverse of $\mu|_{E_{CD}}$ is lower hemi-continuous. Then $M \geqq_F \mathbb{R}^{nT}$.

Proof of Theorem 4. Since the process $\langle \mathcal{M}, X(m), V(m), Y(m) \rangle$ is privacy preserving and non-wasteful on E_{CD}, it is clear that the process $\langle \mathcal{M}, X(m), V(m) \rangle$ is also privacy preserving and non-wasteful on the transformed economies E'_{CD}. Also $Y(m)$ must be produced in an efficient way by Pareto efficiency. Thus there are only T independent public goods (i.e., each firm essentially produces only one public good in the presence of by-products). Then E'_{CD} is homeomorphic to the nT-dimensional Euclidean space $\mathbb{R}_{nT}$. Therefore, by Sato's (1981) Theorems 1 and 2 (with a special case where the production functions are known to the designer),[9] we know that $M \geqq_F \mathbb{R}^{nT}$. Q.E.D.

Corollary 1. Under the assumption of Theorem 4, if M is a separable metric space, then $\dim M \geqq nT$.

Remark 4. The above theorem holds even in the case of IRS. This does not contradict Calsamiglia's (1977) result since we assume that technologies are known to the designer. We only concentrate on informational requirements when preferences are unknown to the designer. When production technologies display IRS and are unknown to the designer (but preferences and endowments are assumed to be known to the designer), Calsamiglia (1977) proved that there is no non-wasteful and privacy preserving mechanism which has a message space of finite dimension.

The following theorem shows that the lower bound given in Theorem 4 can actually be attainable by the GRE process not only for realization but also for implementation, i.e., the message space required for realization and implementation of the GRE correspondence is minimal among mechanisms that realize or/and implement Pareto-optimal allocations.

Theorem 5. For public goods economies E specified in section 2, which contains E_{CD} as a subset, suppose the production functions and individual endowments are known to the designer. Then the minimal dimension required for decentralized realization and implementation of the GRE correspondence is nT, and thus, it is informationally the most efficient one among all resource allocation processes on E which are privacy preserving and non-wasteful.

Proof. For each public goods economy $e \in E$, the corresponding e' has one private good and T public goods which are produced with the CRS technologies without the presence of by-products. Then, from the mechanisms proposed by Walker (1981) and Tian (1990, 1991), we know that the

[9]Sato (1981) gave his results by assuming that endowments are known to the designer.

dimension for implementation of the Lindahl correspondence on E' is nT, which is also the lower bound of informational requirements for the realization of resources allocation processes that is privacy preserving and non-wasteful by Theorem 4. From Theorem 2.4 of Reichelstein and Reiter (1988), we know that Nash implementation is always at least as costly, in message space size, as decentralized realization. Hence the minimal dimension required for decentralized realization and implementation of the Lindahl correspondence on E' is nT, and thus it is informationally the most efficient one among all privacy preserving and non-wasteful resource allocation processes on E'.

Now for any mechanism $\langle \mathcal{M}, \mu, X(m), V(m)\rangle$ [resp. mechanism $\langle \mathcal{M}, X(m), V(m)\rangle$] which realizes [resp. implements] the Lindahl correspondence on E' and has a message space of minimal dimension that is nT, let $Y(m) = f(V(m))$. Then the mechanism $\langle \mathcal{M}, \mu, X(m), V(m), Y(m)\rangle$ [resp. mechanism $\langle \mathcal{M}, X(m), V(m), Y(m)\rangle$] has the same dimension as the mechanism $\langle \mathcal{M}, \mu, X(m), V(m)\rangle$ [resp. $\langle \mathcal{M}, X(m), V(m)\rangle$], and thus has a minimal dimension which is nT. Further, by the TE-Principle, the mechanism $\langle \mathcal{M}, \mu, X(m), V(m), Y(m)\rangle$ [resp. $\langle \mathcal{M}, X(m), V(m), Y(m)\rangle$] realizes [resp. implements] the GRE correspondence. Thus the GRE correspondence is informationally the most efficient one among all resource allocation processes which are privacy preserving and non-wasteful on E. Q.E.D.

Remark 5. In fact, Tian and Li (1994b) gave such a mechanism which implements the GRE correspondence and whose dimension is nT and thus has a minimal dimension by Theorem 5.

For public goods economies without the presence of by-products, from the results of Sato (1981, Theorem 3) and Reichelstein and Reiter (1988, p. 668), one knows that the Lindahl mechanism is also informationally the most efficient one among all resource allocation processes. From this fact and the above theorem, we conclude that, unlike the competitive (Walrasian) allocation process, the Lindahl process is not *uniquely* informationally the most efficient one.

Remark 6. Since producing one public good automatically produces other public goods in the case where a firm has a by-products technology, and each consumer has only one ratio to share the production cost of each firm, it is quite intuitive that requiring one signal for each of the public goods is quite redundant: one signal for each firm (as GRE equilibrium) will do. One may very naturally conjecture that this is also true for the Lindahl equilibrium, i.e., in the presence of by-products, the 'degrees of freedom' of the variables of the Lindahl equilibrium, like the GRE equilibrium, is also less than that in the case without the presence of by-products, so that the

minimal dimension required for decentralized realization and implementation of the Lindahl correspondence is also lower. The following example reveals that this intuitive conjecture is not true by showing that the personalized prices of public goods for the Lindahl equilibrium are independent in the presence of by-products.[10] Thus the independence of personalized price vectors makes the minimal dimension required for decentralized realization and implementation of the Lindahl correspondence on E be at least as big as nK (the product of the number of public goods and the number of individuals), since a higher dimension Euclidean space nK cannot be homomorphic to a lower dimensional Euclidean space nT (by noting that $T<K$). So the minimal dimension required for decentralized realization and implementation of the GRE correspondence is lower than that for the Lindahl correspondence on E. Therefore, the Lindahl process is not informationally efficient.

Example 1. Consider a class of economies with n agents who consumer one private good x and two public goods (y^1, y^2). The production function allows for the presence of by-products and is given by $(y^1, y^2)=(v, \delta v)$, where v is the input which can produce v units of public good y^1 and δv units of public good y^2. Thus the feasible constraint is $\sum_{i=1}^n x_i + C(y) = \sum_{i=1}^n x_i + \max\{y^1, (y^2/\delta)\} \leqq \sum_{i=1}^n w_i$. Suppose that the utility function of agent i is given by $u_i(x_i, y^1, y^2) = x_i^{\alpha_i}(y^1)^{\beta_i^1}(y^2)^{\beta_i^2}$ with $(\alpha_i + \beta_i^1 + \beta_i^2 = 1)$. The budget constraint then is $x_i + q_i^1 y^1 + q_i^2 y^2 = w_i$. It can be easily verified that the Lindahl equilibrium is unique and the Lindahl allocation (x^*, y^*) and the personalized prices are given by $x_i^* = \alpha_i w_i$, $y^{*1} = v^* = \sum_{i=1}^n w_i(1-\alpha_i)$, $y^{*2} = \delta y^{*1} = \delta \sum_{i=1}^n w_i(1-\alpha_i)$, $q_i^{*1} = \beta_i^1 w_i / \sum_{i=1}^n w_i(1-\alpha_i)$, $q_i^{*2} = \beta_i^2 w_i / \delta \sum_{i=1}^n w_i(1-\alpha_i)$, $q^{*1} + \delta q^{*2} = 1$, where $q^{*k} = \sum_{i=1}^n q_i^{*k} (k=1,2)$ are the market prices of public goods.

Thus even though the market prices of two public goods are not independent of each other, the personalized price vectors for two public goods $(q_1^{*1}, q_2^{*1}, \ldots, q_n^{*1})$ and $(q_1^{*2}, q_2^{*2}, \ldots, q_n^{*2})$ are independent of each other as long as agents are not identical.

It is interesting to note that the GRE allocation for the above economy is the same as the Lindahl allocation. But this does not mean the GRE equilibrium is the same as the Lindahl equilibrium since ratios of agents are given by

$$r_i^* = \frac{(\beta_i^1 + \beta_i^2) w_i}{\sum_{i=1}^n (\beta_i^1 + \beta_i^2) w_i} = q_i^1 + \delta q_i^2,$$

which is different from the personalized prices for the Lindahl equilibrium.

[10]One may easily see this fact by noting that the determination of equilibrium personalized prices not only depends the level of public goods produced but also depend on the marginal utilities of individuals.

4. Concluding remarks

In this paper we used the GRE correspondence as a desirable social choice rule and obtained some interesting results on incentive and information aspects. We generalized Hurwicz's (1979a) characterization results on implementable social choice correspondences that selects Pareto efficient and individually rational allocations to the cases of non-constant returns and autarkic individual rationality. Furthermore, we prove that the GRE process is informationally the most efficient process among informationally decentralized and Pareto efficient resource allocation processes even in the presence of by-product technologies, and thus the minimal dimension required for the GRE correspondence is never larger than, and in the presence of by-product technologies actually lower than, that needed for the Lindahl correspondence. Thus the paper in this way shows that some results of Hurwicz, Jordan, Mount and Reiter, Walker, etc. that establish the informational superiority of the competitive mechanism in pure private goods exchange economies do not hold for Lindahl allocations in economies with public goods.

In summary, there are at least three advantages in using the GRE correspondence compared to the Lindahl correspondence: (1) the cost to operate the GRE social system may be lower; (2) one can reach Pareto-efficient and autarkically individually rational allocations by implementing the GRE allocations even if IRS exists; and (3) the mechanisms implementing the GRE correspondence can be the same for any kind of technology – CRS, DRS, or IRS. This is indeed a very desirable property since the production technology usually varies over the level of output.

In ending this paper, we should mention that the model and results presented in this paper deal only with public goods economies with one private good. By the same way given in Tian and Li (1994a), one can define GRE equilibrium with any number of private goods. All results in the paper remain the same for any number of private goods.

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