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Transfer continuities, generalizations of the Weierstrass and maximum theorems: a full characterization

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Abstract

This paper gives necessary and sufficient conditions for (1) a function to attain its maximum on a compact set (2) the set of maximum points of a function on a compact set to be non-empty and compact, and (3) the maximum (marginal) correspondence to be closed. We do so by introducing a class of *transfer continuities* which characterize the essence of topological structures of functions and correspondences for extreme points and significantly weaken the conventional continuities. Thus our results generalize the classical Weierstrass theorem and the Maximum Theorem of Berge (*Espaces topologiques et fonctions multivoques*, Donod, Paris, 1959; *Topological Spaces*, Macmillan, New York, 1963, p. 116), by giving necessary and sufficient conditions. Furthermore, we generalize the maximum theorem of Walker, (International Economic Review, 1979, 20, 267–270) by relaxing the openness of the graph of preference correspondences and the lower semi-continuity of the feasible action correspondence. By applying our maximum theorems to game theory and economics, we can generalize many of the existence theorems on Nash equilibrium of games and equilibrium of the generalized games (the so-called abstract economies) in the literature.

Keywords: Transfer continuities; Weierstrass theorem; Maximum theorem; Optimization; Existence of equilibrium

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1. Introduction

Let E and Y be two topological spaces and 2^Y be the collection of all subsets of Y. Let u: $E \times Y \to \mathbb{R}$ be a real-valued function and F: $E \to 2^Y$ be a non-empty-valued correspondence. Consider a family of maximization problems that depend on the parameter e, where

$$\psi(e) = \sup\{u(e, y) : y \in F(e)\}$$

is called the marginal (or performance, value) function and

$$M(e) = \{y \in F(e) : u(e, y) = \psi(e)\}$$

is called the maximum (marginal) correspondence.

Berger (1959) first proved the theorem: If u is a continuous function and F is a non-empty compact-valued continuous correspondence, then the maximum correspondence $M(e)$ is non-empty compact-valued and upper semi-continuous, and the marginal function $\psi(e)$ is continuous. Since then, this theorem, called Berge's Maximum Theorem, has become one of the most useful and powerful theorems in economics, optimization theory, and game theory. Walker (1979) extended Berge's Maximum Theorem to optimization with respect to preference relations where u is replaced by a preference correspondence $P : E \times Y \to 2^Y$. Besides their own importance in the decision situation, the maximum theorems can also be used together with other theorems to prove the existence of Nash equilibrium for conventional games and equilibrium for the generalized games which were first defined by Debreu (1953), and in turn can be used to prove the existence of competitive equilibrium. [1] The hypotheses in the classical existence theorems of Nash equilibrium and equilibrium for the generalized games of these type (e.g. in Nash, 1950, 1951; Debren, 1952; Glicksberg, 1952; Fan, 1953; Nikaido and Isoda, 1955), however, typically assumed *continuity* for both the payoff functions and the feasible strategy correspondences, and *quasiconcavity* for the payoff functions (in addition to the regular convexity and compactness assumptions on the values of the feasible strategy correspondences). It is known that in many important economic games the payoff functions and the feasible strategy correspondences are discontinuous (see Dasgupta and Maskin, 1986, and the references therein). On the other hand, the non-existence of Nash equilibrium in simple economic models was noted a long time ago by Edgeworth (1925) in his critique of Bertrand's (1883) analysis of the price-setting duopolists. For these reasons, economists continually strive to weaken the conditions that guarantee the existence of equilibrium. Dasgupta and Maskin (1986) first studied this problem. They proved

[1] The generalized game is also called the abstract economy or the social equilibrium in the economics literature.

the existence of Nash equilibrium for games wherein the payoff function is upper semi-continuous and graph-continuous – a notion introduced by them.

In our recent study of economic games, we are challenged by some economic game models where the payoff functions are neither upper semi-continuous nor lower semi-continuous and the feasible strategy correspondences are not lower semi-continuous, since the extant existence theorems fail to determine the existence of an equilibrium for these games. Thus new existence theorems for these types of games must be developed. Looking for mathematical tools, we turn to the maximum theorems. It then becomes clear that both Berge's Maximum Theorem and Walker's Maximum Theorem must be generalized and the corresponding continuity assumptions must be relaxed. An interesting question to ask is whether or not one can find minimal possible conditions under which the conclusions of the maximum theorems still hold. In other words, what are the minimal possible conditions under which (1) a function attains its maximum on a compact set (2) the set of maximum points of a function on a compact set is non-empty and compact, and (3) the maximum correspondence is closed? It is these mathematical problems that motivate our work in this paper. To answer these questions, a class of generalized continuity conditions, called *transfer continuities*, are introduced, which characterize the essence of topological structures of functions and correspondences for extreme points and significantly weaken the conventional continuities. The basic idea behind the transfer continuities' is very simple. For a function u or a preference correspondence P to have, say, maximum points, given $u(x) > u(y)$ (or $y \in P(x)$), the conventional continuity conditions describe topological behavior or relations between x and a neighborhood of y. However, to characterize the existence of maximum points for a function u or a preference correspondence P, the topological structure of u (or P) below the level of $u(y)$ (or y) is irrelevant and only the topological structure of u above the level of $u(y)$ (or y) is important. Therefore, we do not have to know the topological relations between x and a neighborhood of y. We only need to know the topological behavior or relations between a neighborhood of y and an element x' in its 'upper' part (so x can be transferred to a certain element x' in the 'upper' part of a neighborhood of y). Conditions describing the topological relations between a neighborhood of y and an element in its 'upper' part are then called *transfer (upper) continuities*. For instance, to characterize the existence of maximal elements for a function u, for given $u(x) > u(y)$, we really do not have to know the topological relations between x and a neighborhood $\mathcal{N}(y)$ of y. All we need to know is the topological relations between a neighborhood of y and a point x' in the upper part of $u(y)$, i.e. whether x can be transferred to x', a point in the upper part of $u(y)$ such that $u(x') > (\geq) u(y'): y' \in \mathcal{N}(y)$, and if so u is said to be *transfer (weakly) upper continuous on X*. Some basic concepts on transfer continuity are provided in Section 2.

Thus, with the concepts of transfer continuities, in Section 3, we successfully characterize the maximum correspondence by generalizing the well-known Weierstrass theorem with necessary and sufficient conditions. We prove that a function f attains its maximum on a compact set if and only if it is transfer weakly upper continuous, and that the set of maximum points of f on a compact set is non-empty compact if and only if f is transfer upper continuous. In Section 4, we generalize the Maximum Theorem of Berge (1959, 1963, p.116) by characterizing the closedness of the maximum correspondence with a necessary and sufficient condition. The notion of transfer continuities also has wide applications in optimization with respect to preference relations. We also generalize the Maximum Theorem of Walker (1979) by relaxing the openness of the graph of preference correspondences and the lower semi-continuity of the feasible action correspondences. It is worthwhile indicating that if f takes values in a topological space X, then Theorem 1 below still holds even when the preference relation $\succ$ in X is pseudotransitive, and thus Theorem 1 generalizes the results of Campbell and Walker (1990). Also, the sufficiency part of Theorem 2 holds when the preference relation $\succ$ in X is acyclic, and the necessity part holds when the preference relation $\succ$ in Y is fully transitive. These results are stated in Propositions 1 and 2. It may be remarked that the existence of maximum elements for a given preference relation relies both on its topological structure and on its transitivity structure, while our recent study in Zhou and Tian (1992) shows that the transfer weak upper continuity condition describes just the topological structure.

By applying our maximum theorems to game theory and economics, in Section 5, we prove the existence of equilibrium of generalized games and Nash equilibrium of games where the payoff functions are neither lower semi-continuous nor upper semi-continuous, and where the feasible strategy correspondences are not lower semi-continuous. Thus our results generalize many of the existence theorems on Nash equilibrium of (the generalized) games in the literature such as those in Nash (1950, 1951), Debreu (1952), Glicksberg (1952), Nikaido and Isoda (1955), and Dasgupta and Maskin (1986). Concluding remarks are offered in Section 6.

It seems to us that the concepts of 'transfer' (one point is transferred to another point) characterize the essence of extreme points of binary relations. The results in Tian (1993a) and Zhou and Tian (1992) have generalized the concepts of transfer continuities to transfer convexities and transfer transitivities by giving necessary and sufficient conditions. Baye et al. (1993) also used this transfer method to characterize the existence of Nash equilibrium and dominant Nash equilibrium of games in terms of transfer quasiconcavity and transfer continuities.

2. Notation and definitions

Let X be a subset of a topological space. For each $x \in X$, denote $\mathcal{N}(x)$ a neighborhood of $x \in X$.

A function f: $X \to \mathbb{R} \cup \{-\infty\}$ is said to be *upper semi-continuous* if for each point $x' \in X$, we have

$$\limsup_{x \to x'} f(x) \leq f(x'),$$

or equivalently, its epigraph $\text{epi} f \equiv \{(x, a) \in X \times \mathbb{R}: f(x) \geq a\}$ is a closed subset of $X \times \mathbb{R}$. A function f: $X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semi-continuous* if $-f(x)$ is upper semi-continuous. A function f: $X \to \mathbb{R} \cup \{-\infty\}$ is said to be *weakly upper continuous* (cf. Campbell and Walker, 1990) if for points x, $y \in X$, $f(y) < f(x)$ implies that there exists a neighborhood $\mathcal{N}(y)$ of y such that $f(z) \leq f(x)$ for all $z \in \mathcal{N}(y)$. [2]

Now we introduce some concepts of generalized continuities for functions and correspondences, which are significantly weaker than the conventional continuities.

Definition 1. A function f: $X \to \mathbb{R} \cup \{-\infty\}$ is said to be *transfer upper continuous on* X if for points x, $y \in X$, $f(y) < f(x)$ implies that there exists a point $x' \in X$ and a neighborhood $\mathcal{N}(y)$ of y such that $f(z) < f(x')$ for all $z \in \mathcal{N}(y)$.

Definition 2. A function f: $X \to \mathbb{R} \cup \{-\infty\}$ is said to be *transfer weakly upper continuous on* X if for points x, $y \in X$, $f(y) < f(x)$ implies that there exists a point $x' \in X$ and a neighborhood $\mathcal{N}(y)$ of y such that $f(z) \leq f(x')$ for all $z \in \mathcal{N}(y)$.

Definition 3. A function f: $X \to \mathbb{R} \cup \{-\infty\}$ is said to be *quasi-transfer upper continuous on* X if for points x, $y \in X$, $f(y) < f(x)$ implies that there exists a neighborhood $\mathcal{N}(y)$ of y such that for any $z \in \mathcal{N}(y)$, there exists a point $x' \in X$ satisfying $f(z) < f(x')$.

Definition 4. A function f: $X \to \mathbb{R} \cup \{+\infty\}$ is said to be *(quasi-)transfer (weakly) lower continuous on* X if $-f(x)$ is (quasi-)transfer (weakly) upper continuous on X.

Remark 1. It can be seen that f is transfer upper continuous on X if it is upper semi-continuous by choosing $x' = x$; f is transfer weakly upper continuous on X if it is weakly upper continuous on X or it is transfer upper continuous; f is quasi-transfer upper continuous if f is transfer upper continuous. The converse statements may not be true. Furthermore, if $f(y) < f(x)$ and f is upper semi-continuous at y, one can simply choose $x' = x$. Thus, to verify the transfer upper

[2] The weak upper continuity was called the weak lower continuity in Campbell and Walker (1990). The reason we call it 'upper' is that when the statement '$f(z) \leq f(x)$ for all $z \in \mathcal{N}(y)$' is replaced by the statement $f(z) < f(x)$ for all $z \in \mathcal{N}(y)$, it becomes the usual upper semi-continuity.

continuity of a function on a domain, we only need to check those points at which it is not upper semi-continuous.

Remark 2. Let $\succ$ be a *strict* (i.e. *irreflexive*) *preference relation* on X and $\succcurlyeq$ the completion of $\succ$, i.e. $x \succcurlyeq y$ means that $y \succ x$ does not hold. Denote the strictly upper contour sets of $\succ$ by, for each $x \in X$, $P(x) = \{y \in X : y \succ x\}$. An element $x^* \in X$ is said to be a *maximal element* of the binary relation '$\succ$' on X if $x^* \succcurlyeq y$ for all $y \in X$, or equivalently, $P(x*) = \emptyset$. Then, if we replace '$>$' ('$\geq$') by '$\succ$' ('$\succcurlyeq$'), all the transfer continuities for functions given in the paper can be similarly defined for preference relations.

Let X and Y be two topological spaces. A correspondence G: $X \to 2^Y$ is said to be *upper semi-continuous* (in short, u.s.c.) if the set $\{x \in X : G(x) \subset V\}$ is open in X for every open set of V of Y. A correspondence G: $X \to 2^Y$ is said to be *lower semi-continuous* (in short, l.s.c.) if the set $\{x \in X : G(x) \cap V \neq \emptyset\}$ is open in X for every open set V of Y. A correspondence G: $X \to 2^Y$ is said to be *continuous* if it is both u.s.c. and l.s.c. A correspondence G: $X \to 2^Y$ is said to have *open lower sections* if the set $G^{-1}(y) = \{x \in X : y \in G(x)\}$ is open in X for every $y \in Y$. A correspondence G: $X \to 2^Y$ is said to have *open upper sections* if for every $x \in X$, $G(x)$ is open in Y. A correspondence G: $X \to 2^Y$ is said to be *closed (open)* if its graph is closed (open).

Remark 3. If a correspondence G is open, it has open upper and lower sections, but the converse statement may not be true (cf. Bergstrom et al., 1976, pp. 265–266). Furthermore, Yannelis and Prabhakar (1983, p. 237) showed that G is lower semi-continuous if it has open lower sections, but the converse statement may not be true.

Definition 5. A correspondence G: $X \to 2^Y$ is said to be *transfer closed-valued on* X if for every $x \in X$, $y \notin G(x)$ implies that there exists $x' \in X$ such that $y \notin \mathrm{cl}G(x')$.

Remark 4. It is clear that for any function f: $X \to \mathbb{R} \cup \{-\infty\}$, the correspondence G: $X \to 2^Y$ defined by $G(x) = \{y \in X : f(y) \geq f(x)\}$ for all $x \in X$ is transfer closed-valued on X if and only if f is transfer upper continuous on X. Note that a correspondence is transfer closed-valued if it is closed-valued.

Definition 6. A correspondence P: $Y \to 2^X$ is said to have *transfer open lower sections on* X if for every $x \in X$ and $y \in Y$, $x \in P(y)$ implies that there exists a point $x' \in X$ and a neighborhood $\mathcal{N}(y)$ of y such that $x' \in P(y')$ for all $y' \in \mathcal{N}(y)$.

Remark 5. Observe that a correspondence has transfer open lower sections if it has open lower sections by choosing $x' = x$. Also a correspondence P: $Y \to 2^X$ has transfer open lower sections on X if and only if the correspondence G: $X \to 2^Y$, defined by, for every $x \in X$,

$$G(x) = Y \setminus P^{-1}(x),$$

is transfer closed-valued on X.

Let S be a (subspace) subset of topological space T and $D \subset S$. Denote by $\text{int}_S D$ the interior of the set D in the subspace S.

3. Characterizations of the maximum points of f

The classical Weierstrass theorem states that any upper (lower) semi-continuous function reaches its maximum (minimum) on a compact set. This section gives two theorems which generalize the Weierstrass theorem by relaxing the upper (lower) semi-continuity of a function. These theorems completely characterize the existence of maximum (minimum) points of a function on a compact set.

Theorem 1. Let X be a compact subset of a topological space and let f: $X \to \mathbb{R} \cup \{-\infty\}(\mathbb{R} \cup \{+\infty\})$ be a function. Then f attains its maximum (minimum) on X if and only if f is transfer weakly upper (lower) continuous on X.

Proof. Since any maximum point of a function f is a minimum point of the function $-f$, and f is transfer weakly upper continuous if and only if $-f$ is transfer weakly lower continuous on X, we only need to consider the case of the existence of maximal elements of the function f.

Sufficiency. By contradiction, suppose f does not attain its maximum on X. Then for each $y \in X$, there exists $x \in X$ such that $f(x) > f(y)$. By the transfer weak upper continuity of f, there exists $x' \in X$ and a neighborhood $\mathcal{N}(y)$ such that $f(x') \geq f(y')$ for all $y' \in \mathcal{N}(y)$. It follows that $X = \bigcup_{y \in X} \mathcal{N}(y)$. Since X is compact, there exist finitely many points $\{y_1, y_2, \ldots, y_n\}$ such that $X = \bigcup_{i=1}^n \mathcal{N}(y_i)$. Let x'_i be an associated point such that $f(x'_i) \geq f(y')$ for all $y' \in \mathcal{N}(y_i)$. For the finite subset $\{x'_1, x'_2, \ldots, x'_n\}$ f has a greatest point, say x'_1, i.e. $f(x'_1) \geq f(x'_i)$ for $i = 1, 2, \ldots, n$. Since f has no maximum point on X by the contrapositive hypothesis, x'_1 is not a maximum point of f on X, thus there is $x \in X$ such that $f(x) > f(x'_1)$. However, since $X = \bigcup_{i=1} \mathcal{N}(y_i)$, there exists j such that $x \in \mathcal{N}(y_j)$ and thus $f(x'_j) \geq f(x)$. Therefore $f(x) > f(x'_1) \geq f(x'_j) \geq f(x)$, a contradiction. Hence f attains its maximum on X.

Necessity. Trivial. Just let x' be any maximum point of f. Then $f(x') \geq f(y')$ for all $y' \in X$. Q.E.D.

Remark 6. Even though the transfer weak upper continuity is a necessary condition for f to have a maximum on a choice set X, like the conventional continuity, it is *not* a sufficient condition unless one assumes that the set X is compact. For instance, consider a very simple function: $y = f(x) = 2x$ defined on $X = [0, T)$, where T is a finite or infinite positive number. f clearly satisfies the transfer weak upper continuity condition on X, but there is no maximum point on X.

In many cases one hopes that the set of maximum points is not only non-empty but also compact. For instance, when one uses Berge's Maximum Theorem to prove the existence of equilibria of (generalized) games, the maximum correspondence is required to be non-empty compact-valued. Such a result is given in Theorem 2 below. Before doing so, we first prove the following lemma:

Lemma 1. Let X and Y be two topological spaces, and let G: $X \to 2^Y$ be a correspondence. Then $\bigcap_{x \in X} \mathrm{cl}G(x) = \bigcap_{x \in X} G(x)$ if and only if G is transfer closed-valued on X.

Proof.

Sufficiency. We need to prove $\bigcap_{x \in X} \mathrm{cl}G(x) = \bigcap_{x \in X} G(x)$ if G is transfer closed-valued on X. It is clear that $\bigcap_{x \in X} G(x) \subset \bigcap_{x \in X} \mathrm{cl}G(x)$. So we only need to show $\bigcap_{x \in X} \mathrm{cl}G(x) \subset \bigcap_{x \in X} G(x)$. Suppose, by way of contradiction, that there is some y in $\bigcap_{x \in X} \mathrm{cl}G(x)$ but not in $\bigcap_{x \in X} G(x)$. Then $y \notin G(z)$ for some $z \in X$. Since G is transfer closed-valued on X, there exists some $z' \in X$ such that $y \notin \mathrm{cl}G(z')$ and then $y \notin \mathrm{cl}G(z')$, a contradiction.

Necessity. Suppose $\bigcap_{x \in X} \mathrm{cl}G(x) = \bigcap_{x \in X} G(x)$. We need to show that G is transfer closed-valued on X. If $y \notin G(x)$, then $y \notin \bigcap_{x \in X} \mathrm{cl}G(x) = \bigcap_{x \in X} G(x)$ and thus $y \notin \mathrm{cl}G(x')$ for some $x' \in X$. Thus G is transfer closed-valued on X. Q.E.D.

Remark 7. Note that, if we define a correspondence G: $X \to 2^X$ by, for each $x \in X$, $G(x) = \{y \in X : f(y) \geq f(x)\}$, we know that f is transfer upper continuous on X if and only if the correspondence G is transfer closed-valued on X (cf. Remark 4). Thus, by Lemma 1, $\bigcap_{x \in X} \mathrm{cl}G(x)$ characterizes the set of maximum points to be closed.

Theorem 2. Let X be a compact subset of a topological space and let f: $X \to \mathbb{R} \cup \{-\infty\}(\mathbb{R} \cup \{+\infty\})$ be a function. Then the set of maximum (minimum) points of f on X is non-empty and compact if and only if f is transfer upper (lower) continuous on X.

Proof. Again we only need to consider the case of maximum.

Necessity. Suppose that the set of maximum points of f on X is non-empty and compact. For any x, $y \in X$, if $f(y) < f(x)$, then y is not a maximum point of f on X. By the compactness of the set of maximum points, there exists a neighborhood $\mathcal{N}(y)$ of y, which contains no maximum point of f on X. Let x' be any maximum point of f on X, then $f(z) < f(x')$ for all $z \in \mathcal{N}(y)$. Thus, f is transfer upper continuous on X.

Sufficiency. Define a correspondence G: $X \to 2^X$ by, for each $x \in X$, $G(x) = \{y \in X : f(y) \geq f(x)\}$. Since f is transfer upper continuous on X, the correspondence G is transfer closed-valued on X. Thus, by Lemma 1, $\bigcap_{x \in X} \mathrm{cl}G(x) = \bigcap_{x \in X} G(x)$.

Now for any finite subset $\{x_1, x_2, \ldots, x_m\} \subset X$, f has a greatest element, say, x_1, on the finite set, i.e. $f(x_1) \geq f(x_i)$ for $i = 1, \ldots, m$. Then $x_1 \in G(x_i)$ for $i = 1, \ldots, m$. Therefore, $\emptyset \neq \bigcap_{i=1}^{m} G(x_i) \subset \bigcap_{i=1}^{m} \mathrm{cl}G(x_i)$. Hence the family of sets $\{\mathrm{cl}G(x) : x \in X\}$ has the finite intersection property on X. Also, since $\{\mathrm{cl}G(x) : x \in X\}$ is a family of closed subsets in the compact set X, $\emptyset \neq \bigcap_{x \in X} \mathrm{cl}G(x) = \bigcap_{x \in X} G(x)$ which means that there exists a point $x^* \in X$ such that $f(x^*) \geq f(x)$ for all $x \in X$. Since the set of maximum points $\bigcap_{x \in X} \mathrm{cl}G(x)$ is a closed subset of the compact X, it is compact. Q.E.D.

Remark 8. Note that the conclusion of Theorem 2 holds if the compactness of the set X is relaxed by assuming that there exist finitely many points $\{x_{01}, \ldots, x_{0m}\}$ such that $\bigcap_{i=1}^{m} \mathrm{cl}G(x_{0i})$ is compact where $G(x_{0i}) = \{y \in X : f(y) \geq f(x_{0i})\}$. This is true by noting that the family of sets $\{\mathrm{cl}G(x) \cup [\bigcap_{i=1}^{m} \mathrm{cl}G(x_{0i})] : x \in X\}$ has the finite intersection property on the set $\bigcap_{i=1}^{m} \mathrm{cl}G(x_{0i})$.

Remark 9. Note that, while upper semi-continuity and weak upper continuity are entirely 'local' and thus can be inherited on compact subsets, [3], transfer continuities cannot be inherited on compact subsets unless a function has extreme points on every non-empty compact subset. This is not because of the problem of the notion of transfer continuities but because of the structure of functional forms. The reason this happens is that the domain of a function contains a non-empty compact subset on which the function does not have a maximum point (so that conventional continuities and any other continuity conditions which guarantee the existence of a maximal element on a compact set must fail). If this does not happen, then transfer weak upper continuity-like conventional continuities become entirely 'local' and can be inherited on compact subsets. Thus, when a function has a maximum point on a compact set but does not have a maximum point on some non-empty compact subset, there does not exist any 'local' continuity condition which is necessary and sufficient and can be preserved on passage to subsets. Therefore, if one wants to characterize the existence of maximal elements for this situation, one must give up looking for a 'local' continuity condition. Thus the transfer (weak) upper continuities completely characterize the existence of maximum points on a compact set in this general situation where the 'local' continuity conditions fail or do not fail to

[3] That is, if a maximum point exists on a compact set, then a maximum point exists on every non-empty compact subset.

work. Of course, as we mentioned above, if a function has a maximum point on every non-empty closed subset of a compact set, then our transfer weak upper continuity condition like the conventional continuity condition becomes entirely 'local' and is satisfied on every non-empty closed subset of a compact set. Thus, it has a maximum point on every non-empty compact subset of the compact set; consequently, it has a maximal element on the compact set. Summarizing the above discussion, we have the following corollary to Theorem 1.

Corollary 1. Let X be a topological space and let f: $X \to \mathbb{R} \cup (-\infty)$ *be a function. Then a necessary and sufficient condition for f to have a maximum on every compact subset of X is that every point in the domain has a neighborhood upon which it is transfer weakly upper continuous.*

Remark 10. It may be remarked that the existence of a maximum point in Theorem 2 can also be obtained by applying Theorem 1. The reason we provide another proof is that the proof of Theorem 2 depends less on transitivity than that of Theorem 1. One can see from these two propositions below that the existence of maximal elements for a preference relation $\succ$ on X relies both on its topological structure and on its transitivity structure, and that the conclusion of Theorem 1 still holds even when preference relations $\succ$ in X are pseudotransitive, [4] a concept introduced by Campbell and Walker (1990). Also, the sufficiency part of Theorem 2 holds when the preference relation $\succ$ in X is acyclic, and the necessity part holds when the preference relation $\succ$ in X is fully transitive. [5] Thus Propositions 1 and 2 generalize the Bergstrom–Walker Theorem (see Bergstrom, 1975 and Walker, 1977) and Theorem 1 in Campbell and Walker (1990). We state these two results below as propositions without proof (the proofs are essentially the same as those of Theorems 1 and 2). Our recent study in Zhou and Tian (1992) shows that it is the *transfer weak upper continuity* that describes the *topological structure* for a preference relation to attain its maximal elements.

Proposition 1. Let $\succ$ *be a pseudotransitive preference relation on a compact set X of a topological space. Then* $\succ$ *attains its maximum on X if and only if it is transfer weakly upper continuous on X.*

Proposition 2. Let $\succ$ *be a preference relation on a compact set X of a topological space.*

(1) *Assume that the preference relation* $\succ$ *is transfer upper continuous and*

[4] That is, $x_1 \succ x_2 \succcurlyeq x_3 \succ x_4$ implies $x_1 \succ x_4$ when $x_2 \neq x_3$.

[5] A preference relation $\succ$ is said to be acyclic on X if, for any m, $x_1 \succ x_2 \succ, \ldots, x_m$ implies $x_1 \succcurlyeq x_m$, and is said to be fully transitive if its completion $\succcurlyeq$ becomes an ordering (i.e. reflexive, transitive, and complete) preference relation.

acyclic on X. Then the set of all maximal elements of $\succ$ on X is non-empty and compact.

(2) *Assume that the binary relation $\succ$ on X is fully transitive. Then the set of all maximal elements on X is non-empty and compact on X if and only if $\succ$ is transfer upper continuous on X.*

Remark 11. The transfer (weak) upper continuities characterize only the topological structure for a preference relation to have maximal elements. If the transitivity or convexity of a preference relation is too 'weak', it may not have a maximal element even though it satisfies the (transfer) continuity conditions. In fact, Campbell and Walker (1990) constructed an example in which the binary relation is weakly transitive [6] and weakly upper continuous, but it fails to have a maximal element on a non-empty and compact set. So the transfer weak upper continuity is only a necessary but not sufficient condition for the existence of a maximal element of a general preference. Under the transfer weak upper continuity, any transitivity condition proposed must be stronger than the weak transitivity condition to guarantee the existence.

The following examples show how our results characterize existence of maximal elements in functions that do not satisfy the conditions of existing theorems.

Example 1. Consider a function $f(x)$ defined on the interval $X=[0, 1]$ by

$$f(x)=\begin{cases}1+x, & \text{if } x \text{ is a rational number},\\ x, & \text{otherwise}.\end{cases} \tag{1}$$

We can see easily that f is not upper semi-continuous. In order to see f is transfer upper continuous, for a neighborhood $\mathcal{N} \subset [0, 1]$, we may choose any rational number x' such that $\sup\{x\colon x \in \mathcal{N}\} < x' \leq 1$. Thus, by Theorem 2, we know the set of maximal elements is non-empty and compact. In fact $x=1$ is a unique maximum point of f on $[0, 1]$.

Example 2. Consider the so-called Dirichlet function $f(x)$ defined on the interval $X=[0, 1]$ by

$$f(x)=\begin{cases}1, & \text{if } x \text{ is a rational number},\\ 0, & \text{otherwise}.\end{cases} \tag{2}$$

Note that f defined by (2) is clearly not transfer upper continuous on X. However, by Definition 2, it is transfer weakly upper continuous on X by choosing x' as any rational number. We can also see that the set of maximum points of f on $[0, 1]$ is given by all rational numbers, and thus it is not compact.

[6] That is, $x_1 \succ x_2 \succ x_3$ implies $x_1 \succ x_3$.

Example 3. Now if a function $f(x)$ is defined on the interval $X=[0, 1]$ by

$$f(x)=\begin{cases} x, & \text{if } 0 \le x < 1, \\ 0, & x=1, \end{cases} \tag{3}$$

then f is not transfer weakly upper continuous on X. This is because for $y=1$ and $x \in (0, 1)$, we have $f(y) < f(x)$. But we cannot find any $x' \in X$ and neighborhood $\mathcal{N}(y)$ of y such that $f(z) \le f(x')$ for all $z \in \mathcal{N}(y)$. In fact, we can see that f does not have a maximum point on [0, 1].

The following two examples are due to Professor W. Shafer for which we are grateful.

Example 4. Consider a function $f(x)$ defined on the interval $X=[0, 3]$ by

$$f(x)=\begin{cases} a-ax, & \text{if } 0 \le x \le 1, \\ x-1, & \text{if } 1 < x < 2, \\ 0, & 2 \le x \le 3. \end{cases} \tag{4}$$

We can see that only at $x=2$ is f not u.s.c., and f is transfer upper continuous if and only if $a \ge 1$, and thus it is transfer upper continuous if and only if 0 is a maximal element. Note that 0 is very far away from 2, and the function is not transfer weakly upper continuous on some compact subset which contains the point 2, and thus does not have a maximal element on this compact subset. Thus, no 'local' continuity condition can characterize the existence of maximal elements for this function (see Remark 9).

Example 5. Consider a function $f(x)$ defined on the interval $X=[0, 2]$ by

$$f(x)=\begin{cases} 1, & \text{if } x=0, \\ 2, & \text{if } 0 < x < 1, \\ 4-2x, & 1 \le x \le 2. \end{cases} \tag{5}$$

We can see that only at $x=0$ is f not u.s.c., and that f fails the Campbell–Walker weak upper continuity condition ($f(1.1) > f(0)$, but $f(1.1) < f(x)$ for x arbitrarily close to 0). Even though f is not transfer upper continuous either ($f(1) > f(0)$, but there is no $x' \in X$ and neighborhood $\mathcal{N}(y)$ of y such that $f(z) \le f(x')$ for all $z \in \mathcal{N}(0)$), it is transfer weakly upper continuous on every non-empty compact subset of [0,2] (so that the transfer weak upper continuity is a 'local' continuity condition for this function) and thus has a maximal element on every non-empty compact subset. Consequently, we know that there exists a maximal element on the interval [0,2]. In fact, we can see that the set of maximal elements is given by (0, 1] which is not compact.

4. The generalizations of the maximum theorem

The well-known Berge's Maximum Theorem states that if the objective function u is continuous and the feasible correspondence F is a non-empty compact-valued continuous correspondence, then the maximum correspondence M is non-empty compact-valued and u.s.c. Walker (1979) extended Berge's Maximum Theorem to optimization with respect to preference relations where u is replaced by a preference correspondence P. He proved that if the feasible correspondences are continuous and the preference correspondence is open then the maximum correspondence M is compact-valued (but possibly empty) and u.s.c. In this section we use the transfer continuities to characterize both types of maximum theorems. Our maximum theorems below generalize the maximum theorems of Berge (1963), Walker (1979), and Tian and Zhou (1992) by relaxing the continuities of the payoff function u and the feasible choice correspondence F. Namely, Theorem 3 below generalizes Berge's Maximum Theorem and Tian and Zhou (1992) by relaxing upper and lower semi-continuity of the payoff function and the feasible correspondences. And Theorem 4 below generalizes Walker's Maximum Theorem by relaxing upper and lower semi-continuity of the feasible correspondences and the openness of the graph of the preference correspondence.

Accordingly, we introduce

Definition 7. Let E and Y be two topological spaces. Let F: $E \to 2^Y$ be a correspondence. A function u: $E \times Y \to \mathbb{R} \cup \{-\infty\}$ is said to be *quasi-transfer upper continuous in* (e, y) *with respect to* F if, for every $(e, y) \in E \times Y$ with $y \in F(e)$, $u(e, z) > u(e, y)$ for some $z \in F(e)$ implies that there is some neighborhood $\mathcal{N}(e, y)$ of (e, y) such that for any $(e', y') \in \mathcal{N}(e, y)$ with $y' \in F(e')$, there exists some $z' \in F(e')$ satisfying

$$u(e', z') > u(e', y').$$

The following definition is a natural generalization of transfer upper continuity.

Definition 8. Let E and Y be two topological spaces. Let F: $E \to 2^Y$ be a correspondence. A function u: $E \times Y \to \mathbb{R} \cup \{-\infty)$ is said to be *transfer upper continuous in* y *on* F if, for every $(e, y) \in E \times Y$ with $y \in F(e)$, $u(e, z) > u(e, y)$ for some $z \in F(e)$ implies that there is a point $z' \in Y$ and a neighborhood $\mathcal{N}(y)$ of y such that for any $y' \in \mathcal{N}(y)$ with $y' \in F(e)$, $u(e, z') > u(e, y')$ and $z' \in F(e)$.

Theorem 3. Let E *and* Y *be two topological spaces. Let* F: $E \to 2^Y$ *be a non-empty compact-valued and closed correspondence and let* u: $E \times Y \to \mathbb{R} \cup \{-\infty\}$ *be a real-valued function. Then the maximum correspondence* M: $E \to 2^Y$ *defined, for each* $e \in E$, *as*

$$M(e) = \{y \in F(e) : u(e, y) \geq u(e, x), \quad \forall x \in F(e)\},$$

is non-empty compact-valued and closed if and only if u is transfer upper continuous in y on $F(e)$ for every $e \in E$ and u is quasi-transfer upper continuous in (e, y) with respect to F. If, in addition, F is u.s.c., then M is u.s.c.

Proof. Since $F(e)$ is non-empty compact-valued for every $e \in E$, then, by Theorem 2, $M(e)$ is non-empty compact-valued if and only if u is transfer upper continuous in y on $F(e)$.

Now we show that when F is closed, the correspondence M is closed, i.e. the graph of M

$$\text{Graph}(M) = \{(e, y) \in E \times Y : y \in M(e)\}$$

is closed if and only if u is quasi-transfer upper continuous in (e, y) with respect to F.

Sufficiency. To show M is closed, we only need to show that, if $(e, y) \notin \text{Graph}(M)$, there exists some neighborhood $\mathcal{N}(e, y)$ of (e, y) such that

$$\mathcal{N}(e, y) \cap \text{Graph}(M) = \emptyset .$$

Indeed, if $(e, y) \notin \text{Graph}(M)$, i.e. $y \notin M(e)$, then either $y \notin F(e)$, or $y \in F(e)$ but there exists $z \in F(e)$ such that $u(e, z) > u(e, y)$. In the case of $y \notin F(e)$, since F is closed (i.e. its graph is closed), there exists a neighborhood $\mathcal{N}(e, y)$ of (e, y) such that

$$\mathcal{N}(e, y) \cap \text{Graph}(F) = \emptyset ,$$

and thus

$$\mathcal{N}(e, y) \cap \text{Graph}(M) = \emptyset .$$

In the case where $y \in F(e)$ but there exists $z \in F(e)$ such that $u(e, z) > u(e, y)$, since u is quasi-transfer upper continuous in (e, y) with respect to F, there is some neighborhood $\mathcal{N}(e, y)$ of (e, y) such that for any $(e', y') \in \mathcal{N}(e, y)$ with $y' \in F(e')$, there exists some $z' \in F(e')$ satisfying

$$u(e', z') > u(e', y'),$$

which means $y' \notin M(e')$. Now for those $(e', y') \in \mathcal{N}(e, y)$ with $y' \notin F(e')$, it is clear that $y \notin M(e')$. Thus $y' \notin M(e')$ for all $(e', y') \in \mathcal{N}(e, y)$. Therefore

$$\mathcal{N}(e, y) \cap \text{Graph}(M) = \emptyset ,$$

and thus M is closed.

Necessity. We want to show that u is quasi-transfer upper continuous in (e, y) with respect to F if M is closed. For any $(e, y) \in E \times Y$ with $y \in F(e)$, if $u(e, z) > u(e, y)$ for some $z \in F(e)$, then $(e, y) \notin \text{Graph}(M)$, and thus, by the closedness of M, there exists some neighborhood $\mathcal{N}(e, y)$ of (e, y) such that

$$\mathcal{N}(e, y) \cap \text{Graph}(M) = \emptyset .$$

Thus, for any $(e', y') \in \mathcal{N}(e, y)$ with $y' \in F(e')$, there exists some $z' \in F(e')$ satisfying

$$u(e', z') > u(e', y'),$$

which means u is quasi-transfer upper continuous in (e, y) with respect to F.

Finally, if F is u.s.c. as well, since $M(e) = M(e) \cap F(e)$ and F is compact-valued, then, by Theorem 6.1.7. in Berge (1963, p. 112), M is u.s.c. Q.E.D.

Remark 12. From the proof of the theorem, we know that M is non-empty compact-valued if and only if u is transfer upper continuous in y on $F(e)$ for every $e \in E$, and the condition that u is quasi-transfer upper continuous in (e, y) with respect to F is necessary and, in conjunction with the closedness of F, also sufficient for M to be closed.

Remark 13. Observe that u is transfer upper continuous in y on $F(e)$ for every $e \in E$ if u is upper semi-continuous in y by taking $z' = z$, and u is quasi-transfer upper continuous in (e, y) with respect to F if either u is continuous and F is l.s.c. or u is u.s.c. and FPT l.s.c. in e w.r.t. F. [7] Thus Theorem 3 generalizes the Maximum Theorem of Berge (1959, 1963) and Tian and Zhou (1992) by relaxing lower and upper semi-continuities of u and F. Also, the conclusion of Theorem 3 still holds if the compactness of the set $F(e)$ is relaxed by assuming that for each $e \in E$, there exists finitely many points $\{y_{01}, \ldots, y_{0m}\} \in F(e)$ such that $\bigcup_{i=1}^{m} \mathrm{cl}G(e, y_{0i})$ is compact, where $G(e, y) = \{y \in F(e): u(e, y) \geq u(e, y)\}$.

Remark 14. We can see that a sufficient condition for u to be not only transfer upper continuous in y on $F(e)$ for every $e \in E$ but also quasi-transfer upper continuous in (e, y) with respect to F is that u is transfer upper continuous in (e, y) with respect to F, which is defined as follows:

Definition 9. Let E and Y be two topological spaces. Let F: $E \to 2^Y$ be a correspondence. A function $u(e, y)$: $E \times Y \to \mathbb{R} \cup \{-\infty\}$ is said to be *transfer upper continuous in* (e, y) *with respect to* F if, for every $(e, y) \in E \times Y)$ with $y \in F(e)$, $u(e, z) > u(e, y)$ for some $z \in F(e)$ implies that there is a point $z' \in Y$ and a neighborhood $\mathcal{N}(e, y)$ of (e, y) such that for any $(e', y') \in \mathcal{N}(e, y)$ with $y' \in F(e')$, $u(e', z') > u(e', y')$ and $z' \in F(e')$.

[7] A function $u(e, y): E \times Y \to \mathbb{R}$ is said to be *Feasible Path Transfer Lower Semi-continuous* (in short, FPT l.s.c.) *in* e *with respect to* F if for each $(e, y) \in E \times Y$ with $y \in F(e)$, there exists some neighborhood $\mathcal{N}(e)$ of e such that $\forall e' \in \mathcal{N}(e)$, $\exists y' \in F(e')$ (a feasible path) satisfying $[u,(e, y) \leq \liminf_{e' \to e} u(e', y')$, or, equivalently, for each $(e, y) \in E \times Y$ with $y \in F(e)$ and $\forall \epsilon > 0$, $\exists \mathcal{N}(e)$ such that $\forall e' \in \mathcal{N}(e)$, $\exists$ (feasible path) $y' \in F(e')$ satisfying $u(e, y) < u(e', y') + \epsilon$. This notion was introduced by Tian and Zhou (1992).

Corollary 2. Let E and Y be two topological spaces. Let F: $E \to 2^Y$ be a non-empty compact-valued and closed correspondence, and let u: $E \times Y \to \mathbb{R} \cup \{-\infty\}$ be a real-valued function. Suppose u is transfer upper continuous in (e, y) with respect to F. Then the correspondence M: $E \to 2^Y$ defined, for each $e \in E$, as

$$M(e) = \{y \in F(e) : u(e, y) \geq (e, x), \quad \forall x \in F(e)\},$$

is non-empty compact-valued and closed. If, in addition, F is u.s.c., then M is u.s.c.

Now we generalize Walker's Maximum Theorem. We first introduce some transfer concepts for preference correspondences.

Definition 10. Let E and Y be two topological spaces. Let F: $E \to 2^Y$ be a correspondence. A correspondence P: $E \times Y \to 2^Y$ is said to have *quasi-transfer open lower sections in (e, y) with respect to F* if, for every $(e, y) \in E \times Y$ with $y \in F(e)$, $z \in P(e, y)$ for some $z \in F(e)$ implies that there is some neighborhood $\mathcal{N}(e, y)$ of (e, y) such that for any $(e', y') \in \mathcal{N}(e, y)$ with $y' \in F(e')$, there exists some $z' \in F(e')$ satisfying $z' \in P(e', y')$.

Definition 11. Let E and Y be two topological spaces. Let F: $E \to 2^Y$ be a correspondence. A correspondence P: $E \times Y \to 2^Y$ is said to have *transfer open lower sections in y on $F(e)$* for every $e \in E$ if, for every $(e, y) \in E \times Y$ with $y \in F(e)$, $z \in P(e, y)$ for some $z \in F(e)$ implies that there is a $z' \in Y$ and a neighborhood $\mathcal{N}(y)$ of y such that for any $y' \in \mathcal{N}(y)$ with $y' \in F(e)$, $z' \in P(e, y')$ and $z' \in F(e)$.

Definition 12. Let E and Y be two topological spaces. Let F: $E \to 2^Y$ be a correspondence. A correspondence P: $E \times Y \to 2^Y$ is said to have *transfer open lower sections in (e, y) with respect to F* if, for every $(e, y) \in E \times Y$ with $y \in F(e)$, $z \in P(e, y)$ for some $z \in F(e)$ implies that there is a $z' \in Y$ and a neighborhood $\mathcal{N}(e, y)$ of (e, y) such that for any $(e', y') \in \mathcal{N}(e, y)$ with $y' \in F(e')$, $z' \in P(e', y')$ and $z' \in F(e')$.

Theorem 4. Let E and Y be two topological spaces. Let F: $E \to 2^Y$ be a non-empty compact-valued and closed correspondence and let P: $E \times Y \to 2^Y$ be a (preference) correspondence. Suppose (i) *either P is lower semi-continuous in y or it has transfer open lower sections in y on $F(e)$ for every $e \in E$ and* (ii) *P has quasi-transfer open lower sections in (e, y) with respect to F. Then the correspondence M: $E \to 2^Y$ defined, for each $e \in E$, as*

$$M(e) = \{y \in F(e) : P(e, y) \cap F(e) = \emptyset\},$$

is compact-valued (but possibly empty [8]*) and closed. If, in addition, F is u.s.c. then M is u.s.c.*

Proof. We first show that M is compact-valued. Since F is compact-valued, we only need to show that $M(e)$ is closed for every $e \in E$. Thus, it suffices to show that its complement $\{y \in F(e): P(e, y) \cap F(e) \neq \emptyset\}$ is open in $F(e)$. If P is l.s.c. in y, then it is l.s.c. on the subset $F(e) \subset Y$ and thus $\{y \in F(e): P(e, y) \cap F(e) \neq \emptyset\}$ is open in $F(e)$ by the definition of the lower semi-continuity of P.

Now suppose P has transfer open lower sections in y on $F(e)$. Let $P^{-}(z; e) = \{y \in F(e): z \in P(e, y)\}$. We show that $\bigcup_{z \in F(e)} \text{int}_{F(e)} P^{-}(z; e) = \bigcup_{z \in F(e)} P^{-}(z; e)$ for all $e \in E$. It is clear that $\bigcup_{x \in F(e)} \text{int}_{F(e)} P^{-}(z; e) \subset \bigcup_{z \in F(e)} P^{-}(z; e)$. So we only need to show $\bigcup_{z \in F(e)} P^{-}(z; e) \subset \bigcup_{z \in F(e)} \text{int}_{F(e)} P^{-}(z; e)$. Let $y \in \bigcup_{z \in F(e)} P^{-}(z; e)$, then $y \in F(e)$ and $z \in P(e, y)$ for some $z \in F(e)$. Since P has transfer open lower sections in y on $F(e)$, there is a $z' \in Y$ and a neighborhood $\mathcal{N}(y)$ of y such that for any $y' \in \mathcal{N}(y)$ with $y' \in F(e)$, $z' \in P(e, y')$ and $z' \in F(e)$. Then $y \in \text{int}_{F(e)} P^{-}(z'; e)$ and thus

$$\bigcup_{z \in F(e)} P^{-}(z; e) \subset \bigcup_{z \in F(e)} \text{int}_{F(e)} P^{-}(z; e).$$

Hence we have

$$\begin{aligned} \{y \in F(e): P(e, y) \cap F(e) \neq \emptyset\} &= \bigcup_{z \in F(e)} P^{-}(z; e), \\ &= \bigcup_{z \in F(e)} \text{int}_{F(e)} P^{-}(z; e), \qquad (6) \end{aligned}$$

which means that the set $\{y \in F(e): P(e, y) \cap F(e) \neq \emptyset\}$ is open in $F(e)$ and thus $M(x)$ is closed for every $e \in E$.

Finally, we show that M is a closed correspondence, i.e. the graph of M:

$$\text{Graph}(M) = \{(e, y) \in E \times Y: y \in M(e)\}$$

is closed. That is, we only need to show that, if $(e, \text{i}y) \notin \text{Graph}(M)$, there exists some neighborhood $\mathcal{N}(e, y)$ of (e, y) such that

$$\mathcal{N}(e, y) \cap \text{Graph}(M) = \emptyset.$$

Indeed, if $(e, y) \notin \text{Graph}(M)$, i.e. $y \notin M(e)$, then either $y \notin F(e)$, or $y \in F(e)$

[8] Tian (1993a) gave a necessary and sufficient condition for the existence of maximal elements of a preference correspondence when it has transfer open lower sections. This condition can be used to show the non-emptyness of $M(e)$ when P has transfer open lower sections in y on $F(e)$ for every $e \in E$.

but there exists $z \in F(e)$ such that $P(e, y) \cap F(e) \neq \emptyset$. In the case of $y \notin F(e)$, since F is closed (i.e. its graph is closed), there exists a neighborhood $\mathscr{N}(e, y)$ of (e, y) such that

$$\mathscr{N}(e, y) \cap \text{Graph}(F) = \emptyset,$$

and thus

$$\mathscr{N}(e, y) \cap \text{Graph}(M) = \emptyset.$$

In the case where $y \in F(e)$ but there exists $z \in F(e)$ such that $z \in P(e, y)$, since P has quasi-transfer open lower sections in (e, y) with respect to F, there is some neighborhood $\mathscr{N}(e, y)$ of (e, y) such that for any $(e', y') \in \mathscr{N}(e, y)$ with $y' \in F(e')$, there is some $z' \in F(e')$ satisfying $z' \in P(e', y')$, which means $y' \notin M(e')$. Now for those $(e', y') \in \mathscr{N}(e, y)$ with $y' \notin F(e')$, it is clear $y' \notin M(e')$. Thus $y' \notin M(e')$ for all $(e', y') \in \mathscr{N}(e, y)$. Therefore

$$\mathscr{N}(e, y) \cap \text{Graph}(M) = \emptyset,$$

and thus M is closed. Q.E.D.

Remark 15. The conditions above are satisfied if P has open lower sections on $F(e)$ for every $e \in E$ (i.e. $P^{-}(z; e)$ is open in $F(e)$ for all $z \in F(e)$) and F is l.s.c. Also the condition that P has quasi-transfer open lower sections in (e, y) with respect to F is a necessary, and under the closedness of F, sufficient condition for M to be closed.

Similarly, since the condition that P is transfer open-valued in (e, y) with respect to F implies that P has transfer open lower sections in y on $F(e)$ for every $e \in E$ and P has quasi-transfer open lower sections in (e, y) with respect to F, we have the following corollary.

Corollary 3. Let E and Y be two topological spaces. Let F: $E \to 2^Y$ be a non-empty compact-valued and closed correspondence and let P: $E \times Y \to 2^Y$ be a (preference) correspondence. Suppose P has transfer open lower sections in (e,y) with respect to F. Then the correspondence M: $E \to 2^Y$ defined, for each $e \in E$, as

$$M(e) = \{ y \in F(e) : P(e, y) \cap F(e) = \emptyset \}$$

is compact-valued (but possibly empty) and closed. If, in addition, Y is compact or F is u.s.c., then M is u.s.c.

Remark 16. Note that P has transfer open lower sections in (e, y) with respect to F if P and F have open lower sections.

5. The existence of equilibria of games

In this section we consider the existence of equilibria of (generalized) games as possible applications of our main results given above. Accordingly, let I be the set of agents which is any countable or uncountable set. Each agent i chooses a strategy x_i in a choice set X_i of a locally convex topological vector space. Denote by X the (Cartesian) product $\prod_{j \in I} X_j$ and X_{-i} the product $\prod_{j \in I \setminus \{i\}} X_j$. Denote by x and x_{-i} elements of X and X_{-i}. Each agent i has a payoff (utility) function u_i: $X \to \mathbb{R} \cup \{-\infty\}$. Given x_{-i} (the strategies of others), the choice of the ith agent is restricted to a non-empty set $F_i(x_{-i}) \subset X_i$, the *feasible strategy set*; the ith agent chooses $x_i \in F_i(x_{-i})$ so as to maximize $u_i(x_{-i}, x_i)$ over $F_i(x\_i)$.

Definition 13. A generalized game (an abstract economy) $\Gamma = (X_i, F_i, u_i)_{i \in I}$ is defined as a family of ordered triples (X_i, F_i, u_i). A vector $x^* \in X$ is said to be an *equilibrium of a generalized game* if, $\forall i \in I$,

(i) $x_i^* \in F_i(x_{-i}^*)$ and

(ii) x_i^* maximizes $u_i(x_{-i}^*, x_i)$ over $F_i(x_{-i}^*)$.

If $F_i(x_{-i}) \equiv X_i$, $\forall i \in I$, the generalized game reduces to the conventional game $\Gamma = (X_i, u_i)$ and the equilibrium is called a *Nash equilibrium*.

5.1. The existence of equilibria of generalized games

Debreu (1952) proved that, for a finite number of players and the Euclidean space, an equilibrium of the generalized game exists if (i) X_i is a contractable polyhedron, (ii) F_i is closed, (iii) u_i is continuous such that the function $x_{-i} \to \max\{u_i(x_{-i}, x_i)$: $x_i \in F_i(x_{-i})\}$ is continuous, and (iv) the maximum set $M(x_{-i}) = \{x_i \in F_i(x_{-i}) : u_i(x_{-i}, x_i) \geq u_i(x_{-i}, z_i), \forall z_i \in F_i(x_{-i})\}$ is contractable. Since then, this classical result has been extended in many ways such as the results in Shafer and Sonnenschein (1975), Borglin and Keiding (1976), Yannelis and Prabhakar (1983), Khan and Vohra (1984), Toussaint (1984), Khan (1986), Yannelis (1987), Khan and Papageorgiou (1987), Kim et al. (1989), Tian (1992a, 1993b), and others. Now we apply our Theorem 3 to prove an existence theorem on equilibria in generalized games where the payoff (utility) functions u_i are neither l.s.c. nor u.s.c. and the feasible strategy correspondences F_i are not l.s.c. either

Theorem 5. For $i \in I$ let X_i be a non-empty convex compact subset in a locally convex topological vector space. If, for each $i \in I$, $x = (x_{-i}, x_i)$,

(i) *$F_i : X_{-i} \to 2^{X_i}$, is non-empty convex compact-valued and u.s.c.,*

(ii) *u_i and F_i satisfy the following three conditions:*

(ii.a) *for every $x_{-i} \in X_{-i}$, $u_i : X \to \mathbb{R} \cup \{-\infty\}$ is transfer upper continuous in x_i on $F_i(x_{-i})$,*

(ii.b) *u_i is quasi-transfer upper continuous in x with respect to F_i,* (ii.c) *either* u_i *is quasiconcave in* x_i *or it has a unique maximum point on* $F_i(x_{-i})$ *for all* $x_i \in F_i(x_{-i})$,

then there exists an equilibrium for the generalized game $\Gamma = (X_i, F_i, u_i)_{i \in I}$.

Proof. For each $i \in I$, define the maximizing correspondence

$$M_i(x_{-i}) = \{x_i \in F_i(x_{-i}) : u_i(x_{-i}, x_i) \geq u_i(x_{-i}, z_i), \quad \forall z_i \in F_i(x_{-i})\}.$$

For these u_i, F_i and M_i, from Theorem 3, we know that M_i is u.s.c. and non-empty compact-valued if and only if conditions (ii.a) and (ii.b) hold. Also, by condition (ii.c) and convexity of $F_i(x_{-i})$, the correspondence $M_i : X_{-i} \to 2^{X_i}$, is convex-valued. Therefore the correspondence

$$M(x) = \prod_{i \in I} M_i(x_{-i})$$

is u.s.c. by Lemma 3 in Fan (1952, p. 124) and non-empty convex compact-valued from $X \to 2^X$. So, by Theorem 1 in Fan (1952, p. 122), there exists $x^* \in X$ such that $x^* \in M(x^*)$. Thus x^* is an equilibrium for the generalized game. Q.E.D

Similarly, if u_i is transfer upper continuous in x with respect to F_i, it implies conditions (ii.a) and (ii.b) in Theorem 5.

5.2. The existence of Nash equilibrium

By definition, when the feasible strategy correspondence F_i is constant set-valued, the generalized game reduces to the conventional game. Nash (1951) proved that an (Nash) equilibrium of the game exists if each $X_i \subset \mathbb{R}^{L_i}$ is compact, convex, and non-empty, and if u_i is continuous on X and (quasi-)concave in x_i. Dasgupta and Maskin (1986) first studied the existence of Nash equilibrium for games with discontinuous payoff functions. They proved that an equilibrium of the game exists if each $Z_i \subset \mathbb{R}^{L_i}$ is compact, convex, and non-empty, and if the $u_i(x_i, x_{-i})$ is quasiconcave in x_i, upper semi-continuous in x and graph-continuous. [9] Here we give an existence theorem on Nash equilibrium which generalizes the results of Nash (1950, 1951), Nikaido and Isoda (1955), Dasgupta and Maskin (1986), and Tian and Zhou (1992).

Definition 14. The payoff function u_i is said to be *quasi-transfer upper continuous in* x *with respect to* X_i if, for every $x \in X$, $u_i(x_{-i}, z_i) > u(x_{-i}, x_i)$ for some $z_i \in X_i$ implies that there is some neighborhood $\mathcal{N}(x)$ of x such that for any $x' \in \mathcal{N}(x)$, there exists some $z'_i \in X_i$ satisfying $u_i(x'_{-i}, z'_i) > u_i(x'_{-i}, x'_i)$.

[9] Dasgupta and Maskin (1986) defined that a payoff function is graph-continuous if for all $\bar{x} \in X$ there exists a function $f_i : X_{-i} \to X_i$ with $f_i(\bar{x}_{-i}) = \bar{x}_i$ such that $u_i(f_i(x_{-i}), x_{-i}))$ is continuous at $x_{-i} = \bar{x}_{-i}$.

Theorem 6. For $i \in I$. let X_i be a non-empty convex compact set of a locally convex topological vector space. If for each $i \in I$, $x = (x_{-i}, x_i)$, the payoff function $u_i : X \to \mathbb{R} \cup \{-\infty\}$ satisfies the following conditions:

(i) *u_i is transfer upper continuous in x_i*;

(ii) *u_i is quasi-transfer upper continuous in x with respect to X_i*,

(iii) *either u_i is quasiconcave in x_i or it has a unique maximum point on X_i, then there exists a Nash equilibrium for the game $\Gamma = (X_i, u_i)_{i \in I}$*.

Proof. For each $i \in N$ and $x_{-i} \in X_{-i}$, let $F_i(x_{-i}) \equiv X_i$ and apply Theorem 5. Q.E.D.

Remark 17. Since conditions (i) and (ii) are necessary and sufficient conditions for the maximum correspondence $M_i(x_{-i}) = \{x_i \in X_i : u_i(x_{-i}, x_i) \geq u_i(x_{-i}, z_i), \forall z_i \in X_i\}$ to be non-empty and closed (see Remark 12), the conditions in Theorem 2 of Dasgupta and Maskin (1986) imply conditions (i)–(iii) of the above theorem.

Similarly, the condition that u_i is transfer upper continuous in x with respect to X_i (i.e. for every $x \in X$, if $u_i(x_{-i}, z_i) > u_i(x_{-i}, x_i)$ for some $z_i \in X_i$, then there is some $z'_i \in X_i$ and some neighborhood $\mathcal{N}(x)$ of x such that for any $x' \in \mathcal{N}(x)$, $u_i(x'_{-i}, z'_i) > u_i(x'_{-i}, x'_i)$) implies conditions (i) and (ii) in the above theorem.

Remark 18. The compactness of strategy spaces of games can be relaxed by assuming that for each $i \in I$ and $x_{-i} \in X_{-i}$, there exist finitely many points $\{x_i^{01}, \ldots, x_i^{0m}\}$ such that $\bigcap_{k=1}^{m} \mathrm{cl} G_i(x_{-i}, x_i^{0k})$ is compact where the set $G_i(x_{-i}, x_i) = \{y_i \in X_i : u_i(x_{-i}, x_i) - u_i(x_{-i}, y_i) \leq 0\}$. The compactness can also be relaxed along the line proposed in Tian (1992c, 1993a) or Tian and Zhou (1992).

6. Conclusion

The results obtained above reveal that the conventional continuities of a function or preference correspondence assumed in the literature on the existence of extreme points can be significantly weakened. In doing so, we have introduced the concepts of various transfer continuities to deal with the existence of extreme points by giving necessary and sufficient conditions. Specifically, we proved that transfer weak upper (lower) continuity and transfer upper (lower) continuity are necessary and sufficient conditions for a function to attain its maximum on a compact set, and for the set of maximum points of a function on a compact set to be non-empty and compact. We also gave necessary and sufficient conditions for the maximum (marginal) correspondence to be closed. Thus our results generalize the classical Weierstrass theorem and the Maximum Theorems of Berge (1959, 1963, p. 116) and Walker (1979) by giving a full characterization. By applying our results to game theory and economics, we can generalize many of the existence theorems on the equilibrium of games and the equilibrium of generalized games (the so-called abstract economies) in the literature.

As a final remark, the transfer continuities introduced in this paper characterize the essence of topological structures for extreme points. The conventional continuities are unnecessarily strong for optimization. Even though this paper only considers generalizations of the Weierstrass and maximum theorems and some possible applications of these results, we believe that the transfer method can be profitably used to generalize many results in microeconomics, dynamic economics, optimization theory, and many other fields.

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