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On Nash-Implementation in the Presence of Withholding

GUOQIANG TIAN*,†

Department of Economics, Texas A&M University, College Station, Texas 77843

AND

QI LI*,‡

Department of Economics, University of Guelph, Guelph, Ontario N1G 2W1, Canada

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In this paper we construct a completely feasible and continuous mechanism whose Nash allocations coincide with Lindahl allocations when both preferences and initial endowments are unknown to the designer and unreported endowments are withheld. This mechanism extends the mechanism of Hong by allowing for semi-positivity of endowments for private goods economies and the mechanism of Tian by allowing for any number of private goods for public goods economies. Thus our mechanism improves all the existing mechanisms that implement Walrasian or Lindahl allocations. *Journal of Economic Literature* Classification Numbers: C72, D61, D78, H41. © 1995 Academic Press, Inc.

1. INTRODUCTION

Economists continue to look for better incentive-compatible mechanisms which could solve the so-called "free-rider" problem in the mechanism design literature. Until recently, the existing mechanisms (e.g., those in Groves and Ledyard (1977, 1987), Hurwicz (1979), Schmeidler (1980),

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Walker (1981), and Hurwicz *et al.* (1984)) had been not completely feasible, or not continuous in the sense that outcome functions may not be continuous, or result in allocations which may not be in the individuals' consumption sets and/or exceed total endowments. Recently Tian (1992) and Tian and Li (1991) gave completely feasible and continuous mechanisms which Nash-implement the constrained Walrasian correspondence for private goods economies and the Lindahl correspondence for public goods economies with an arbitrary number of private and public goods, respectively. These mechanisms, however, assume that the designer knows the true endowments of individuals to guarantee complete feasibility. When the true endowments are unknown to the designer, individuals may have an incentive to underreport their endowments. Hence constructing mechanisms that are robust in the case of withholding is an important research topic. There are only a few papers in the literature which discuss the case of withholding. Hurwicz *et al.* (1984) were the first to give such withholding mechanisms, but their mechanisms are discontinuous. More recently, Hong (1990) successfully constructed a completely feasible and continuous withholding mechanism which Nash-implements the Walrasian correspondence for private goods economies with productions by extending a mechanism in Tian (1992). For public goods economies, Tian (1993) similarly gave such a mechanism which Nash-implements the Lindahl correspondence. However, Hong (1990) assumed that true endowments are strictly positive for all goods, and the designer knows this information. Strict positivity of endowments is an unrealistic assumption. In particular, this is true when the number of goods becomes large. On the other hand, Tian's mechanism only considers the case of one private good while allowing for any number of public goods. The motivations behind designing completely feasible and continuous withholding mechanisms can be found in Tian (1990, 1991, 1992).

This paper shows that both the positive endowments and the single private good assumptions can be relaxed. We give a completely feasible and continuous withholding mechanism which Nash-implements the Lindahl correspondence for public goods economies with semi-positive true endowments and any number of private goods. When there are no public goods in economies, the mechanism reduces to a mechanism which Nash-implements the Walrasian correspondence. Unlike all the existing mechanisms such as those in Hurwicz *et al.* (1984), Tian (1989), and Hong (1990), which deal with the case where endowments are private information and underreported endowments are either destroyed or withheld (consumed), our mechanisms allow individuals to report semi-positive endowments even at equilibrium as long as not all components of reported endowments are zero. In addition, our mechanism requires only that for each good, the designer knows that at least one of the individuals has this private good (but need not know how much). This information requirement for the designer in our mechanism is much less restrictive than previous mechanisms and will still guarantee that the sum of reported endowments is strictly positive. Note also that the mechanism presented here has the advantage that agents are not required to report their true endowments even at equilibrium. This is an important property. The situation where agents are not forced to report their true endowments is quite different from the previous one in which endowments are assumed to be known to the designer. Thus the incentive compatibility problem for reporting endowments is well taken.

The plan of the paper is as follows. A public goods model and the mechanism with the desirable properties are presented in Section 2. The main results and their proofs are given in Section 3. Finally, the concluding remarks are offered in Section 4.

2. Public Goods Model and Mechanism

2.1. *Economic Environments and Lindahl Allocations*

In a public goods economy under consideration, there are n ($n \geqq 3$) agents who consume L private goods x and K public goods y.[1] Throughout this paper, the subscripts are used to index agents while the superscripts are used to index goods. Denote by $N = \{1, 2, \ldots, n\}$ the set of agents. The characteristic of agent i is denoted by $e_i = (\mathring{w}_i, P_i)$, where $\mathring{w}_i$ is the initial endowment vector of private goods with $\mathring{w}_i \geq 0$ and $\mathring{w} \equiv \sum_{i=1}^{n} \mathring{w}_i > 0$, and P_i the strict (irreflexive) preference relation which may be nontotal–nontransitive but is convex,[2] strictly monotone increasing in private goods on $\mathbb{R}^L_{++}$, and further satisfies the condition that every interior private goods consumption is strictly preferred to the boundary private goods consumption, i.e., for all $i \in N$, $(x_i, y)\ P_i\ (x_i', y')$ for all $x_i \in \mathbb{R}^L_{++}$, $x_i' \in \partial\mathbb{R}^L_+$, and $y, y' \in \mathbb{R}^K_+$, where $\partial\mathbb{R}^m_+$ is the boundary of $\mathbb{R}^m_+$. We assume that there are no initial endowments of public goods, but that public goods can be produced from private goods by a production function f: $\mathbb{R}^L_+ \to \mathbb{R}^K_+$ which is assumed to be continuous, strictly quasi-concave, homogeneous of degree one, and increasing. An economy is the full vector $e = (e_1, \ldots, e_n, f(\cdot))$ and the set of all such economies is denoted by E.

An allocation $(x, y) = (x_1, \ldots, x_n, y)$ is *feasible* for an economy e if

[1] As usual, vector inequalities are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \ldots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \ldots, m$.

[2] P_i is convex if, for bundles a, b, c and $0 < \lambda \leqq 1$, $c = \lambda a + (1 - \lambda)^b$, the relation $a\ P_i\ b$ implies $c\ P_i\ b$ on $\mathbb{R}^{L+K}_+$.

$(x, y) \in \mathbb{R}_+^{nL+K}$, $y = f(v)$, and $\sum_{i=1}^n x_i + v \leqq \sum_{i=1}^n \mathring{w}_i$. An allocation (x^*, y^*) is a *Lindahl allocation* for an economy e if it is feasible and there is a price vector $p^* \in \mathbb{R}_+^L$, price vectors $q_i^* \in \mathbb{R}^K$, one for each i, such that

(1) $p^* \cdot x_i^* + q_i^* \cdot y^* \leqq p^* \cdot \mathring{w}_i$ for all $i \in N$;

(2) for all $i \in N$, $(x_i, y)\ P_i\ (x_i^*, y^*)$ implies $p^* \cdot x_i + q_i^* \cdot y > p^* \cdot \mathring{w}_i$;

(3) $q^* \cdot y^* - p^* \cdot v^* = 0$, where $v^* = \sum_{i=1}^n \mathring{w}_i - \sum_{i=1}^n x_i^*$ and $\sum_{i=1}^n q_i^* = q^*$.

Note that condition (3) is the familiar zero-profit condition under constant returns to scale. Denote by $L(e)$ the set of all such allocations.

2.2. *Mechanism*

We now present an individually feasible, balanced, and continuous mechanism which fully Nash-implements the Lindahl correspondence for economies with many private and public goods, when both preferences and endowments are unknown to the designer and withheld endowments are consumed but not destroyed. Unlike previous mechanisms in the literature, our mechanism requires only that for each private good l, the designer know at least one agent has some endowment of the good although he does not know how much. In addition, the designer knows that every agent has some endowments, i.e., $\mathring{w}_i \geq 0$, but does not know for which goods or how much of which goods, i.e., he does not know the agent's true endowment $\mathring{w}_i$. Further, we assume that every agent is asked to report a nonzero endowment vector, denoted by w_i, and a positive amount of some endowment of good l if the designer knows the agent has an endowment for good l. Thus the sum of reported endowments of all goods is positive, i.e., $\sum_{i=1}^n w_i > 0$ with $w_i \neq 0$.

Let M_i denote the ith message domain. Its elements are written as m_i and called messages. Let $M = \prod_{i=1}^n M_i$ denote the message space. The message spaces of agents are defined as follows.

For each $i \in N$, his/her message domain is of the form

$$M_i = (0,1] \times \{0, \mathring{w}_i] \times \mathbb{R}_{++}^L \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}_+^{nL}. \tag{1}$$

Here $\{0, \mathring{w}_i] = \{w_i \in \mathbb{R}_+^L : 0 \leq w_i \leqq \mathring{w}_i\}$. A generic element of M_i is $(\delta_i, w_i, p_i, \phi_i, y_i, (x_{i1}, \ldots, x_{in}))$ whose components have the following interpretations. The component δ_i denotes the ratio of income that agent i is willing to spend on private goods. In particular, when $\delta_i = 1$, agent i wishes that public goods would not be produced. The designer will use the smallest δ_i of all agents to determine the level of public goods (see Eq. (6) below). The component w_i denotes a claim about agent i's endowment, the inequality $0 \leq w_i \leqq \mathring{w}_i$ means that the agent cannot overstate his own endowment; On the other hand, the endowment can be understated, but the claimed endowment w_i must satisfy the assumption made above, i.e., $\sum_{i=1}^n w_i > 0$ with $w_i \neq 0$. Note that although the true endowment is the upper bound of the reported endowment, the designer does not need to know this upper bound. This is because whenever an agent claims an endowment of a certain amount, the designer can ask him to *exhibit* it. The component p_i denotes the price vector of private goods proposed by agent i. The component ϕ_i denotes the price vector of public goods proposed by agent i for use in other agents' budget constraints. The component y_i denotes the proposed level of public goods that agent i is willing to contribute (a negative y_i means the agent wants to receive a subsidy from society). Finally, the component x_{ij} denotes the contribution of private goods that agent i is willing to make to agent j (a negative x_{ij} means agent i wants to get $-x_{ij}$ amount of private goods from agent j).

Define the price vector for the private goods p: $M \to \mathbb{R}^L_{++}$ by

$$p(m) = \begin{cases} \sum_{i=1}^n \frac{a_i}{a} p_i & \text{if } a > 0 \\ \sum_{i=1}^n \frac{1}{n} p_i & \text{if } a = 0 \end{cases}, \tag{2}$$

which is continuous. Here $a_i = \sum_{j,s \in N \setminus \{i\}} \|p_j - p_s\|$, $a = \sum_{i=1}^n a_i$, and $\|\cdot\|$ is the Euclidean norm. Note that even though $p(m)$ is a function of the component $(p_1, \ldots, p_n)$ of the message m only, we can write it as a function of m without loss of generality.

Let the prices $q^k(m)$ of public goods be equal to the minimum-unit-cost functions which are dual to the production functions, f^k ($k = 1, \ldots, K$), and defined by

$$c^k(p(m)) = \min\{p(m) \cdot v : f^k(v) = 1\} \tag{3}$$

which are differentiable, homogeneous of degree one, and concave in $p(m)$. Thus $q^k(m) = c^k(p(m))$ for $k = 1, \ldots, K$. Then, by Shephard's lemma, the input demand functions, $v^k(y, p(m))$, are given by

$$v^k(y, p(m)) = y^k \frac{\partial c^k(p(m))}{\partial p(m)} \qquad \text{for } k = 1, \ldots, K.$$

Note that $v^k(y, p(m))$ is continuous by the Maximum Theorem.

The aggregate input demand function is then given by $v(y, p(m)) = \sum_{k=1}^K v^k(y, p(m)) = Dc(p(m))y$, where $Dc(p(m)) = (\partial c^1(p(m))/\partial p(m), \ldots,$

$\partial c^K(p(m))/\partial p(m))$ is a $L \times K$ matrix. Thus, by the homogeneity of degree one of $c(p(m))$, we have

$$q(m) \cdot y - p(m) \cdot v(y, p(m)) = 0 \tag{4}$$

which means the zero-profit condition holds for all $m \in M$.

Define the personalized price of each public good k for the ith consumer by

$$q_i^k(m) = \frac{1}{n} q^k(m) + \phi_{i+1}^k - \phi_{i+2}^k, \tag{5}$$

where $n + 1$ is regarded as 1 and $n + 2$ is regarded as 2. Observe that, by construction, $\sum_{i=1}^n q_i(m) = q(m)$ for all $m \in M$ and each agent's personalized prices are independent of his own message ϕ_i. Here $q_i(m) = (q_i^1(m), \dots, q_i^K(m))$ and $q(m) = (q^1(m), \dots, q^K(m))$.

Define the correspondence B_y: $M \rightarrow 2^{\mathbb{R}_+^K}$ by

$$\begin{aligned} B_y(m) = \{y \in \mathbb{R}_+^K : (1 - \delta(m))p(m) \cdot w_i - q_i(m) \cdot y \geqq 0 \quad \forall i \in N, \\ (1 - \delta(m))w - v(y, p(m)) \geqq 0\}, \end{aligned} \tag{6}$$

which is clearly a continuous correspondence with non-empty compact convex values. Here $\delta(m) = \min\{\delta_1, \dots, \delta_n\}$.

Define the outcome function for public goods Y: $M \rightarrow B_y$ by

$$Y(m) = \{y: \min_{y \in B_y(m)} \|y - \bar{y}\|\}, \tag{7}$$

which is the closest to $\bar{y}$. Here $\bar{y} = \sum_{i=1}^n y_i$. Then $Y(m)$ is single-valued and continuous on M by the Maximum Theorem.

For each individual i, define the taxing function T_i: $M \rightarrow \mathbb{R}$ by

$$T_i(m) = q_i(m) \cdot Y(m). \tag{8}$$

Then

$$\sum_{i=1}^n T_i(m) = q(m) \cdot Y(m). \tag{9}$$

To determine the level of private goods for each individual i, define a completely feasible correspondence B_x: $M \rightarrow 2^{\mathbb{R}_+^{nL}}$ by

$$B_x(m) = \Bigg\{ x \in \mathbb{R}^{nL}_{+} : \sum_{i=1}^{n} x_i + \hat{v}(m) = w,$$
$$p(m) \cdot x_i + q_i(m) \cdot Y(m) = p(m) \cdot w_i \qquad \forall i \in N, \tag{10}$$
$$x_i \geqq \frac{\delta(m)[p(m) \cdot w_i - q_i(m) \cdot Y(m)]}{p(m) \cdot w - q(m) \cdot Y(m)} (w - \hat{v}(m)) \qquad \forall i \in N \Bigg\},$$

where $\hat{v}(m) = v(Y(m), p(m)) = Dc(p(m))Y(m)$ is the input demand outcome function. Again we can see that B_x is a continuous correspondence with non-empty (by letting $x_i = \{(p(m) \cdot w_i - q_i(m) \cdot Y(m))/(p(m) \cdot w - q(m) \cdot Y(m))\}(w - \hat{v}(m)))$, compact, and convex values.

Let $\tilde{x}_j = \sum_{i=1}^{n} x_{ij}$, which is the sum of contributions that agents are willing to make to agent j, and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$.

The outcome function $X(m)$: $M \rightarrow B_x$ is given by

$$X(m) = \{z \in \mathbb{R}^{nL}_{+} : \min_{z \in B_x(m)} \|z - \tilde{x}\|\}, \tag{11}$$

which is the closest to $\tilde{x}$. Then $X(m)$ is single-valued and continuous on M. Note that $X_i(m) > 0$ by the definition of the constrained correspondence B_x, and the total (final consumption of agent i for private goods is the sum of $X_i(m)$ and $(\mathring{w}_i - w_i)$. That is, it is the sum of the amount of private goods allocated by the mechanism and the unreported amount of his/her own endowments.

Thus the outcome function is continuous on M, $(X(m), Y(m)) \in \mathbb{R}^{nL+K}_{+}$, and

$$\sum_{i=1}^{n} [X_i(m) + \mathring{w}_i - w_i] + v(Y(m), p(m)) = \sum_{i=1}^{n} \mathring{w}_i \tag{12}$$

for all $m \in M$.

From (12), we have

$$\sum_{i=1}^{n} X_i(m) + v(Y(m), p(m)) = \sum_{i=1}^{n} w_i, \tag{13}$$

which means that the sum of the aggregate consumption of the private goods allocated by the mechanism and the inputs used to produce the $Y(m)$ is equal to the sum of endowments reported by agents for all $m \in M$.

Denote by $h : M \rightarrow \mathbb{R}^{nL+K}_{+}$ the outcome function, or more explicitly, $h_i(m) = (X_i(m), Y(m))$. Then the withholding mechanism consists of $\langle M$,

$h\rangle$ defined on E. By the construction of the mechanism, the mechanism $\langle M, h\rangle$ is *individually feasible* (i.e., $(X_i(m) + \mathring{w}_i - w_i, Y(m)) \in \mathbb{R}_+^{L+K}$ for all $i \in N$ and all $m \in M$), *balanced* (i.e., (12) holds for all $m \in M$), and continuous.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is said to be a *Nash equilibrium* of the mechanism $\langle M, h\rangle$ in the presence of withholding for an economy e, if for any $i \in N$ and for all $m_i \in M_i$,

$$\neg(X_i(m^*/m_i, i) + \mathring{w}_i - w_i, Y(m^*/m_i, i))P_i(X_i(m^*) + \mathring{w}_i - w_i^*, Y(m^*)), \tag{14}$$

where $(m^*/m_i, i) = (m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. The outcome $((X_i(m^*) + \mathring{w}_i - w_i^*)_i, Y(m^*))$ is then called a *Nash (equilibrium) allocation* of the mechanism for the economy e. Denote by $V_{M,h}(e)$ the set of all such Nash equilibria and by $N_{M,h}(e)$ the set of all such Nash (equilibrium) allocations. The mechanism $\langle M, h\rangle$ is said to fully *Nash-implement* the Lindall correspondence L on E if, for all $e \in E$, $N_{M,h}(e) = L(e)$.

Remark 1. Observe that for $L = 1$ the part of the mechanism for allocations of private goods is not needed, reducing the mechanism to the one presented in Tian (1993). Also, if there are no public goods (i.e., $K = 0$), public goods economies reduce to private goods economies and the mechanism becomes a mechanism which Nash-implements the Walrasian correspondence. This mechanism extends the mechanism of Hong (1990) by allowing the semi-positivity of endowments.

3. Main Results

The remainder of the paper is devoted to the proof of equivalence between Nash allocations and Lindahl allocations. Theorem 1 below proves that every Nash allocation is a Lindahl allocation. Theorem 2 proves that every Lindahl allocation is a Nash allocation.

Theorem 1. *For every $e \in E$, if the withholding mechanism defined above has a Nash equilibrium m^*, then $((X_i(m^*) + \mathring{w}_i - w_i^*)_i, Y(m^*))$ is a Lindahl allocation with the price system $(p(m^*), q_1(m^*), \ldots, q_n(m^*))$, i.e., $N_{M,h}(e) \subseteq L(e)$ for all $e \in E$.*

Proof. Let m^* be a Nash equilibrium. We prove that $((X_i(m^*) + \mathring{w}_i - w_i^*)_i, Y(m^*))$ is a Lindahl allocation with the price vector system $(p(m^*), q_1(m^*), \ldots, q_n(m^*)) \in \mathbb{R}_+^{L+nK}$. By the construction of the mechanism, we know $p(m^*) \cdot [X_i(m^*) + \mathring{w}_i - w_i^*] + q_i(m^*) \cdot Y(m^*) = p(m^*) \cdot \mathring{w}_i$ and the zero-profit condition holds. Also, by the definition of the mechanism,

$((X_i(m^*) + \mathring{w}_i - w_i')_i, Y(m^*))$ is feasible. So we only need to show that all individuals are maximizing their preferences. Suppose, by way of contradiction, that there exists some agent, say agent 1, and $(x_1, y) \in \mathbb{R}_+^{L+K}$ such that $(x_1, y)\ P_1\ (X_1(m^*), Y(m^*))$ and $p(m^*) \cdot x_1 + q_1(m^*) \cdot y \leqq (m^*) \cdot \mathring{w}_1$. Because of strict monotonicity of preferences, it will be enough to confine ourselves to the case of $p(m^*) \cdot x_1 + q_1(m^*) \cdot y = p(m^*) \cdot \mathring{w}_1$. Let $(x_{1\lambda}, y_\lambda, v_\lambda) = (\lambda x_1 + (1 - \lambda)[X_1(m^*) + \mathring{w}_1 - w_1^*], \lambda y + (1 - \lambda)Y(m^*), \lambda v(y, p(m^*)) + (1 - \lambda)\hat{v}(m^*))$. Then, by convexity of preferences and constant returns to scale, we have $(x_{1\lambda}, y_\lambda)\ P_1\ (X_1(m^*), Y(m^*))$ and $v_\lambda = v(y_\lambda, p(m^*))$ for all $0 < \lambda < 1$. Also, $(x_{1\lambda}, y_\lambda) \in \mathbb{R}_+^{L+K}$ and $p(m^*) \cdot x_{1\lambda} + q_1(m^*) \cdot y_\lambda = p(m^*) \cdot \mathring{w}_1$. Let $\bar{x}_{1\lambda} = x_{1\lambda} - (\mathring{w}_1 - w_1^*)$.

Since $p(m^*) \in \mathbb{R}_{++}^{nL}$ and $X(m^*) \in \mathbb{R}_{++}^{nL}$, we must have $p(m^*) \cdot w_j^* - q_j(m^*) \cdot Y(m^*) > 0$ for all $j \in N$ and $w^* = \hat{v}(m^*) + \sum_{j=1}^n X_j(m^*) > \hat{v}(m^*) + X_1(m^*)$. Then, we have $p(m^*) \cdot w_j^* - q_j(m^*) \cdot y_\lambda > 0$ for all $j \in N$, $\bar{x}_{1\lambda} > 0$ (since it can be sufficiently close to $X_i(m^*)$), and $\bar{x}_{1\lambda} + v_\lambda < w^*$ as λ is a sufficiently small positive number.

Define $\bar{x}_{i\lambda}$ $(i = 2, \ldots, n)$ as follows:

$$\bar{x}_{i\lambda} = \frac{p(m^*) \cdot w_i^* - q_i(m^*) \cdot y_\lambda}{p(m^*) \cdot [w^* - v_\lambda - \sum_{j=1}^{i-1} \bar{x}_{j\lambda}]} \left[w^* - v_\lambda - \sum_{j=1}^{i-1} \bar{x}_{j\lambda} \right]. \tag{15}$$

Note that, when $i = n$, we have

$$\bar{x}_{n\lambda} = \sum_{j=1}^{n} w_j^* - v_\lambda - \sum_{j=1}^{n-1} \bar{x}_{j\lambda}. \tag{16}$$

Then

$$\sum_{j=1}^{n} \bar{x}_{j\lambda} + v_\lambda = \sum_{j=1}^{n} w_j^*$$

and

$$p(m^*) \cdot \bar{x}_{i\lambda} + q_i(m^*) \cdot y_\lambda = p(m^*) \cdot w_i^*$$

for all $i \in N$. Also, since $\bar{x}_{1\lambda} + v_\lambda < w^*$ and

$$\frac{p(m^*) \cdot w_i^* - q_i(m^*) \cdot y_\lambda}{p(m^*) \cdot [w^* - v_\lambda - \sum_{j=1}^{i-1} \bar{x}_{j\lambda}]} = \frac{p(m^*) \cdot w_i^* - q_i(m^*) \cdot y_\lambda}{\sum_{j=i}^{n} [p(m^*) \cdot w_j^* - q_j(m^*) \cdot y_\lambda]} < 1$$

for $i = 2, \ldots, n$, we can easily see that $\bar{x}_{2\lambda}, \ldots, \bar{x}_{i\lambda}, \ldots, \bar{x}_{n\lambda}$ are strictly positive.

Now suppose that player 1 chooses a sufficiently small δ_1, $y_1 = y_\lambda - \sum_{j\neq 1}^{n} y_j^*$ and $x_{1j} = \bar{x}_{j\lambda} - \sum_{k=2}^{n} x_{kj}^*$ for all $j \in N$, and keeps other messages unchanged. Then $\delta(m^*/m_1, 1) = \delta_1$, $y_\lambda \in B_y(m^*/m_1, 1)$, and $\bar{x}_\lambda \in B_x(m^*/m_1, 1)$ by noting that $Y(m^*) \in B_y(m^*)$ and $X(m^*) \in B_x(m^*)$. Therefore, we have $Y(m^*/m_1, 1) = y_\lambda$, $X_1(m^*/m_1, 1) = \bar{x}_{1\lambda}$, and $\hat{v}(m^*/m_1, 1) = v_\lambda$. Hence $([X_1(m^*/m_1, 1) + \mathring{w}_1 - w_1^*], Y(m^*/m_1, 1)) P_1 (X_1(m^*), Y(m^*))$. This contradicts $((X_i(m^*) + \mathring{w}_i - w_i^*)_i, Y(m^*)) \in N_{M,h}(e)$. So $((X_i(m^*) + \mathring{w}_i - w_i^*)_i, Y(m^*))$ is a Lindahl allocation. Q.E.D.

THEOREM 2. *For every $e \in E$, if (x^*, y^*) is a Lindahl allocation with the price system $(p^*, q_1^*, \ldots, q_n^*)$, then there is a Nash equilibrium m^* of the withholding mechanism such that $Y(m^*) = y^*$, $X_i(m^*) = x_i^*$, $q_i(m^*) = q_i^*$, for all $i \in N$, $p(m^*) = p^*$, i.e., $L(e) \subseteq N_{N,h}(e)$ for all $e \in E$.*

Proof. We note that $x^* > 0$ by monotonicity of preferences. We must show that there is a message m^* such that (x^*, y^*) is a Nash allocation. Let $w_i^* = \mathring{w}_i$, $p_i^* = p^*$, $y_i^* = y^*/n$, $x_{ii}^* = x_i^*$, $x_{ij}^* = 0$ for $j \neq i$, and let δ_i^* be sufficiently small so that $(1 - \delta(m^*))p(m^*) \cdot \mathring{w}_i - q_i(m^*) \cdot y^* > 0$ for all $i \in N$, $(1 - \delta(m^*))\mathring{w} - v(y^*, p^*) > 0$, and $x_i^* > (\delta(m)[p^* \cdot \mathring{w}_i - q_i^* \cdot y^*]/(p^* \cdot \mathring{w} - q^* \cdot y^*))(\mathring{w} - v^*)$. Let $(\phi_1^*, \ldots, \phi_n^*)$ be a solution of the linear equations system

$$q_i^{*k} = \frac{l}{n} q^{*k} + \phi_{i+1}^k - \phi_{i+2}^k \tag{17}$$

for $k = 1, \ldots, K$. Then, $p(m^*) = p^*$, $q(m^*) = q^*$, $Y(m^*) = y^*$, and $q_i(m^*) = q_i^*$, $X_i(m^*) = x_i^*$, for all $i \in N$. Note that $(p(m^*/m_i, i), q_i(m^*/m_i, i)) = (p(m^*), q_i(m^*))$ for all $m_i \in M_i$. Indeed, agent i cannot change $p(m^*)$ and $q^k(m^*)$ by changing his proposed price since changing p_i yields $a > 0$ and $a_i = 0$ so that the new p_i cannot change $p(m^*)$ nor $q^k(m^*) = c^k(p(m^*))$. Also, since each agent's personalized prices are independent of his own message ϕ_i, $q_i(m^*/m_i, i) = q_i(m^*)$ for all $m_i \in M_i$. Thus, announcing a different message m_i by agent i may yield an allocation $(X_i(m^*/m_i, i), Y(m^*/m_i, i))$ such that $(X_i(m^*/m_i, i), Y(m^*/m_i, i)) \in \mathbb{R}_+^{L+K}$, and $p(m^*) \cdot X_i(m^*/m_i, i) + q_i(m^*) \cdot Y(m^*/m_i, i) = p(m^*) \cdot \mathring{w}_i$ for all $i \in N$ and $m_i \in M_i$. From $(x^*, y^*) \in L(e)$, we have

$$\neg(X_i(m^*/m_i, i) + \mathring{w}_i - w_i, Y(m^*/m_i, i)) P_i (X_i(m^*), Y(m^*)),$$

for otherwise it contradicts the fact that $(X_i(m^*), Y(m^*))$ is a Lindahl allocation. Q.E.D.

Thus, from the above discussions, we can conclude that for public goods economies E there exists a single-valued, completely feasible, and continuous withholding mechanism which fully Nash-implements the Lindahl correspondence on E.

4. Conclusions

We have presented a simple mechanism which fully implements Lindahl allocations for public goods economies with any number of private and public goods and with semi-positive true endowments. The true endowments are private information, and unreported endowments are consumed but not destroyed. When there are no public goods in economies, the mechanism reduces to a mechanism which Nash-implements the Walrasian correspondence. It should be noted that the mechanism presented in this paper has the advantage that agents are not required to report their true endowments even at equilibrium. This is an interesting and important property. Since we do not need to assume that agents are forced to report their true endowments, our mechanism creates quite a different situation from the usual one in which endowments are assumed to be known to the designer, so the incentive compatibility problem is well taken in our paper. Also, the mechanism is individually feasible, balanced, and continuous, and preferences may be discontinuous and nontotal–nontransitive. Thus our mechanism improves all the existing mechanisms which implement Walrasian allocations or Lindahl allocations. Although this paper considers only Nash implementation of the Walrasian and Lindahl correspondences, some of the techniques developed here may be applied when considering withholding implementation of other important equilibrium allocations such as the ratio equilibrium allocations of Kaneko (1977), cost share equilibrium allocations of Mas-Colell and Silvestre (1989), fair allocations of Thomson (1990), and general social choice correspondences in Maskin (1977), Sajio (1988), and Yamato (1992).

Finally, we mention that the dimension of the message space of this mechanism, although it is finite, is relatively large. It includes agent i's contribution of private goods to agent j. We leave it as an open question as to whether or not there exists a completely feasible and continuous mechanism which implements the Lindahl correspondence in the presence of withholding and has a message space of lower dimension.

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