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Implementation of Lindahl allocations with nontotal–nontransitive preferences

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The purpose of this paper is to present a mechanism the Nash allocations of which coincide with the Lindahl allocations even when preferences are nontransitive or nontotal. This extension to include nontotal–nontransitive preferences is potentially very important since preferences of agents are nontransitive–nontotal in many cases – in particular in the case where the players are groups of individuals. Besides, the mechanism presented here uses outcome functions that are individually feasible, balanced (not merely weakly balanced), continuous, differentiable around Nash equilibria and, furthermore, almost everywhere differentiable. Moreover, the mechanism has a message space of minimal dimension.

1. Introduction

In the incentives literature, the central questions asked were: Given a class of economic environments and a set of socially optimal alternatives (especially Pareto-optimal ones[1]), is there a rule or mechanism that achieves one of the optimal alternatives without destroying participants' incentives? And if so, does the mechanism have some desirable properties? There have been many studies on these problems such as Drèze and de la Vallée-Poussin (1971), Malinvaud (1971), Hurwicz (1972, 1979), Groves and Ledyard (1977, 1987), Schmeidler (1980), Walker (1981), Hurwicz, Maskin and Postlewaite (1984), Postlewaite and Wettstein (1989) and Tian (1987, 1989a, 1990), among others. However, all studies explicitly or implicitly assumed that

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[1]It was noted that much of the literature on the design and evaluation of allocation mechanisms has adopted the Pareto-efficiency correspondence as a primary ideal with which to compare performance, whether players are individuals or groups of individuals. This may be partly because of the known and satisfactory efficiency properties of competitive markets and partly because of the inherent acceptability of the concept of Pareto-efficiency as a minimal welfare criterion.

preferences are transitive and total,[2] until Hurwicz (1986a) generalized the Maskin Theorem on Nash implementation of social choice correspondences. This paper deals with the problem of Nash implementation of the Lindahl correspondence when preferences under consideration may be nontransitive or nontotal. The reason we are interested in Lindahl equilibria is because they still result in Pareto-optimal and individually rational allocations even for nontotal–nontransitive preferences. Besides, the paper continues in the tradition of looking for 'better' mechanisms.

Extending the implementation literature to include preferences that may not be total or transitive is potentially very important since in many cases – in particular in the case where economic entities are composed of more than one individual – it is natural that the preferences for such agents would be nontransitive or nontotal owing to the problem of aggregating the individual's preferences. For instance, as Hurwicz (1986b, p. 75) pointed out, this is the case when the society whose goals are to be implemented consists of groups (communities) whose choices are defined by voting procedures. It is easy to think of examples: the United Nations Security Council, the National Congress whose members represent individual states, a university council whose members represent individual departments. This is particularly true for public goods (projects) since choices of public goods are likely to be determined by communities via voting procedures in the real world. For example, when the National Congress votes for whether a public project should be undertaken, the attitudes (preferences) of groups (individual states) are likely to be nontransitive or nontotal. In fact, because of well-known problems in aggregating preferences of individuals, it may be necessary (or desirable) to represent the preferences of groups (communities) as nontransitive or nontotal.[3] These enable economists to study the behavior of agents with nontotal–nontransitive preferences. There are two approaches to nontransitive–nontotal preferences theory in the literature: one through 'weak' (i.e. reflexive) preferences and the other through 'strict' (i.e. irreflexive) preferences [see Kim and Richter (1986) and Tian (1989b) and the references cited therein]. Kim and Richter (1986) made the connection between the weak preference approach and the strict preference approach. They showed that the two approaches are equally valid: definitions and theorems in one approach correspond to definitions and theorems in the other approach (see footnote 8 below). So, in this paper we only consider the case of strict preferences, but the same theorems can be obtained for weak preferences.

A 'better' mechanism yielding Pareto-optimal allocations should be individually rational, completely feasible (i.e. individually feasible and balanced), continuous, differentiable, and furthermore, have a small-sized message space.

[2]A preference R is said to be total if for any x and y, $x \neq y$ implies xRy or yRx.
[3]We thank an anonymous referee for bringing this to our notice.

These properties are highly desirable if we are to seriously consider accepting the mechanisms – in particular, if such a mechanism which is terminated prior to the attainment of an equilibrium is actually implemented, we would like the allocation at nonequilibrium to be both feasible and close to the equilibrium allocation as long as the message is close to the equilibrium message. Also, from the view point of information, a mechanism with a higher dimensional message space means that a higher informational cost is needed to operate the mechanism [cf. Hurwicz (1972) and Reichelstein and Reiter (1988, p. 661)]. As Hurwicz (1986b, p. 250) pointed out, 'Other things being equal, one would expect that fewer resources will be required to operate the system when the dimension of the message space is smaller.' However, until quite recently the incentives literature has only analyzed conditions for a performance function or social choice correspondence to be implementable in given behavior equilibrium strategies, ignoring the question of information costs. The informational requirements literature, on the other hand, has studied the size of the message space needed for decentralized realization of a given performance, ignoring incentives issues. The design of 'better' mechanisms must, of course, involve both informational and incentive considerations and the trade-offs among them. Recent work by Reichelstein and Reiter (1988) analyses conditions for the dimensional requirement for implementation in Nash-equilibrium strategies.

There have been a number of mechanisms that yield Pareto-optimal allocations for public goods economies. Groves and Ledyard (1977) were the first to propose a mechanism to solve the 'free rider problem' by yielding Pareto-optimal Nash allocations even though their mechanism is neither individually rational nor individually feasible. Hurwicz (1979) and Walker (1981) gave balanced and smooth mechanisms[4] the Nash allocations of which are precisely Lindahl allocations. Their mechanisms, however, are not individually feasible. Hurwicz, Maskin and Postlewaite (1984) presented a mechanism Nash-implementing the constrained Lindahl correspondence.[5] Their mechanism is individually feasible and balanced, but discontinuous, and uses a huge message space. Thus, these mechanisms do not guarantee complete feasibility and/or continuity. Recently, Tian (1990) gave a completely feasible and continuous mechanism with a message space of minimal dimension which implements the Lindahl correspondence for economies with one private and one public good. All of these mechanisms mentioned above assume that preferences are transitive and total.

In this paper we present a parametric mechanism which is individually

[4]Walker's mechanism has the advantage of using a minimal-sized message space.

[5]A constrained Lindahl allocation differs from an ordinary Lindahl allocation (defined below) only in the way that each agent maximizes his preferences not only subject to his budget constraint but also subject to total endowments available to the economy [see Hurwicz (1986c) or Tian (1988)].

feasible, balanced (not weakly balanced) and continuous, and fully Nash implements the Lindahl correspondence for economies with nontransitive–nontotal preferences and with one private good and any number of public goods.[6] Since the mechanism has a message space the dimension of which is the same as the minimal dimension required for decentralized realization of the Lindahl correspondence, it has a message space of minimal dimension from the results of Reichelstein and Reiter (1988). This mechanism is also differentiable around Nash equilibria and almost everywhere differentiable so that we can use the differentiation approach to compute the Nash equilibrium if preferences can be represented by differentiable utility functions. The paper also shows that the mechanism can also be easily extended to the nonparametric mechanism in which the initial endowments are private information. This situation would certainly increase the size of the message space,[7] but would reduce the information requirements on the designer.

The plan of this paper is as follows. Section 2 sets forth a public goods model and presents a mechanism that has the desirable properties mentioned above. Section 3 shows that this mechanism fully implements the Lindahl correspondence. Finally, in section 4 we briefly show how to generalize the mechanism to the case of endowments unknown to the designer.

2. Model and mechanism

2.1. Economic environments

In an economy with public goods, there are n agents (groups, players, or voters) who consume one private good and K public goods, x being private (as a numeraire) and y public. The single private good x can, and probably should, be thought of as a Hicksian composite commodity, and public goods y can be thought of as K public projects. Denote by $N=\{1,2,\ldots,n\}$ the set of agents. Each agent's characteristic is denoted by $e_i=(\hat{w}_i, P_i)$, where $\hat{w}_i$ is the initial endowment of the private good and P_i is the strict (irreflexive) preference defined on $\mathbb{R}_+^{1+K}$, which may be nontotal or nontransitive.[8] We assume that there are no initial endowments of public goods, but that the

[6]For economies with any number of private and public goods, Tian (1989a) gave a feasible (i.e. individual feasible and weakly balanced) and continuous mechanism that fully implements the Lindahl correspondence even though it is not balanced and has a message space of higher dimension.

[7]A theorem by Hurwicz, Maskin and Postlewaite (1984) [see Groves and Ledyard (1987, theorem 4.8)] asserts that in a nonparametric mechanism the message space must depend on initial endowments.

[8]If we define the binary relation P_i^* in the way that aP_i^*b if and only if $\neg bP_ia$, where $\neg$ stands for 'it is not the case that', then P_i^* is the weak (reflexive) preference and is called the 'canonical conjugate' of P_i [see Kim and Richter (1986)]. Let concepts used in this paper such as Nash equilibrium and Lindahl allocations be interpreted in terms of the P_i^*. Then the results obtained in this paper for P_i are in particular valid for the P_i^*.

public goods can be produced from the private good under constant returns to scale. That is, the production function f_k is given by $y^k = f^k(x) = (1/q^k)x$ for each $k = 1, \ldots, K$. Thus each unit of public good y_k requires q^k units of private good. Hence the feasibility constraint becomes

$$\sum_{i=1}^{n} x_i + q \cdot y \leqq \sum_{i=1}^{n} \hat{w}_i, \tag{1}$$

where $q = (q^1, \ldots, q^K) \in \mathbb{R}^K_{++}$.

An economy is the full vector $e = (e_1, \ldots, e_n)$ and the set of all such economies is denoted by E. The following assumptions are made on E.

Assumption 1. $n \geqq 3$.

Assumption 2. $\hat{w}_i > 0$ for all $i \in N$.

Assumption 3. P_i is convex and locally nonsatiated on $\mathbb{R}^{1+K}_+$.[9]

Assumption 4. For all $i \in N$, $(x_i, y)\ P_i\ (x'_i, y')$ for all $(x_i, y) \in \mathbb{R}^{1+K}_{++}$ and $(x'_i, y') \in \partial\mathbb{R}^{1+K}_+$, where $\partial\mathbb{R}^{1+K}_+$ is the boundary of $\mathbb{R}^{1+K}_+$.[10]

Remark 1. Assumption 4 is not needed in the mechanisms presented by Hurwicz (1979) and Walker (1981) since their mechanisms are not individually feasible. However the assumption cannot be dispensed with if we want a mechanism to be feasible. As Hurwicz, Maskin and Postlewaite (1984) pointed out, even with Assumptions 1–3 we must impose further restrictions on E and/or $F(e)$ (like Assumption 4) in order to obtain Nash-implementability of the Lindahl correspondence by the feasible mechanisms. Moreover, Tian (1988, theorem i'a) showed that only under Assumptions 1–3 does the (constrained) Lindahl correspondence violate Maskin's (1977) monotonicity condition and thus cannot be Nash-implemented by a feasible mechanism. Also, convexity of preferences is needed in the feasible mechanism in order to guarantee that the interior Lindahl correspondence (defined below) is monotone.

[9] P_i is convex if, for bundles a,, b, c and $0 < \lambda \leqq 1$, $c = \lambda a + (1-\lambda)b$, the relation aP_ib implies cP_ib. P_i is locally nonsatiated if for any bundle a and any $\varepsilon > 0$, there is some bundle b with $\|a - b\| < \varepsilon$ such that bP_ia.

[10] If Assumption 4 is weakened to Assumption 4': for all $i \in N$, $(x_i, y)\ P_i\ (x'_i, y')$ for all $x_i \in \mathbb{R}_{++}$, y, $y' \in \mathbb{R}^K_+$, and $x'_i \in \partial\mathbb{R}_+$, the mechanism in the paper can be slightly modified so that the modified mechanism is still completely feasible and continuous and implements the Lindahl correspondence which may have boundary points. But this case may increase the dimension of message space [see Tian (1989a)].

2.2. Lindahl allocations

An allocation $(x, y) = (x_1, \ldots, x_n, y)$ is *feasible* for an economy e if $(x, y) \in \mathbb{R}_+^{n+K}$ and (1) holds.

An allocation (x^*, y^*) is a *Lindahl allocation* for an economy e if it is feasible and there are price vectors $q_i^* \in \mathbb{R}_+^K$, one for each i, such that

(1) $x_i^* + q_i^* \cdot y^* \leqq \hat{w}_i$ for all $i \in N$;

(2) for all $i \in N$, there does not exist (x_i, y) such that $(x_i, y)\ P_i\ (x_i^*, y^*)$ and $x_i + q_i^* \cdot y \leqq \hat{w}_i$;

(3) $\sum_{i=1}^n q_i^* = q$.

Denote by $L(e)$ the set of all such allocations.

An allocation (x, y) is *Pareto-optimal* with respect to the strict preference profile $P = (P_1, \ldots, P_n)$ if it is feasible and there does not exist another feasible allocation (x', y') such that $(x_i', y')\ P_i\ (x_i, y)$ for all $i \in N$.

An allocation (x, y) is *individually rational* with respect to P if $\neg\ (w_i, 0)\ P_i\ (x_i, y)$ for all $i \in N$.

2.3. Mechanism

Let F be a social choice rule, i.e. a correspondence from E to the set Z of resource allocations. In the rest of the paper we use the Lindahl correspondence as the social choice rule.

Let M_i denote the ith message (strategy) domain. Its elements are written as m_i and called messages. Let $M = \prod_{i=1}^n M_i$ denote the message (strategy) space. Let $h: M \to Z$ denote the outcome function, or more explicitly, $h_i(m) = (X_i(m), Y(m))$, where $X_i(m)$ is the ith agent's outcome function for the private good and $Y(m)$ the outcome function for the public goods. A mechanism consists of $\langle M, h \rangle$ defined on E. A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is a *Nash equilibrium* (NE) of the mechanism $\langle M, h \rangle$ for an economy e if for all $i \in N$ and for all $m_i \in M_i$,

$$\neg\, h_i(m^*/m_i, i)\, P_i\, h_i(m^*), \tag{2}$$

where $(m^*/m_i, i) = (m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. The $h(m^*)$ is then called a *Nash (equilibrium) allocation*. Denote by $V_{M,h}(e)$ the set of all such Nash equilibria and by $N_{M,h}(e)$ the set of all such Nash (equilibrium) allocations.

Remark 2. The Lindahl and Nash allocations with nontotal–nontransitive preferences that are defined above may not be the same as the conventional ones. So Hurwicz (1986a) called the Nash equilibrium, defined by (2), the Generalized Nash Equilibrium. However, when preferences are transitive and total, the Nash equilibrium and Lindahl allocations defined above reduce to the conventional ones.

A mechanism $\langle M,h\rangle$ fully *Nash-implements* the social choice rule F on E if, for all $e\in E$, $N_{M,h}(e)=F(e)$. A mechanism $\langle M,h\rangle$ is *individually feasible* if $(X_i(m), Y(m))\in\mathbb{R}_+^{1+K}$ for all $i\in N$ and all $m\in M$. A mechanism $\langle M,h\rangle$ is *weakly balanced* if for all $m\in M$

$$\sum_{j=1}^{N} X_j(m)+q\cdot Y(m)\leqq\sum_{j=1}^{n}\hat{w}_j. \tag{3}$$

A mechanism $\langle M,h\rangle$ is *balanced* if eq. (3) holds with equality for all $m\in M$. A mechanism $\langle M,h\rangle$ is *completely feasible* (or *feasible*) if it is individually feasible and balanced (or weakly balanced). Sometimes we say that an outcome function is individually feasible, balanced, or continuous if the mechanism is individually feasible, balanced, or continuous.

In what follows we present a completely feasible and continuous mechanism with a minimal-sized message space that fully Nash-implements the Lindahl correspondence. The mechanism will be simply described as follows. The designer first determines the personalized prices for public goods according to the tax that agents are willing to pay. He then defines a feasible choice correspondence $B(m)$ (defined below) for public goods that can be produced with total endowments and can be purchased by all agents. The outcome $Y(m)$ for public goods will be chosen from $B(m)$ so that it is the closest to the sum of the contributions that each agent is willing to pay. The outcome $X_i(m)$ for the private good will be chosen such that the budget constraint holds with equality.

We now turn to the formal construction of the mechanism. For each $i\in N$, let $M_i=\mathbb{R}^K$ the elements m_i of which are interpreted as the taxes that agent i is willing to pay.

Let the personalized price of each public good k for the ith consumer be of the form

$$q_i^k(m)=b_i^k+\sum_{j=1}^{n} a_{ij}^k m_j^k, \tag{4}$$

where $\sum_{i=1}^n b_i^k=q^k$, $\sum_{i=1}^n a_{ij}^k=0$, $a_{ii}^k=0$, and $\sum_{j=1}^n |a_{ij}^k|>0$ for $i\in N$ and $k=1$ to K. In addition, a_{ij}^k are chosen so that the linear equations system (11) below has a unique solution for $(m_1,\ldots,m_n)$. Notice that by construction $\sum_{i=1}^n q_i(m)=q$ for all $m\in M$. Here $q_i(m)=(q_i^1(m),\ldots,q_i^K(m))$.

Remark 3. Note that the personalized prices of agent i defined in (4) are independent of his own messages m_i. It is also a general form of the

personalized price vector and concludes the personalized prices specified by Hurwicz (1979) and Walker (1981) as special cases.

Define the feasible choice correspondence $B: M \to\to \mathbb{R}^K_+$ for public goods by

$$B(m) = \{y \in \mathbb{R}^K_+ : \hat{w}_i - q_i(m) \cdot y \geqq 0 \text{ for all } i \in N\}, \tag{5}$$

which is nonempty, compact, and convex for all $m \in M$.

The outcome function of the public goods, $Y(m): M \to \mathbb{R}^K_+$, is given by

$$Y(m) = \left\{ y: \min_{y \in B(m)} \|y - \tilde{m}\| \right\}, \tag{6}$$

which is the closest to $\tilde{m}$. Here $\tilde{m} = \sum_{i=1}^n m_i$ and $\|\cdot\|$ is the Euclidian norm. Then $Y(m)$ is single-valued and continuous on M.[11] Define the tax-outcome function $T_i(m): M \to \mathbb{R}$ by

$$T_i(m) = q_i(m) \cdot Y(m). \tag{7}$$

The ith agent's outcome function for the private good, $X_i(m): M \to \mathbb{R}_+$ is given by

$$X_i(m) = \hat{w}_i - q_i(m) \cdot Y(m). \tag{8}$$

The outcome function is clearly continuous on M. Also, since $(X_i(m), Y(m)) \in \mathbb{R}^{1+K}_+$ and

$$\sum_{i=1}^n X_i(m) + q \cdot Y(m) = \sum_{i=1}^n \hat{w}_i \tag{9}$$

for all $m \in M$, the mechanism is completely feasible.

3. Main results

The remainder of this paper is devoted to the proof of equivalence between Nash allocations and Lindahl allocations, which is stated in Theorem 1 and Theorem 2 below. Lemmas 1 and 2 are preliminary results used to prove Theorem 1, which states that every Nash allocation is a Lindahl allocation.

[11]This is because $Y(m)$ is an upper hemicontinuous correspondence by Berge's Maximum Theorem [see Debreu (1959, p. 19)] and single-valued [see Mas-Colell (1985, p. 28)].

Lemma 1. If $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, *then* $(X_i(m^*), Y(m^*)) \in \mathbb{R}^{1+K}_{++}$ *for all* $i \in N$.

Proof. Suppose, by way of contradiction, that $(X_i(m^*), Y(m^*)) \in \partial\mathbb{R}^{1+K}_{+}$ for some $i \in N$.

Consider the quadratic equation

$$t = \frac{\hat{w}}{2a(t+c)}, \tag{10}$$

where $\hat{w} = \min_{i \in N} \hat{w}_i$, $a = \max_{i,j \in N \& k \in \{1,2,\ldots,K\}} |a^k_{ij}|$, $c = b^*/a + 2\sum_{k=1}^{K}\sum_{i=1}^{n} |m_i^{*k}|$ with $b^* = \max_{i \in N, k \in \{1,2,\ldots,K\}} |b^k_i|$. Then the larger root (denoted by $\tilde{t}$) of the above equation is positive. Let $\tilde{m} = (1/K)(\tilde{t}, \tilde{t}, \ldots, \tilde{t})'$. Suppose that player i chooses his message $m_i = \tilde{m} - \sum_{j \neq i}^{n} m_j^*$. Then $\tilde{m} = m_i + \sum_{j \neq i}^{n} m_j^*$ and

$$\begin{aligned}
&\hat{w}_j - q_j(m^*/m_i, i) \cdot \tilde{m} \\
&= \hat{w}_j - \frac{1}{K}\sum_{k=1}^{K}\left[b^k_j + \sum_{s \neq i}^{n} a^k_{js} m_s^{*k} + a^k_{ji}\left(\tilde{t} - \sum_{s \neq i}^{n} m_s^{*k}\right)\right]\tilde{t} \\
&\geqq \hat{w}_j - a\left(\tilde{t} + \frac{b^*}{a} + 2\sum_{s=1}^{n} |m_s^{*k}|\right)\tilde{t} \\
&= \hat{w}_j - \frac{\hat{w}}{2} \geqq \frac{\hat{w}_j}{2} > 0
\end{aligned}$$

for all $j \in N$. So $\tilde{m} \in B(m^*/m_i, i)$ and thus $Y(m^*/m_i) = \tilde{m}$. Since $X_j(m^*/m_i, i) = \hat{w}_j - q_j(m^*/m_i, i) \cdot Y(m^*/m_i, i) = \hat{w}_j - q_j(m^*/m_i, i) \cdot \tilde{m} > 0$ for all $j \in N$ and $Y(m^*/m_i, i) = \tilde{m} > 0$, we have $(X_i(m^*/m_i, i), Y(m^*/m_i, i))\ P_i\ (X_i(m^*), Y(m^*))$ by Assumption 4. This contradicts the hypothesis $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

Lemma 2. If $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, *then* $\tilde{m}^* = \sum_{i=1}^{n} m_i^* \in \operatorname{Int} B(m^*)$ *and therefore* $Y(m^*) = \sum_{i=1}^{n} m_i^*$. *Here* $\operatorname{Int} B(m^*)$ *is the interior of* $B(m^*)$.

Proof. Suppose, by way of contradiction, that $\tilde{m}^* \notin \operatorname{Int} B(m^*)$. Then $Y(m^*) \in$

$\partial B(m^*)$ and thus $Y(m^*) \in \partial \mathbb{R}^K_+$ or $X_j(m^*) = \hat{w}_j - q_j(m^*) \cdot Y(m^*) = 0$ for some $j \in N$. Both cases contradict $(X(m^*), Y(m^*)) \in \mathbb{R}^{1+K}_{++}$ by Lemma 1. Q.E.D.

Remark 4. From Lemma 2, we know that the outcome function is differentiable on some neighborhhod of any Nash equilibrium and thus we can use the differentiation approach to find Nash equilibrium points if utility functions exist and are differentiable.

We now prove the main results of this paper in the following theorems.

Theorem 1. Under Assumptions 1–4, if the mechanism has a Nash equilibrium m^*, then $(X(m^*), Y(m^*))$ is a Lindahl allocation, with $(q_1(m^*), \ldots, q_n(m^*))$ as the Lindahl price vector, i.e. $N_{M,h}(e) \subseteq L(e)$.

Proof. Let m^* be a Nash equilibrium. Now we prove that $(X(m^*), Y(m^*))$ is a Lindahl allocation with $(q_1(m^*), \ldots, q_n(m^*))$ as the Lindahl price vector. Since the mechanism is completely feasible and $\sum_{i=1}^n q_i(m^*) = q$ as well as $X_i(m^*) + q_i(m^*) \cdot Y(m^*) = \hat{w}_i$ for all $i \in N$, we only need to show that each individual is maximizing his/her preferences. Suppose, by way of contradiction, that there is some $(x_i, y) \in \mathbb{R}^{1+K}_+$ such that $(x_i, y)\ P_i\ (X_i(m^*), Y(m^*))$ and $x_i + q_i(m^*) \cdot y \leqq \hat{w}_i$. Because of local nonsatiation of preferences, it will be enough to confine ourselves to the case of $x_i + q_i(m^*) \cdot y = \hat{w}_i$. Let $(x_{\lambda i}, y_\lambda) = (\lambda x_i + (1-\lambda) X_i(m^*),\ \lambda y + (1-\lambda) Y(m^*))$. Then by convexity of preferences we have $(x_{\lambda i}, y_\lambda)\ P_i\ (X_i(m^*), Y(m^*))$ for any $0 < \lambda < 1$. Also $(x_{\lambda i}, y_\lambda) \in \mathbb{R}^{1+K}_+$ and $x_{\lambda i} + q_i(m^*) \cdot y_\lambda = \hat{w}_i$. Suppose that player i chooses his/her message $m_i = y_\lambda - \sum_{j \neq i}^n m_j^*$. Since $Y(m^*) = \sum_{j=1}^n m_i^*$ by Lemma 2, $m_i = y_\lambda - Y(m^*) + m_i^*$. Thus as $\lambda \to 0$, $y_\lambda \to Y(m^*)$, and therefore $m_i \to m_i^*$. Since $X_j(m^*) = \hat{w}_j - q_j(m^*) \cdot Y(m^*) > 0$ for all $j \in N$ by Lemma 1, we have $\hat{w}_j - q_j(m^*/m_i, i) \cdot y_\lambda > 0$ for all $j \in N$ as λ is a sufficiently small positive number. Therefore, $Y(m^*/m_i, i) = y_\lambda$ and $X_i(m^*/m_i, i) = \hat{w}_i - q_i(m^*)$. $Y(m^*/m_i, i) = \hat{w}_i - q_i(m^*) \cdot y_\lambda = x_{i\lambda}$. From $(x_{i\lambda}, y_\lambda)\ P_i\ (X_i(m^*), Y(m^*))$, we have $(X_i(m^*/m_i, i), Y(m^*/m_i, i))\ P_i\ (X_i(m^*), Y(m^*))$. This contradicts the hypothesis that $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

Remark 5. In the above proof, convexity of preferences is used to show that if an outcome is not a maximum element for some agent, the agent can change his own messages so that the new outcome (which is completely feasible by the mechanism) is arbitrarily close and preferred to the original one.

Theorem 2. Under Assumptions 1–3, if (x^*, y^*) is a Lindahl allocation with the Lindahl price vector $q^* = (q_1^*, \ldots, q_n^*)$, then there is a Nash equilibrium m^* of the mechanism such that $X_i(m^*) = x_i^*$, $q_i(m^*) = q_i^*$, for all $i \in N$, $Y(m^*) = y^*$, i.e. $L(e) \subseteq N_{M,h}(e)$.

Proof. We need to show that there is a message m^* such that (x^*, y^*) is a Nash allocation. Let $m^* = (m_1^*, \ldots, m_n^*)$ be the solution of the following linear equations system:

$$\sum_{i=1}^{n} m_i = y^*, \qquad \sum_{j=1}^{n} a_{ij}^k m_j^k = q_i^{*k} - b_i^k. \tag{11}$$

Then $X_i(m^*) = x_i^*$, $Y(m^*) = y^*$ and $q_i(m^*) = q_i^*$ for all $i \in N$. Thus, from $(X(m^*/m_i, i),\ Y(m^*/m_i, i)) \in \mathbb{R}_+^{1+K}$ and $X_i(m^*/m_i, i) + q_i(m^*) \cdot Y(m^*/m_i, i) = \hat{w}_i$ for all $i \in N$ and $m_i \in M_i$, we know that

$$\neg(X_i(m^*/m_i, i), Y(m^*/m_i, i))\ P_i\ (X_i(m^*), Y(m^*)),$$

for otherwise it contradicts the fact that $(X_i(m^*), Y(m^*))$ is a Lindahl allocation. Q.E.D.

Since Lindahl allocations are Pareto optimal and individually rational, the mechanism yields Pareto-optimal and individually rational allocations. On the other hand, from Theorem 2.4 of Reichelstein and Reiter (1988), we know that Nash implementation is always at least as costly, in message space size, as (decentralized) realization. Because the minimal dimension required for decentralized realization of the Lindahl correspondence is $(L+K-1)n$ (which is equal to nK when $L=1$), where L is the number of private goods [see, for example, Hurwicz (1986b)], the mechanism has a message space of minimal dimension (since the dimension of the message space of the mechanism also is nK). Thus, the mechanism is informationally efficient.

Summarizing the above discussions, we conclude that for one private and K public goods economies E satisfying Assumptions 1–4, there exists a completely feasible and continuous mechanism with a message space of minimal dimension which fully Nash-implements the Lindahl correspondence.

4. Concluding remarks

In the above sections we have given a simple mechanism that extends the implementation literature to include nonstandard preferences and which is potentially very important since in many cases – in particular in the case where economic entities (agents) are composed of more than one individual – the preferences of agents are nontotal or nontransitive. Besides, the mecha-

nism is well-behaved in the sense that the mechanism is individually feasible, balanced, continuous, and has a message space of minimal dimension. Moreover, the mechanism is almost everywhere differentiable on the message space and differentiable on some neighborhood of every Nash equilibrium. In this section we mention some of the possible extensions of the mechanism.

The mechanism presented above only considers the case where initial endowments are known. Yet, similar to those mechanisms proposed by Hurwicz, Maskin and Postlewaite (1984), Postlewaite and Wettstein (1989) and Tian (1989a), the mechanism can be easily extended to allow for endowments to be unknown to the designer. Such a mechanism can be obtained by modifying the message space to

$$M_i = (0, \hat{w}_i] \times \mathbb{R}^K. \tag{12}$$

A generic element of M_i is $m_i = (w_i, y_i)$, where w_i denotes a profession of agent i's endowment and the inequality $0 < w_i \leqq \hat{w}_i$ means that the agent cannot overstate his own endowment; on the other hand, the endowment can be understated, but the claimed endowment w_i must be positive. Then the modified mechanism is the same as before except for replacing $\hat{w}_i$ by w_i so that the mechnism is feasible and continuous. Under the assumption of local nonsatiation of preferences, we can easily see that, at Nash equilibrium, $w_i^* = \hat{w}_i$ for all $i \in N$. Then the proof of equivalence of Nash allocations of the modified mechanism and the Lindahl allocations also remains the same.

Secondly, the mechanism deals only with public goods economies with one private good. However, by using similar techniques given in Tian and Li (1991), the mechanism presented above can be generalized to include economies with an arbitrary number of private goods. The resulting mechanism is completely feasible and continuous and implements the Lindahl correspondence.

Finally, the mechanism allows the agents to consist of groups of individuals. In this case we have only addressed Pareto-efficiency with respect to the nonordered preferences of groups and not efficiency with respect to the preferences of the individuals who have been aggregated to get these preferences. The paper does not deal with the relation of welfare analysis with respect to preferences of groups and the welfare of the underlying individuals even though it may be potentially an interesting question.

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