On the Informational Requirements of Decentralized Pareto-Satisfactory Mechanisms in Economies with Increasing Returns *

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Abstract

This paper investigates the dimension requirements of informationally decentralized Pareto-satisfactory processes in production economies with increasing returns to scale or more general types of non-convexities. We show that the marginal cost pricing (MCP) mechanism is informationally efficient over the class of non-convex production economies where MCP equilibrium allocations are Pareto efficient. We then discuss the informational requirements of realizing Pareto efficient allocations for a general class of non-convex production economies. We do so by examining the dimension of the message space of the marginal cost pricing mechanism with transfers. Since the set of marginal cost pricing equilibrium allocations with transfers contains Pareto efficient allocations as a subset for every economy under consideration, Pareto efficient allocations can be realized through the MCP mechanism with transfers, which is informationally decentralized and has a finite-dimensional message space. This result is sharply contrasted to the impossibility result given in Calsamigla (1977).

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1 Introduction

1.1 Motivation

This paper studies the informational requirements of resource allocation mechanisms that select Pareto optimal allocations for economies with increasing returns to scale or more general types of non-convexities. The importance of this research is motivated from the following three aspects: (1) the failure of the competitive model and Walrasian equilibrium principle in the presence of increasing returns , (2) regulation and pricing in some sectors (typically in public sector) that usually are connected with imperfect competition, and (3) increasing returns to knowledge creation are used to model long-term economic growth evidenced in the so-called "new economy" and studied in the endogenous growth theory.

Non-convexities in production, which can often arise from technical progress, imperfection of markets, fixed costs, increasing returns to scale, or indivisibilities, have their practical importance. The non-convex firms can be thought of as privately owned public utilities, which are regulated (cf. Brown and Heal, 1983, 1985). This type of market structure is common in the United States, and various pricing rules such as marginal cost, average cost, loss-free, voluntary trading, and quantity-taking pricing rules have been proposed in the literature. Since public utilities are privately owned, any pricing rule imposed by a regulator should be informationally decentralized. Also, beginning in the mid-1980s, Paul Romer (1986) formalized the relationship between the economics of ideas and long term economic growth. According to Romer, an inherent characteristic of ideas is that they are non-rivalry. If ideas are non-rival, then the economy faces increasing returns to scale; implying there is imperfect competition. This is evidenced by the present "new economy". Increasing returns to scale in aggregation are common features in some sectors such as high-tech or biotech industries due to market imperfection or technical progress, and they are the main sources of long-term economic growth. Study of this relationship results in the "new growth theory" (cf. Romer, 1986; Barro and Sala-i-Martin, 1995; and Jones, 2002).

A formal study of the informational requirements and informational optimality of resource allocation processes was initiated by Hurwicz (1960). The interest in such a study was greatly stimulated by the "socialist controversy" — the debate over the feasibility of central planning between Mises-Hayek and Lange-Lerner (von Hayek, 1935, 1945; Lange, 1936-7, 1944; Lerner, 1944). In the Mises-Hayek-Lange-Lerner debate, the marginal cost pricing doctrine was proposed in response to Mises-Hayek's criticism of a socialist system's information problem, a centrally planned system has to use immense information (infinite dimension of message space) to make production decisions. In line with the prevailing tradition, interest in this area was focused on the design of Pareto-satisfactory (non-wasteful) and privacy-preserving mechanisms, i.e., mechanisms that result in Pareto efficient allocations and use informationally decentralized decision making processes. Allocative efficiency and informational efficiency are two highly desired properties for an economic system to have. The importance of Pareto optimality is attributed to what may be regarded as a minimal welfare property. Pareto optimality requires resources be allocated efficiently. If an allocation is not efficient, there is a waste in allocating resources and thus at least one agent is better off without making others worse off under given resources. Informational efficiency requires an economic system have the minimal informational cost of operation. The informational requirements depend upon two basic components: the class and types of economic environments over which a mechanism is supposed to operate and the particular outcomes that a mechanism is required to realize.

A mechanism can be viewed as an abstract planning procedure; it consists of a message space in which communication takes place, rules by which the agents form messages, and an outcome function which translates messages into outcomes (allocations of resources). Mechanisms are imagined to operate iteratively. Attention, however, may be focused on mechanisms that have stationary or equilibrium messages for each possible economic environment. A mechanism realizes a prespecified welfare criterion (also called performance, social choice rule, or social choice correspondence) if the outcomes given by the outcome function agree with the welfare criterion of the stationary messages. The realization theory studies the question of how much communication must be provided to realize a given performance, or more precisely, the minimal informational cost of operating a given performance in terms of the size of the message space. It determines which economic system or social choice rule is informationally the most efficient in the sense that the minimal informational cost is used to operate the system.

1.2 Related Literature

Since the pioneering work of Hurwicz (1960), there has been a lot of work on studying the informational requirements of decentralized resource allocation mechanisms over various classes of economies such as those in Hurwicz (1972, 1977, 1999), Mount and Reiter (1974), Calsamiglia (1977), Walker (1977), Sato (1981), Calsamiglia and Kirman (1993), Tian (1990, 1994, 2003, 2004, 2006), Ishikida and Marschak (1996) among others.

One of the well-known results in this literature establishes the minimal usage of information in the competitive (Walrasian) mechanism for pure exchange economies. Hurwicz (1972, 1986), Mount and Reiter (1974), Walker (1977) among others proved that, for pure exchange private goods economies, the competitive allocation process is the most informationally efficient process. Any smooth, informationally decentralized allocation mechanism which achieves Pareto optimal allocations must use information at least as large as the competitive mechanism.[1] Thus, the competitive allocation process has a message space of minimal dimension among a certain class of resource allocation processes that are privacy-preserving and non-wasteful. For brevity, this result has been referred to as the Efficiency Theorem. Jordan (1982) further proved that the competitive allocation process is uniquely informationally efficient among mechanisms that realize Pareto efficient and individually rational allocations. Calsamiglia and Kirman (1993) proved the equal income Walrasian mechanism is uniquely informationally efficient among all resource allocation mechanisms that realize fair allocations. The work of these researches provides the Uniqueness Theorem for pure exchange economies. Recently, Tian (2006) further proved the informational optimality and uniqueness of the competitive mechanism in using information efficiently for convex private ownership production economies. These efficiency and uniqueness results are of fundamental importance from the view point of political economy. They show the uniqueness of the competitive market mechanism in terms of allocative efficiency and informational efficiency, provided the economies have only convex production sets.

Although there is great appeal to the presence of non-convexities in production, most work mentioned above only considers the issue of informational requirements of a decentralized resource allocation mechanism for pure exchange economies or economies with convex production possibility sets. The only exception was Calsamiglia (1977) who considered the economic environments with increasing returns. However, his result is an impossibility result. He showed that in production economies with unbounded increasing returns to scale there exists no smooth privacy-preserving and non-wasteful process that uses a finite-dimensional message space. Because of this impossibility result, economists generally believe that there is no hope to have a smooth privacy-preserving and non-wasteful process that uses a finite-dimensional message space.

1.3 The Results of the Paper

The purpose of this paper is to show that Calsamiglia's result is too pessimistic. To do so, we first establish a lower bound of information, as measured by the size of the message space,

[1] A mechanism is called *smooth* if the stationary message correspondence is either locally threaded or if the inverse of the stationary message correspondence has a Lipschizian-continuous selection in a subset. This terminology was used by Hurwicz (1999). We will give the definition of the local threadedness below.

that is required to guarantee an informationally decentralized mechanism that realizes Pareto efficient allocations over the class of economies that include non-convex production technologies. We show that any informationally decentralized smooth mechanism that realizes Pareto efficient allocations over the class of production economies has a message space of dimension no smaller than $(L-1)I+LJ$, where I is the number of consumers, J is the number of producers, and L is the number of commodities. We then establish the informational optimality of the marginal cost pricing (MCP) mechanism over the class of non-convex production economies where all MCP equilibrium allocations are Pareto efficient. We show that the lower bound is exactly the size of the message space of the MCP mechanism, and thus any smooth informationally decentralized mechanism that realizes Pareto efficient allocations has a message space whose topological size is greater than or equal to that of the MCP mechanism. Thus, the MCP mechanism is an informationally efficient process among privacy-preserving and non-wasteful resource allocation mechanisms over the class of non-convex production economies where every MCP equilibrium allocation is Pareto efficient.

We then study the informational requirements for realizing Pareto efficient allocations for a general class of non-convex production economies. We establish our possibility result that Pareto efficient allocations can be realized by an informationally decentralized mechanism with a finite-dimensional message space for the general class of non-convex production economies. Since Pareto efficient allocations can be characterized by marginal cost pricing equilibrium allocations with transfers (MCPT), the positive results are obtained by examining the upper bound of the message space of the MCPT allocation mechanism, defined in the paper, that is privacy-preserving and then obtain the upper bound of the message space of the MCPT mechanism which has finite dimension $L(I+J)$ for general non-convex production economies. Thus, in sharp contrast to Calsamigla (1977), we show that it is possible to have a non-wasteful informationally decentralized mechanism that has a finite-dimensional message space.

What leads us to obtain such a different result? As it will be seen, the local threadedness on the entailed class of economic environments is probably an unduly strong requirement to impose on processes defined over classes of economies displaying production non-convexities. Calsamiglia's impossibility theorem is based on the condition that the message correspondence is locally threaded for every economy under consideration. In our opinion, to exclude the possibility of smuggling information arbitrarily, it is enough only to assume the message correspondence is locally threaded for some economy as assumed such as those in Calsamiglia and Kirman (1993) and Tian (1990, 1994, 2003, 2004, 2006), but not assume that it is locally threaded for every economy since the marginal cost pricing process is not locally threaded for all economies under consideration.[2] Thus, our possibility result is not only positive, but also more reasonable and realistic.

The remainder of this paper is organized as follows. In Section 2, we give a basic setting for the framework used in the paper. We specify economic environments with non-convex production economies, and provide notation and definitions on resource allocation, the MCP, the MCPT, social choice correspondence, outcome function, allocation mechanism, etc. Section 3 establishes a lower bound of the size of the message space which is required to guarantee that an informationally decentralized mechanism realizes Pareto efficient allocations for a class of non-convex production economies. Section 4 gives an Efficiency Theorem on the allocative and informational efficiency for the MCP mechanism for the class of non-convex production economies in which all MCP equilibrium allocations are Pareto efficient. Section 5 investigates informational requirements for realizing Pareto efficient allocations for the general class of non-convex production economies by examining the dimension of the message space of the MCPT mechanism. Concluding remarks are presented in section 6. The proofs of Lemmas are in the Appendix.

2 The Setup

In this section we give notation, definitions, and provide the basic settings needed in the paper.

2.1 Economic Environments

Consider production economies with L private goods, I consumers (characterized by their consumption sets, preferences, and endowments), and J firms (characterized by their production sets). Throughout this paper, subscripts are used to index consumers or firms, and superscripts are used to index goods unless otherwise stated. By an agent, we will mean either a consumer or a producer, thus there are $N := I + J \geqq 2$ agents.[3] Characteristics of agents are unknown to the designer. For the ith consumer, his characteristic is denoted by $e_i = (X_i, w_i, \succcurlyeq_i)$, where $X_i \subset \mathbb{R}^L$ is his consumption set, w_i is his initial endowment vector, and $\succcurlyeq_i$ is his preference ordering that is assumed to be convex[4], continuous on X_i, and strictly monotone on the set of

[2] It is easily seen that the competitive process is not locally threaded for all economic environments.

[3] As usual, vector inequalities, $\geqq$, $\geq$, and $>$, are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \dots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \dots, m$.

[4] $\succcurlyeq_i$ is convex if for bundles a, b, c with $0 < \lambda \leqq 1$ and $c = \lambda a + (1 - \lambda)b$, the relation $a \succ_i b$ implies $c \succ_i b$. Note that the term "convex" is defined as in Debreu (1959), not as in some recent textbooks.

interior points of X_i. Let $\succ_i$ be the strict preference (asymmetric part) of $\succcurlyeq_i$.

For producer j, her characteristic is denoted by $e_j = (\mathcal{Y}_j)$ where $\mathcal{Y}_j \subset \mathbb{R}^L$ is her production possibility set. We assume that, for $j = I+1, \ldots, N$, $\mathcal{Y}_j$ is nonempty, closed, contains 0 (possibility of inaction), and $\mathcal{Y}_j - \mathbb{R}^L_+ \subseteq \mathcal{Y}_j$ (free-disposal). Note that we do not assume that $\mathcal{Y}_j$ is convex so that production technologies may exhibit increasing returns to scale. It is also important to note that, under these assumptions, $\partial\mathcal{Y}_j$, the boundary of the production set $\mathcal{Y}_j$, is exactly the set of (weakly) efficient production plans of the jth producer, that is,

$$\partial\mathcal{Y}_j = \{y_j \in \mathcal{Y}_j : \nexists y'_j \in \mathcal{Y}_j, y'_j > y_j\}.$$

We assume that there are no externalities or public goods. An economy is the full vector $e = (e_1, \ldots, e_I, e_{I+1}, \ldots, e_N)$ and the set of all such production economies is denoted by E which is assumed to be endowed with the product topology.

Let x_i denote the consumption vector of consumer i and let $z_i = x_i - w_i$ denote the net increment in commodity holdings (net trade vector). Let $x = (x_1, \ldots, x_I)$ and $z = (z_1, \ldots, z_I)$ denote respectively the I-tuples of consumption and net trades. x_i or z_i is said to be *individually feasible* if $x_i = z_i + w_i \in X_i$. Similarly, let y_j denote producer j's (net) output vector that has positive components for outputs and negative ones for inputs, and y_i is said to be *individually feasible* if $y_j \in \mathcal{Y}_j$. Denote by $y = (y_{I+1}, \ldots, y_N)$.

An allocation of the economy e is a vector $(z, y) \in \mathbb{R}^{NL}$. It is said to be *balanced* if $\sum_{i=1}^I z_i = \sum_{j=I+1}^N y_j$, and is said to be *feasible* if it is balanced and individually feasible for every individual. Denote by $\mathcal{A} = \{(z, y) \in \mathbb{R}^{L(I+J)} : z_i + w_i \in X_i,\ y_j \in \mathcal{Y}_j,\ \sum_{i=1}^I z_i = \sum_{j=I+1}^N y_j\}$ the set of all such feasible allocations.

An allocation (z, y) is said to be *Pareto efficient* if the following two conditions are satisfied:

(i) it is feasible;

(ii) there does not exist another feasible allocation (z', y') such that $(z'_i + w_i) \succcurlyeq_i (z_i + w_i)$ for all $i = 1, \ldots, I$ and $(z'_i + w_i) \succ_i (z_i + w_i)$ for some $i = 1, \ldots, I$.

Denote by $P(e)$ the set of all such allocations.

2.2 Marginal Cost Pricing Equilibrium

It is generally recognized that the standard behavioral assumption of profit maximization is non-applicable in the presence of increasing returns to scale, and thus one needs to adopt alternative rules of firms' behavior which cover not only convex production economies, but also non-convex production economies. The marginal cost pricing equilibrium principle is such an alternative equilibrium principle. It is a natural generalization of the Walrasian equilibrium principle to extend to economies involving non-convex production technologies. An advantage of using the MCP principle is that unlike the other pricing rules, such as loss-free, average cost, or voluntary-trading pricing equilibrium principle, a MCP equilibrium may result in a Pareto efficient allocation for a certain class of non-convex economies. This may be why the MCP doctrine is the earliest pricing rule proposed in the literature for general production economies involving increasing returns to scale.

To define marginal cost pricing equilibrium for non-convex economies, we first give the notion of the Clarke tangent and normal cones. The Clarke tangent normal cone to $\mathcal{Y}$ is a generalization of the notion of the marginal rate of transformation in the absence of smoothness and convexity assumptions (cf. Clarke (1975)). The formal definition of the Clarke normal cone requires the notion of the Clarke tangent cone. For a non-empty set $Y \subseteq \mathbb{R}^L$ and $y \in Y$, the tangent cone of Y is given by $T_Y(y) = \{x \in \mathbb{R}^L$: for every sequence $y^k \in Y$ with $y^k \to y$ and every sequence $t^k \in (0, \infty)$ with $t^k \to 0$, there exists a sequence $x^k \in \mathbb{R}^L$ with $x^k \to x$ such that $y^k + t^k x^k \in Y$ for all k $\}$. The Clarke normal cone is then given by $N_Y(y) = \{x \in \mathbb{R}^L : (z, x) \leqq 0 \ \forall z \in T_Y(y)\}$.

The important properties of the Clarke normal cone are: (1) it coincides with the standard normal cone when Y is convex or when the boundary of Y is differentiable; (2) it is convex and never reduces to the null vector for any boundary point of Y, and (3) the correspondence $y \to N_Y(y)$ has a closed graph. These properties are well adapted to economic problem of optimization and fixed points (cf. Cornet (1990), Quinzii (1992)).

For the marginal cost pricing rule, one usually considers a private ownership economy so that consumer i's wealth function is $r_i(w, p, y) = p \cdot w_i + \sum_{i=1}^{I} \theta_{ij} p \cdot y_j$ where $\theta_{ij} \in \mathbb{R}_+$ are the profit shares of private firms j, $j = I+1, \ldots, N$, satisfying $\sum_{i=1}^{I} \theta_{ij} = 1$. In this case the MCP equilibrium reduces to the Walrasian equilibrium for convex production sets. However, the MCP rule in general will result in losses rather than profits. Thus, a typical wealth function of consumer i is given by $r_i(w, p, y) = p \cdot w_i + \theta_i p \cdot y$ with $w_i = \theta_i w$. This implies Guesnerie's fixed structure of revenues condition, i.e. $r_i(w, p, y) = \theta_i p \cdot (w + y)$, is satisfied. In this case, the lump sum taxation to cover the losses of the firm is implicit in the formation of the budget constraint, i.e., $r_i(p, y) = \theta_i p \cdot (w + y)$ should be interpreted as "after-tax" income. The informational efficiency result obtained in this paper, however, is valid for a general form of wealth map specified below.

The i-th consumer's wealth function is then assumed to be a function $(w, p, y) \to r_i(w, p, y)$ on $\mathbb{R}_+^{IL} \times \mathbb{R}_+^{L} \times \mathbb{R}^{JL}$, which is continuous, $\sum_{i=1}^{I} r_i(w, p, y) = \sum_{i=1}^{I} w_i + \sum_{j=I+1}^{N} p \cdot y_j$, $r_i(w, tp, y) =$

$tr_i(w_i, p, y)$ for all $t > 0$ and $\sum_{i=1}^{I} w_i + \sum_{j=I+1}^{N} p \cdot y_j > 0$ implies that $r_i(w, p, y) > 0$. This abstract wealth structure clearly encompasses the case of private ownership structure of $r_i(w, p, y) = p \cdot w_i + \sum_{i=1}^{n} \theta_{ij} p \cdot y_j$ as well as the case of Guesnerie's fixed structure of $r_i(w, p, y) = \theta_i p \cdot (w + y)$.

A marginal cost pricing equilibrium for a given wealth map $(r_1(w, p, y), \ldots, r_I(w, p, y))$ and for an economy e is then a list of consumption plans (x_i^*), a list of production plans (y_j^*), and a price vector p^* such that (a) every consumer maximizes his/her preferences subject to his/her budget constraint, (b) firms's production plans satisfy the first-order necessary conditions for profit maximization, i.e., at the given production plans the market prices lie in the Clarke normal cone, and (c) the excess demand over supply is zero. The main difference with the Walrasian model is condition (b), in which firms may not maximize profits but instead behave according to the marginal cost pricing rule. Thus, the concept of a MCP equilibrium is a natural extension of the concept of a competitive equilibrium for economies in which some firms have non-convex production sets. Formally, we have the following definition.

Definition 1 An allocation $(x^*, y^*) = (x_1^*, x_2^*, \ldots, x_I^*, y_1^*, y_2^*, \ldots, y_J^*) \in \mathbb{R}_+^{nL} \times \mathcal{Y}$ is a *marginal cost pricing equilibrium allocation* for an economy e if it is feasible and there is a price vector $p^* \in \mathbb{R}_+^L$ such that

(1) $p^* \cdot x_i^* \leqq p^* \cdot r_i(w_i, p^*, y^*)$ for all $i = 1, \ldots, I$;

(2) for all $i = 1, \ldots, I$, $x_i \succ_i x_i^*$ implies $p^* \cdot x_i > r_i(w_i, p^*, y^*)$; and

(3) $y_j^* \in \partial \mathcal{Y}_j$ and $p^* \in MC_j(y_j) \equiv N_{\mathcal{Y}_j}(y^*)$ for $j = I + 1, \ldots, N$.

Denote by $MCP(e)$ the set of all such marginal cost pricing equilibrium allocations and $\mathcal{MCP}(e)$ the set of all such marginal cost pricing equilibria (x^*, y^*, p).

The detailed discussions on the settings of the model in economies with increasing returns and the existence of a pricing equilibrium in general and the marginal cost pricing equilibrium in particular can be found in Beato (1982), Brown and Heal (1982), Cornet (1988, 1989, 1990), Bonnisseau (1988), Bonnisseau and Cornet (1988), Kamiya (1988), Vohra (1988), Brown (1991), Quinzii (1991, 1992), and Brown, Heller, and Starr (1992).

Remark 1 The term of MCP equilibrium is, strictly speaking, inappropriate since it is not always true that the price of a good is set at its marginal cost. $p \in N_{\mathcal{Y}_j}$ implies equality between the price of a good and its marginal cost only if the input requirement sets are convex. It has been adopted because it is suggestive. With this qualification in mind we retain it.

Remark 2 From the above homogeneity assumptions of r_i and the strict monotonicity of preferences, we may assume the equilibrium price vector p^* belongs to the $L-1$ dimensional unit simplex $\Delta^{L-1} = \{p \in \mathbb{R}^L_{++} : \sum_{i=1}^L p^l = 1\}$.

Remark 3 When all firm's transformation functions $T_j(y)$ are smooth, the MCP rule becomes simpler and given by

$$p = \gamma_j \nabla T_j(y_j) \text{ for some } \ \gamma_j > 0$$

where $\nabla T_j(y_j)$ denotes the gradient of T_j at y_j

Remark 4 Also, if the production set $\mathcal{Y}_j$ is convex, then condition (2) implies that y_j^* maximizes firm j's profit at price p, i.e.,

$(2)'$ for $j = I+1, \ldots, N$, $p^* \cdot y_j^* \geqq p^* \cdot y_j$ for all $y_j \in \partial \mathcal{Y}_j$.

2.3 Marginal Cost Pricing Quasi-Equilibrium with Transfers and the Second Welfare Theorem

The notion of the marginal cost pricing quasi-equilibrium with transfers (MCPQT in short) can be used to characterize Pareto efficient allocations for production economies with increasing returns. To see this, let us first give a geometric interpretation to the two conditions for Pareto efficiency.

Let $\hat{w} = \sum_{i=1}^I w_i$ and $\hat{\mathcal{Y}} = \sum_{j=I+1}^N \mathcal{Y}_j$ be respectively the aggregate initial endowment and the aggregate production set. Let

$$U_i(x_i) = \{x_i' \in X_i : x_i' \succcurlyeq_i \ x_i\} \tag{1}$$

be the weak upper contour set of consumer i and let

$$U(x) = \sum_{i=1}^I U_i(x_i) \tag{2}$$

the aggregate weak upper contour set. The boundary of $U(x)$ is known as a social indifference curve or a Scitovski contour through the point $\hat{x} = \sum_{i=1}^I x_i$.

Condition (i) for Pareto efficiency implies that $\sum_{i=1}^I x_i \in \{\hat{w}\} + \hat{\mathcal{Y}}$ and hence the set $U \cap [\{\hat{w}\} + \hat{\mathcal{Y}}]$ contains at least $\sum_{i=1}^I x_i$. Condition (ii) implies that $\{\hat{w}\} + \hat{\mathcal{Y}}$ does not intersect the interior of U. Hence, the sets U and $\{\hat{w}\} + \hat{\mathcal{Y}}$ must be "tangent" at the point $\sum_{i=1}^I x_i$.

When the aggregate weak upper contour set U and the aggregate production set $\hat{\mathcal{Y}}$ are both convex (this is true if preferences $\succcurlyeq_i$ and production sets $\mathcal{Y}_j$ are convex) , this tangency condition implies the existence of a hyperplane which separates the two sets U and $\{\hat{w}\} + \hat{\mathcal{Y}}$. The vector normal to this hyperplane is the vector of prices which supports the Pareto efficient allocation. When $\hat{\mathcal{Y}}$ is non-convex, this separation property will not in general hold. However, there still exists a hyperplane tangent to U and $\{\hat{w}\} + \hat{\mathcal{Y}}$ and a vector p orthogonal to the two sets if the boundaries of the two sets are smooth. Thus, in both cases, such a vector p is the supporting prices for the Pareto efficient allocation and is called the efficient price vector in the literature.

Notice that the smoothness of U and $\partial\hat{\mathcal{Y}}$ is not needed to obtain the existence of a price vector which supports an efficient allocation. When production sets are convex, the existence of a supporting price vector follows from the separation theorem applied to the convex sets U and $\partial\hat{\mathcal{Y}}$. The same theorem also implies the existence of a cone of normals at each point of the boundary of these sets so that one can obtain the Second Welfare Theorem that characterizes Pareto efficient allocations by decentralized competitive markets. When production sets are non-convex, it is also possible to generalize the Second Welfare Theorem using the MCPQT.

Definition 2 An allocation $(x^*, y^*) = (x_1^*, x_2^*, \ldots, x_I^*, y_1^*, y_2^*, \ldots, y_J^*) \in \mathbb{R}_+^{NL} \times \mathcal{Y}$ is a *marginal cost pricing quasi-equilibrium allocation* with transfers for an economy e if it is feasible and there is a price vector $p^* \in \mathbb{R}_+^L$ such that

(1) for every $i = 1, \ldots, I$, $p^* \cdot x_i \geqq p^* \cdot x_i^*$ implies $x_i \succcurlyeq_i x_i^*$, i.e., x_i^* minimizes $p \cdot x_i$ over the weak upper contour set $U_i(x_i^*) = \{x_i \in \mathbb{R}_+^L : x_i^* \succcurlyeq_i x_i^*\}$,

(2) for $j = 1, \ldots, J$, $p^* \in N_{\mathcal{Y}_j}(y^*)$.

Denote by $MCPQT(e)$ the set of all such marginal cost pricing quasi-equilibrium allocations with transfers.

Remark 5 It is well known that if $p \cdot x_i^* > 0$, then condition (1) reduces to

(1)$'$ for every $i \in N$, $x_i \succ_i x_i^*$ implies $p^* \cdot x_i > p^* \cdot x_i^*$, i.e., x_i^* is a greatest element for $\succcurlyeq_i$ in the budget set $\{x_i \in \mathbb{R}_+^L : p^* \cdot x_i \leqq p^* \cdot x_i^*\}$.

The resulting equilibrium that satisfies the feasibility, conditions (1)$'$ and (2) is then called the marginal cost pricing equilibrium with transfers. Denote by $MCPT(e)$ the set of all such marginal cost pricing equilibrium allocations with transfers.

An important characterization of Pareto optimal allocations is associated with the following version of the Second Theorem of Welfare Economics for non-convex production economies.

Lemma 1 *Suppose e is an economy such that (1) preferences $\succcurlyeq_i$ are continuous, convex and monotonic, (2) production sets $\mathcal{Y}_j$ are closed and satisfy free disposal property. Let $(x^*, y^*) = (x_1^*, x_2^*, \ldots, x_I^*, y_1^*, y_2^*, \ldots, y_J^*) \in \mathbb{R}_+^{(I+J)L} \times \mathcal{Y}$ be Pareto optimal. Then there exists a price vector $p \in \mathbb{R}_+^L$ with $p \geq 0$ such that (x^*, y^*, p) is a MCPQT, and thus $P(e) \subseteq MCPQT(e)$ for all $e \in E$.*

The proof of this lemma can be found in Cornet (1990) and Quinzii (1992).

Remark 6 Let (x, y) be a Pareto efficient allocation for a production economy e. If p is a supporting price vector for this allocation and $p \cdot x_i > 0$ for consumers $i = 1, \ldots, I$, then (p, x, y) is a MCP equilibrium for any wealth map $(r_1(w, p, y), \ldots, r_I(p, y))$ satisfying $r_i(w, p, y) = p \cdot x_i$ by Lemma 1 and Remark 5.

In general, a MCP equilibrium does not result in Pareto efficient allocations in the presence of increasing returns. To have a marginal cost pricing equilibrium allocation be Pareto efficient, one needs to restrict economic environments so that the economy has efficient allocations for all income maps $(r_1(w, p, y), \ldots, r_I(w, p, y))$. Brown and Heal (1975), Guesnerie (1980), Dierker (1986), Cornet (1990), and Quinzii (1991, 1992) provide various conditions that guarantee the existence of a MCP equilibrium allocation that is Pareto optimal in the presence of non-convex production economies.

Let E^* be the maximal class of economies over which all MCP equilibrium allocations are Pareto efficient for a general wealth map $(r_1(w, p, y), \ldots, r_I(w, p, y))$. The main purpose of the paper is to prove the informational efficiency of the MCP mechanism among the class of allocation mechanisms that result in Pareto efficient allocations on E^*. Notice that since every MCP equilibrium allocation is Pareto efficient for a convex production economy, the set of convex production economies is a subset of E^*.

To ensure a MCP equilibrium to be Pareto efficient, non-convex economies should be "well-behaved". This is true if there is a one to one map between the utility possibility frontier and the feasible allocations which lead to these utilities. Guesnerie (1980) and Quinzii (1992) showed that in a non-convex economy, if there is a unique allocation corresponding to every point on the utility frontier, then for every income map, there exists a MCP equilibrium which is Pareto efficient. In other words, if the choice of a production plan is unique once the social objectives have been fixed then the choice of an income map does not affect the efficiency of the resulting allocation. The formal result can be stated precisely as follows.

Lemma 2 *Let e be a production economy satisfying the following conditions:*

(i) The utility functions $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ are continuous, quasi-concave, and strictly monotonic.

(ii) The production possibility sets $\mathcal{Y}_j$ are nonempty, closed, contain 0, and $\mathcal{Y}_j - \mathbb{R}^L_+ \subseteq \mathcal{Y}_j$.

(iii) $[\hat{w} + \hat{\mathcal{Y}}] \cap \mathbb{R}^L_+$ is compact with a non-empty interior, where $\hat{w} = \sum_{i=1}^I w_i$ and $\hat{\mathcal{Y}} = \sum_{j=I+1}^N \mathcal{Y}_j$.

and let $P(e)$ be the set of Pareto efficient allocations. Suppose there is a one-to-one utility mapping $T : P(e) \to \mathbb{R}^I$, i.e., $(x, y) \to (u_1(x_1), \ldots, u_I(x_I))$. Then for every wealth map $(r_1, \ldots, r_I)$, there exists a MCP equilibrium which is Pareto efficient.

The proof of this lemma can be found in Quinzii (1992).

In Lemma 2, the assumption that the map T is one to one is not placed directly on the characteristics of individuals and it cannot guarantee all MCP equilibria are Pareto efficient. What restrictions must be placed on the characteristics to ensure that there is only one allocation associated with a Pareto optimal utility level? We know from Lemma 1 that an allocation (x, y) is Pareto efficient if the social indifference curve associated with the levels of utility $(u_1(x_1), \ldots, u_I(x_I))$ is tangent to the aggregate feasible consumptions $\hat{w} + \hat{\mathcal{Y}}$. Thus, there is a unique feasible allocation which gives the Pareto optimal utility level $(u_1(x_1), \ldots, u_I(x_I))$ if and only if the social indifference curve corresponding to $(u_1(x_1), \ldots, u_I(x_I))$ is tangent to $\hat{w} + \hat{\mathcal{Y}}$ at only one point. This requires restrictions on the curvature of the frontiers of the sets $\hat{w} + \hat{\mathcal{Y}}$ and $P(x)$. Quinzii (1988, 1992) provided such conditions based on the characteristics of individuals in terms of the elasticities of demand and of marginal cost, which ensure that the uniqueness property of Lemma 2 is satisfied so that every MCP equilibrium allocation is Pareto efficient (cf. Theorems 4.3 and 4.4, 4.6 and 4.7 in Quinzii (1992),).

2.4 Allocation Mechanism

Let F be a social choice rule (correspondence) from E to $\mathcal{A}$. Following Mount and Reiter (1974), a *message process* is a pair $\langle M, \mu \rangle$, where M is a set of abstract messages called the message space, and $\mu : E \to\to M$ is a message correspondence that assigns to every economy e the set of stationary (equilibrium) messages. An *allocation mechanism (process)* is a triple $\langle M, \mu, h \rangle$ defined on E, where $h : M \to \mathcal{A}$ is the outcome function that assigns to every equilibrium message $m \in \mu(e)$ the corresponding trade $(z, y) \in \mathcal{A}$.

Definition 3 An allocation mechanism $\langle M, \mu, h\rangle$, defined on E, *realizes* the social choice rule F, if for all $e \in E$, $\mu(e) \neq \emptyset$ and $h(m) \in F(e)$ for all $m \in \mu(e)$.

Assume the social choice rule is restricted to the one that yields Pareto efficient outcomes. Let $\mathcal{P}(e)$ be a subset of Pareto efficient allocations for $e \in E$. An allocation mechanism $\langle M, \mu, h\rangle$ is said to be *non-wasteful* on E with respect to $\mathcal{P}$ if for all $e \in E$, $\mu(e) \neq \emptyset$ and $h(m) \in \mathcal{P}(e)$ for all $m \in \mu(e)$. If an allocation mechanism $\langle M, \mu, h\rangle$ is non-wasteful on E with respect to P, the set of all Pareto efficient outcomes, then it is said to be non-wasteful on E. The concept of non-wastefulness was first introduced by Hurwicz (1960).

Definition 4 An allocation mechanism $\langle M, \mu, h\rangle$ is said to be *privacy-preserving or informationally decentralized* on E if there exist individual message correspondences $\mu_i : E_i \rightarrow\rightarrow M$, one for each i, such that $\mu(e) = \bigcap_{i=1}^{N} \mu_i(e_i)$ for all $e \in E$.

Thus, when a mechanism is privacy-preserving, each individual's response to a message is only based on that person's private information on his/her own characteristic, but not based on characteristics of the other individuals. The privacy-preserving property is an important property for a mechanism. Under any type of institution or ownership structure, only the manager or the owner of a firm has better information about her own production set, and only a consumer knows her own preferences and initial endowments.

Remark 7 This important feature of the communication process implies that the so called "crossing condition" has to be satisfied. Mount and Reiter (Lemma 5, 1974) showed that an allocation mechanism $\langle M, \mu, h\rangle$ is privacy-preserving on E if and only if for every i and every e and e' in E, $\mu(e) \cap \mu(e') = \mu(e'_i, e_{-i}) \cap \mu(e_i, e'_{-i})$, where $(e'_i, e_{-i}) = (e_1, \ldots, e_{i-1}, e'_i, e_{i+1}, \ldots, e_N)$, i.e., the ith element of e is replaced by e'_i. Thus, if two economies have the same equilibrium message, then any "crossed economy" in which an agent from one of the two initial economies is "switched" with the agent from the other must have the same equilibrium message. Hence, for a given mechanism, if two economies have the same equilibrium message m, the mechanism leads to the same outcome for both, and further, this outcome must also be the outcome of the mechanism for any of the crossed economies because of the crossing condition.

Definition 5 Let $\langle M, \mu, h\rangle$ be an allocation mechanism on E. The stationary message correspondence μ is said to be locally threaded at $e \in E$ if it has a locally continuous, single-valued selection at e. That is, there is a neighborhood $N(e) \subset E$ and a continuous function $f : N(e) \rightarrow M$ such that $f(e') \in \mu(e')$ for all $e' \in N(e)$. The stationary message correspondence μ is said to be locally threaded on E if it is locally threaded at every $e \in E$.

The notion of local threadedness was first introduced into the realization literature by Mount and Reiter (1974). This regularity condition is used mainly to exclude the possibility of intuitive smuggling information. Many continuous selection results have been given in the mathematics literature since Micheal (1956).

It will be seen that the requirement that μ is locally threaded at every $e \in E$ is not only too strong but it also results in too pessimistic results such as those in Calsamiglia (1977) and Hurwicz (1999) who showed the non-existence of a smooth finite-dimensional message space mechanism that realize Pareto efficient allocations in a certain class of economies with increasing returns and economies with production externalities that result in non-convex production sets.

2.5 The Marginal Cost Pricing Process

We now define the MCP allocation process that is a privacy-preserving process and realizes the marginal cost pricing correspondence MCP, and in which messages consist of prices and trades of all agents. In defining the MCP allocation process, it is assumed that the wealth map $(r_1, \ldots, r_I)$ is common knowledge for all the agents.

Define the excess demand correspondence of consumer i $(i = 1, \ldots, I)$ $D_i : \Delta^{L-1} \times \mathbb{R}^{IL}_+ \times \mathbb{R}^{JL} \times E_i \to\to \mathbb{R}^L$ by

$$\begin{aligned} D_i(p, w, y, e_i) \quad &= \quad \{z_i : z_i + w_i \in X_i, p \cdot (z_i + w_i) = r_i(w, p, y) \\ & \qquad (z_i' + w_i) \succ_i (z_i + w_i) \text{ implies } p \cdot (z_i' + w_i) > r_i(w, p, y)\}. \end{aligned} \tag{3}$$

Define the supply correspondence of firm j $(j = I+1, \ldots, N)$ $S_j : \Delta^{L-1} \times E_j \to\to \mathbb{R}^L$ by

$$S_{i\text{J}}(p, e_j) = \{y_j \in \partial\mathcal{Y}_j : p \in MC_j(y_j)\}. \tag{4}$$

Note that (p, z, y) is a MCP equilibrium for economy e with respect to a wealth distribution map $(r_1, \ldots, r_I)$ if $p \in \Delta^{L-1}$, $z_i \in D_i(p, w, y, e_i)$ for $i = 1, \ldots, I$, $y_j \in S_j(p, e_j)$ for $j = I = 1, \ldots, N$, and the allocation (z, y) is balanced.

The MCP process $\langle M_{mc}, \mu_{mc}, h_{mc} \rangle$ is then defined as follows.

Define $M_{mc} = \Delta^{L-1} \times \mathcal{A}$.

Define $\mu_{mc} : E \to\to M_{mc}$ by

$$\mu_{mc}(e) = \cap_{i=1}^N \mu_{mci}(e_i), \tag{5}$$

where $\mu_{mci} : E_i \to\to M_{mc}$ is defined as follows:

(1) For $i = 1, \ldots, I$, $\mu_{mci}(e_i) = \{(p, z, y, v) : p \in \Delta^{L-1}, z_i \in D_i(p, v, y, e_i),\ v_i = w_i$ and $\sum_{i=1}^I z_i = \sum_{j=I+1}^N y_j\}$.

(2) For $i = I+1, \ldots, N$, $\mu_{mci}(e_i) = \{(p, z, y, v) : p \in \Delta^{L-1}, y_i \in S_i(p, e_i)$ and $\sum_{i=1}^{I} z_i = \sum_{j=I+1}^{N} y_j\}$.

Thus, we have $\mu_{mc}(e) = \mathcal{MCP}(e)$ for all $e \in E^{MC}$.

Finally, the MCP outcome function $h_{mc} : M_{mc} \to \mathcal{A}$ is defined by

$$h_{mc}(p, z, y) = (z, y), \tag{6}$$

which is an element in $MCP(e)$.

Thus, the MCP correspondence associates with every economy $e \in E^{MC}$ the corresponding set of MCP equilibrium allocations, while the MCP message correspondence associates with every economy the corresponding set of MCP equilibria.

The MCP process can be viewed as a formalization of a resource allocation, which is non-wasteful on the class of production economies E^*. The marginal cost pricing message process is privacy-preserving by the construction of the marginal cost pricing process.

Remark 8 For a given wealth distribution map $(r_1, \ldots, r_I)$, since an element, $m = (p, z_1, \ldots, z_I, y_{I+1}, \ldots, y_N, v_1, \ldots, v_I) \in \mathbb{R}^L_{++} \times \mathbb{R}^{IL} \times \mathbb{R}^{JL} \times \mathbb{R}^{IL}$, of the MCP message space M_{mc} satisfies the conditions $\sum_{l=1}^{L} p^l = 1$, $\sum_{i=1}^{I} z_i = \sum_{j=I+1}^{N} y_j$, $p \cdot (z_i + w_i) = r_i(w_i, p, p \cdot y_{I+1}, \ldots, p \cdot y_N)$ $(i = 1, \ldots, I)$, $v_i = w_i$ for $i = 1, \ldots, I$, and one of these equations is not independent by Walras Law, any MCP message is contained within a Euclidean space of dimension $(L+IL+JL+IL)-(1+L+I)-IL+1 = (L-1)I+LJ$ and thus, an upper bound on the Euclidean dimension of M_{mc} is $(L-1)I+LJ$, which has the same upper bound as the Walrasian process. Notice that this upper bound holds for any marginal cost pricing equilibria under all income maps regardless if it results in Pareto efficient allocations.

2.6 Informational Size of Message Spaces

Informational size can be considered as a concept that characterizes the relative sizes of topological spaces used to convey information in the resource allocation process. It would be natural to consider that a space, say S, has more information than another space T whenever S is topologically "larger" than T. This suggests the following definition, which was introduced by Walker (1977).

Definition 6 Let S and T be two topological spaces. The space S is said to have as much information as the space T by the Fréchet ordering, denoted by $S \geqq_F T$, if T can be embedded homeomorphically in S, i.e., if there is a subspace of S' of S that is homeomorphic to T.

Definition 7 Let S and T be two topological spaces and let $\psi : T \to S$ be a correspondence. The correspondence ψ is said to be injective if $\psi(t) \cap \psi(t') \neq \emptyset$ implies $t = t'$ for any $t, t' \in T$. That is, the inverse, $(\psi)^{-1}$, of ψ is a single-valued function.

A topological space M is an *n-dimensional manifold* if it is locally homeomorphic to $\mathbb{R}^n$.

Definition 8 An informationally decentralized non-wasteful mechanism $\langle M, \mu, h \rangle$ is said to be *informationally efficient* on E if the size of its message space M is the smallest one among all other informationally decentralized non-wasteful mechanisms defined on E.

2.7 Cobb-Douglas-Quadratic Economies

To establish the informational efficiency of the MCP mechanism, we will adopt a standard approach that is widely used in the realization literature. For a set of admissible economies and a smooth informationally decentralized mechanism realizing a social choice correspondence, if one can find a (parametrized) subset (test family) with dimension n, and the stationary message correspondence is injective, then the size of the message space required for an informationally decentralized mechanism to realize the social choice correspondence cannot be lower than n on the subset. Thus, it cannot be lower than n for any superset of the subset, and in particular, for the entire class of economies. It is this result that was used by Hurwicz (1977), Mount and Reiter (1974), Walker (1977), Rsana (1978), Sato (1981), Calsamiglia and Kirman (1993), Tian (2003, 2004, 2006) among others to show the minimal informational size and thus informational efficiency of the competitive mechanism, Lindahl mechanism, the equal-income Walrasian mechanism, and the distributive Lindahl mechanism over the various classes of economic environments. It is also this result that was used by Calsamiglia (1977) and Hurwicz (1999) to show the non-existence of a smooth finite-dimensional message space mechanism that realizes Pareto efficient allocations in a certain class of economies with increasing returns and economies with production externalities that result in non-convex production sets. It is the same result that will be used in the present paper to establish a lower bound of the size of the message space required for an informationally decentralized and non-wasteful smooth mechanism on the test family that we will specify below, and consequently over the entire class of economies with general non-convex production sets.

The test family, denoted by $E^{cq} = \prod_{i=1}^{N} E_i^{cq}$, is a special class of economies, where preference orderings are characterized by Cobb-Douglas utility functions, and efficient production technology are characterized by quadratic functions. It will be showed that E^{cq} is a subset of E^* in Lemma 2.

For $i = 1, \ldots, I$, consumer i's admissible economic characteristics in E_i^{cq} are given by the set of all $e_i = (X_i, w_i, \succcurlyeq_i)$ such that $X_i = \mathbb{R}_+^L$, $w_i > 0$, and $\succcurlyeq_i$ is represented by a Cobb-Douglas utility function $u(\cdot, a_i)$ with $a_i \in \Delta^{L-1}$ such that $u(z_i + w_i, a_i) = \prod_{l=1}^{L} (z_i^l + w_i^l)^{a_i^l}$.

For $i = I+1, \ldots, N$, producer i's admissible economic characteristics are given by the set of all $e_i = \mathcal{Y}_i = \mathcal{Y}(b_i)$ such that

$$\begin{aligned} \mathcal{Y}(b_i) &= \{y_i \in \mathbb{R}^L : b_i^1 y_i^1 + \sum_{l=2}^{L} (y^l + \frac{b_i^l}{2}(y^l)^2) \leqq 0 \\ &\qquad -\frac{1}{b_i^l} \leqq y_i^l \leqq 0 \text{ for all } l \neq 1\}, \end{aligned} \tag{7}$$

where $b_i = (b_i^1, \ldots, b_i^L)$ with $b_i^l > \frac{J}{w_i^l}$. It is clear that any economy in E^{cq} is fully specified by the parameters $a = (a_1, \ldots, a_I)$ and $b = (b_{I+1}, \ldots, b_N)$. Furthermore, production sets are nonempty, closed, and convex by noting that $0 \in \mathcal{Y}(b_j)$ and their efficient points are represented by quadratic production functions in which $(y_i^2, \ldots, y_i^L)$ are inputs and y_i^1 is possibly an output.

Remark 9 When we defined $\mathcal{Y}_i$ as given in (7), we have assumed that $y_i^2, \ldots, y_i^L$ are inputs. Then, y_i^1 may be an output (when $y_i^1 \geqq 0$). If we, instead, assume that $0 \leqq y_i^l \leqq \frac{1}{b_i^l}$ for $2, \ldots, L$, and y_i^1 is an input, all of the results given in the paper remains true. For instance, we can similarly prove Lemma 4 on the uniqueness of the marginal coast pricing equilibrium on E^{cq}.

Given an initial endowment $\bar{w} \in \mathbb{R}_{++}^{LI}$, define a subset $\bar{E}^{cq}$ of E^{cq} by $\bar{E}^{cq} = \{e \in E^{cd} : w_i = \bar{w}_i \; \forall i = 1, \ldots, I\}$. That is, endowments are constant over $\bar{E}^{cq}$.

A topology is introduced to the class $\bar{E}^{cq}$ as follows. Let $\|\cdot\|$ be the usual Euclidean norm on $\mathbb{R}^L$. For each consumer i, $(i = 1, \ldots, I)$, define a metric d on $\bar{E}_i^{cq}$ by $d[u(\cdot, a_i), u(\cdot, \bar{a}_i)] = \|a_i - \bar{a}_i\|$. Note that, since endowments are fixed over $\bar{E}_i^{cq}$, this defines a topology on $\bar{E}_i^{cq}$. Similarly, for each producer i, $(i = I+1, \ldots, N)$, define a metric d on $\bar{E}_i^{cq}$ by $d[\mathcal{Y}(b_i), \mathcal{Y}(\bar{b}_i)] = \|b_i - \bar{b}_i\|$. We may endow $\bar{E}^{cq}$ with the product topology of the $\bar{E}_i^{cq} (i = 1, \ldots, N)$ and we call this the parameter topology, which will be denoted by $\mathcal{T}_p$. Then it is clear that the topological space $(\bar{E}^{cq}, \mathcal{T}_p)$ is homeomorphic to the $(L-1)I + LJ$ dimensional Euclidean space $\mathbb{R}^{(L-1)I+LJ}$.

It may be remarked that we had used the class of economies E^{cq} in Tian (2006) in the context of convex economies with production over which we have studied the informational efficiency of the competitive process.

3 The Lower Bound of Informational Requirements of Mechanisms

In this section we establish a lower bound (the minimal amount) of information, as measured by the size of the message space, that is required to guarantee that an informationally decentralized mechanism realizes Pareto efficient allocations on E; the class of production economies.

To make the problem nontrivial, as usual, the assumption of interiority has to be made.[5] Indeed, a mechanism that gives everything to a single individual yields Pareto efficient outcomes and no information about prices is needed. Thus, given a class of economies E that includes E^{cq}, we define an optimality correspondence $\mathcal{P} : E \rightarrow\rightarrow \mathcal{A}$ such that the restriction $\mathcal{P}|E^{cq}$ associates with $e \in E^{cq}$ the set $\mathcal{P}(e)$ of all the Pareto efficient allocations that assign strictly positive consumption to every consumer.

Lemma 3, which is based on the special class of Cobb-Douglas–Quadratic economies $\bar{E}^{cq}$ is central to finding the lower bound of informational requirements of resource allocation processes.

Lemma 3 *Suppose $\langle M, \mu, h\rangle$ is an allocation mechanism on the special class of economies $\bar{E}^{cq}$ such that:*

(i) it is informationally decentralized;

(ii) it is non-wasteful with respect to $\mathcal{P}$.

Then, the stationary message correspondence μ is injective on $\bar{E}^{cq}$. That is, its inverse is a single-valued mapping on $\mu(\bar{E}^{cq})$.

Theorem 1 establishes a lower bound informational size of messages spaces of any smooth allocation mechanism that is informationally decentralized and non-wasteful over any class of economies that includes E^{cq}.

Theorem 1 (Informational Boundedness Theorem) *Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism on any class of production economies E that includes E^{cq} such that:*

(i) it is informationally decentralized;

(ii) it is non-wasteful with respect to $\mathcal{P}$;

(iii) M is a Hausdorff topological space;

(iv) μ is locally threaded at some point $e \in \bar{E}^{cq}$.

[5] A stronger condition that can guarantee interior outcomes is that a mechanism is individually rational.

Then, the size of the message space M is at least as large as $\mathbb{R}^{(L-1)I+LJ}$, that is, $M \geqq_F M_c =_F \mathbb{R}^{(L-1)I+LJ}$.

Proof. As was noted above, $\bar{E}^{cq}$ is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$. Hence, it suffices to show $M \geqq_F \bar{E}^{cq}$.

By injectiveness shown in Lemma 3, we know that the restriction $\mu|_{\bar{E}^{cq}}$ of the stationary message correspondence μ to $\bar{E}^{cq}$ is an injective correspondence. Since μ is locally threaded at $e \in \bar{E}^{cq}$, there exists a neighborhood $N(e)$ of e and a continuous function $f : N(e) \to M$ such that $f(e') \in \mu(e')$ for all $e' \in N(e)$. Then f is a continuous injection from $N(e)$ into M. Since μ is an injective correspondence from $\bar{E}^{cq}$ into M, f is a continuous one-to-one function on $N(e)$.

Since $\bar{E}^{cq}$ is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$, there exists a compact set $\bar{N}(e) \subset N(e)$ with nonempty interior point. Also, since f is a continuous one-to-one function on $N(e)$, f is a continuous one-to-one function from the compact space $\bar{N}(e)$ onto a Hausdorff topological space $f(\bar{N}(e))$. Hence, it follows that the restriction $f|_{\bar{N}(e)}$ is a homeomorphic imbedding on $\bar{N}(e)$ by Theorem 5.8 in Kelley (1955, p. 141). Choose an open ball $\mathring{N}(e) \subset \bar{N}(e)$. Then $\mathring{N}(e)$ and $f(\mathring{N}(e))$ are homeomorphic by a homeomorphism $f|\mathring{N}(e) : \mathring{N}(e) \to f(\mathring{N}(e))$. This, together with the fact that $\bar{E}^{cq}$ is homeomorphic to its open ball $\mathring{N}(e)$, implies that $\bar{E}^{cq}$ is homeomorphic to $f(\mathring{N}(e)) \subset M$. Hence, it follows that $M \geqq_F \bar{E}^{cd} =_F \mathbb{R}^{(L-1)I+LJ}$. Q.E.D.

4 Informational Efficiency of the MCP Process

In the previous section, we found that the lower bound informational size of message spaces for smooth allocation mechanisms that are privacy-preserving and non-wasteful over the class E of production economies that includes $\bar{E}^{cq}$ is the $(L-1)I + LJ$-dimensional Euclidean space $\mathbb{R}^{(L-1)I+LJ}$. In this section we assert that the lower bound is exactly the size of the message space of the MCP mechanism, and thus the MCP mechanism is informationally efficient among all smooth resource allocation mechanisms that are informationally decentralized and non-wasteful over the set E^*.

By Theorem 1, to show this result, we need to show that M_{mc} is homeomorphic to the $(L-1)I + LJ$-dimensional Euclidean space $\mathbb{R}^{(L-1)I+LJ}$. To do so, note that, from Remark 2, we know that the upper bound dimension of the message space of the MCP mechanism is $(L-1)I+LJ$. As a result, if we can show that this upper bound can be reached on the restriction of the message space of the MCP mechanism for a given wealth distribution map $(r_1, \ldots, r_I)$ to the test family $\bar{E}^{cq}$ of Cobb-Douglas–Quadratic economies, i.e., if we can show that, for a given wealth map $(r_1, \ldots, r_I)$, $\mu_{mc}|_{\bar{E}^{cq}}$ is homeomorphic to the $(L-1)I + LJ$-dimensional Euclidean space $\mathbb{R}^{(L-1)I+LJ}$, then we know that the size of the message space of the MCP mechanism is $(L-1)I + LJ$ and thus the MCP mechanism is informationally efficient among all resource allocation mechanisms that are informationally decentralized and non-wasteful over the class of economies E^* in which MCP equilibria result in Pareto efficient allocations. Hence, to show the informational efficiency of the MCP mechanism, it suffices for us to show that this upper bound can be actually reached on the test family of economies for the MCP mechanism with a given wealth distribution map $(r_1, \ldots, r_I)$.

We will first state the following lemmas that shows that the MCP mechanism is single-valued and continuous so that it is locally threaded on the test family set $\bar{E}^{cq}$ of Cobb-Douglas–Quadratic economies.

Lemma 4 *For the wealth distribution map $(r_1, \ldots, r_I)$ with $r_i(w, p, y) = p \cdot w_i + \sum_{i=1}^{n} \theta_{ij} p \cdot y_j$ for $l = 1, \ldots, L$, every economy in $\bar{E}^{cq}$ has a unique MCP equilibrium, i.e., $MCP(e)$ is a single-valued mapping from $\bar{E}^{cq}$ to $\mathcal{A}$.*

Lemma 5 *Let μ_{cq} be the MCP message correspondence on $\bar{E}^{cq}$. The μ_{cq} is a continuous function.*

Lemma 6 *μ_{mc} defined as the MCP message correspondence on E^* is is homeomorphic to $\bar{E}^{cq}$.*

From the above lemmas and Theorem 1, we have the following theorem that establishes the informational efficiency of the MCP mechanism within the class of all smooth resource allocation mechanisms which are informationally decentralized and non-wasteful over the class of production economies E^*.

Theorem 2 (Informational Efficiency Theorem) *The MCP allocation mechanism $\langle M_{mc}, \mu_{mc}, h_{mc} \rangle$ is informationally efficient among all allocation mechanisms $\langle M, \mu, h \rangle$ defined on E^* that*

(i) are informationally decentralized;

(ii) are non-wasteful with respect to $\mathcal{P}$;

(iii) have Hausdorff topological message spaces;

(iv) satisfy the local threadedness property at some point $e \in \bar{E}^{cq}$.

That is, $M_{mc} =_F \mathbb{R}^{(L-1)I+LJ} \leqq_F M$.

Proof. First note that, since $MCP(e) \neq \emptyset$ for all $e \in E^*$, the MCP mechanism is well defined on E^*. $\langle M_{mc}, \mu_{mc}, h_{mc} \rangle$ is also privacy-preserving as shown in Section 2. Since u_i is strictly monotone on E by assumption, we know z is Pareto efficient by the First Theorem of Welfare Economics. Thus, the MCP process $\langle M_{mc}, \mu_{mc}, h_{mc} \rangle$ is privacy-preserving and non-wasteful over E^*. Furthermore, by Lemmas 2 and 3, we know that μ_{mc} is a single-valued and continuous function on $\bar{E}^{cq}$. Therefore, the MCP allocation mechanism $\langle M_{mc}, \mu_{mc}, h_{mc} \rangle$ satisfies Conditions (i) - (iv).

Also, since $\mu_{mc}(E^*)$ is homeomorphic to $\bar{E}^{cq}$ by Lemma 6 and $\bar{E}^{cq}$ is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$ as noted above, then M_{mc} is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$. Thus, by Theorem 1, we have $M \geqq_F M_{mc} =_F \mathbb{R}^{(L-1)I+LJ}$. Hence, the MCP allocation mechanism $\langle M_{mc}, \mu_{mc}, h_{mc} \rangle$ is informationally efficient among all allocation mechanisms that satisfy Conditions (i) -(iv). Q.E.D.

5 Dimension Requirement of the MCPT Process

Since a MCP equilibrium in general does not result in Pareto efficient allocations in the presence of increasing returns, the set E^*, which is the maximal class of economies over which every MCP equilibrium is Pareto efficient for any given wealth distribution map $(r_1(w,p,y), \ldots, r_I(w,p,y))$, is relatively restrictive. A question is then whether or not the Pareto efficient performance can be realized by an informationally decentralized mechanism that has a finite-dimensional message space.

There are two types of inefficiencies that may arise in a MCP equilibrium be inefficient. The first type of inefficiency is productive inefficiency that is linked to the failure of price to coordinate firms' productions. One way to avoid productive inefficiency is to aggregate individual production sets and then prove the existence of a MCP equilibrium for the aggregate production set. The second type of inefficiency is product mix inefficiency that arises from a lack of coordination between the consumption and the production sector and the way income is distributed among the agents at equilibrium. Guesnerie (1975) and Brown and Heal (1979) have constructed examples that show that all MCP equilibria are inefficient for a given income map. Since there are income maps for which at least one MCP equilibrium is efficient, their examples also show that in non-convex economies, efficiency of MCP equilibria and choice of income distribution are not separate questions as they are in convex economies. In a convex Arrow-Debreu economy, every distribution of income give rise to at least one equilibrium and every equilibrium allocation is efficient for any given distribution of income. The only relevant criterion for comparing two income maps is equity since both maps give rise to efficient allocations. Thus, in the case of convex economies, the search for efficiency through decentralized competitive markets and the search for equity through redistributing endowments are two separate issues.

However, as shown by Brown and Heal (1979), when a production exhibits increasing returns to scale, the situation is quite different. Although production may be efficient and prices correctly reflect marginal costs, an equilibrium associated with a particular distribution of income may not be Pareto optimal. In other words, the efficiency of an allocation in general depends on the equity of an allocation. Thus, to ensure a MCP equilibrium is Pareto efficient, one should choose a suitable wealth distribution map.

Indeed, by Lemma 1, for any economy e specified in the paper, we know that every Pareto efficient allocation (x, y) can be supported by a MCPQT allocation. Further, by Remark 5, if p is a supporting price vector for this allocation and $p \cdot x_i > 0$ (this is true as long as $x \neq 0$ under strict monotonicity of preferences since $p > 0$ at equilibrium) for consumers $i = 1, \ldots, I$, then (p, x, y) is a MCPT or it is a MCP equilibrium for any wealth map $(r_1(w, p, y), \ldots, r_I(p, y))$ satisfying $r_i(w, p, y) = p \cdot x_i$. Thus the investigation of the dimension requirement for realizing Pareto efficient allocations over a general class of production economies with non-convex production sets is reduced to the investigation of the dimension requirement for realizing the MCPT process defined below.

To define the MCPT process, define consumer i's excess demand correspondence with transfers $DT_i : \Delta^{L-1} \times E_i \rightarrow\rightarrow \mathbb{R}^L$ by

$$\begin{aligned} DT_i(p, e_i) &= \{z_i : z_i + w_i \in X_i, \\ &\qquad (z_i' + w_i) \succ_i (z_i + w_i) \text{ implies } p \cdot (z_i' + w_i) > p \cdot (z_i + w_i)\} \end{aligned} \tag{8}$$

for $i = 1, \ldots, I$.

Define firm j's supply correspondence $S_j : \Delta^{L-1} \times E_j \rightarrow\rightarrow \mathbb{R}^L$ by

$$S_j(p, e_j) = \{y_j \in \partial \mathcal{Y}_j : p \in MC_j(y_j)\} \tag{9}$$

$j = I + 1, \ldots, N$.

Then (p, z, y) is a marginal cost pricing equilibrium with transfers for economy e if $p \in \Delta^{L-1}$, $z_i \in DT_i(p, e_i)$ for $i = 1, \ldots, I$, $y_j \in S_j(p, e_j)$ for $j = I = 1, \ldots, N$, and the allocation (z, y) is balanced.

The MCPT process $\langle M_{mct}, \mu_{mct}, h_{mct} \rangle$ is then defined as follows.

Define $M_{mct} = \Delta^{L-1} \times \mathcal{A}$.

Define $\mu_{mct} : E \to\to M_{mct}$ by

$$\mu_{mct}(e) = \cap_{i=1}^{N} \mu_{mcti}(e_i), \tag{10}$$

where $\mu_{mcti} : E_i \to\to M_{mct}$ is defined as follows:

(1) For $i = 1, \ldots, I$, $\mu_{mcti}(e_i) = \{(p, z, y) : p \in \Delta^{L-1}, z_i \in DT_i(p, e_i)$, and $\sum_{i=1}^{I} z_i = \sum_{j=I+1}^{N} y_j\}$.

(2) For $i = I+1, \ldots, N$, $\mu_{mcti}(e_i) = \{(p, z, y) : p \in \Delta^{L-1}, y_i \in S_i(p, e_i)$ and $\sum_{i=1}^{I} z_i = \sum_{j=I+1}^{N} y_j\}$.

Thus, we have $\mu_{mct}(e) = \mathcal{MCPT}(e)$ for all $e \in E$. Notice that, $DT_i(z, y, e_i)$ is not single-valued because the budget is free to change, and thus $\mu_{mct}(e) = \mathcal{MCPT}(e) = P(e) \setminus \{0\}$ by Lemma 1 and Remark 5, which cannot be singled.

Finally, the MCPT outcome function with transfers $h_{mct} : M_{mct} \to \mathcal{A}$ is defined by

$$h_{mct}(p, z, y) = (z, y), \tag{11}$$

which is an element in $MCPT(e)$.

Thus, the MCPT correspondence associates with every economy $e \in E$ the corresponding set of marginal cost pricing equilibrium allocations with transfers, while the MCPT message correspondence associates with every economy the corresponding set of marginal cost equilibria with transfers.

The MCPT process then can be viewed as a formalization of resource allocation, which covers the non-wasteful performance as a subprocess on the entailed class of production economies E under consideration. The MCPT process is clearly privacy-preserving by construction.

Since an element, $m = (p, z_1, \ldots, z_I, y_{I+1}, \ldots, y_N) \in \mathbb{R}^L_{++} \times \mathbb{R}^{IL} \times \mathbb{R}^{JL}$, of the message space for the MCPT process M_{mct} satisfies the conditions $\sum_{l=1}^{L} p^l = 1$, $\sum_{i=1}^{I} z_i = \sum_{j=I+1}^{N} y_j$, and one of these equations is not independent by Walras Law, any MCPT message is contained within a Euclidean space of dimension $(L + IL + JL) - (1 + L) + 1 = (I + J)L$ and thus, an upper bound on the Euclidean dimension of M_{mct} is $(I + J)L$, which higher than the upper bound as the marginal cost pricing (or Walrasian) process and is increased by I. The reason that the upper bound for the MCPT process is I-dimension higher is that each consumer minimizes his expenditure for each given level of utility and thus is not restricted by his/her initial endowment. Hence each consumer has one more degree of freedom in choosing his/her consumption bundle. Thus, for any Pareto efficient allocation (x, y) (with $x \neq 0$) to be realized by an informationally decentralized mechanism, we can realize the allocation (x, y) by the MCPT mechanism. Thus, the dimension requirement of realizing a Pareto efficient allocation is finite.

Notice that our result, which shows that it is possible to have a non-wasteful informationally decentralized mechanism that has a finite-dimensional message space, is sharply contrast to the result of Calsamiglia (1977), which establishes a proposition that in classes of production economies within which at least one firm operates under unbounded increasing returns to scale, it is impossible to have a non-wasteful informationally decentralized mechanism that has a finite-dimensional message space. Calsamiglia first established for a class of production economies with a single produced good, a single primary good and two firms, one of which can operate under arbitrary increasing returns to scale. The same conclusion is then derived for a class of economies with two desired goods, one consumer and one producer, where the producer uses one of the goods as an input in the production of the other good under arbitrary increasing returns to scale, while the consumer has the usual quasi-concave and strictly increasing utility function. This impossibility theorem then obviously holds over any class of economies containing either of the two classes of above mentioned economies.

What causes such a sharp difference? As we mentioned in Section 2, the local threadedness on the entailed set E is probably an unduly strong requirement to impose on processes defined over classes of economies displaying production non-convexities. Calsamiglia's impossibility theorem is based on the condition that the message correspondence μ is locally threaded for every economy $e \in E$. If this requirement is dropped, then an inspection of Calsamiglis's proofs indicates that the argument used will no longer work. Analytical reason underlying this observation is, quite simply, that the class of economies that Calsamigla use to obtain the impossibility theorem satisfies the conditions of the Second Welfare Theorem given in Lemma 1. In our opinion, to exclude the possibility of smuggling information arbitrarily, it is enough only to assume the message correspondence is locally threaded at some point is E, but not assume that it is locally threaded everywhere in E since the MCP process is not locally threaded for every point in E. Thus, our possibility result is not only positive, but also more reasonable and realistic.

Indeed, Tian (2005) actually considered the incentive issue of the marginal cost pricing equilibrium allocations with transfers by presenting a mechanism that implements marginal cost pricing equilibrium allocations with transfers and has a finite-dimensional message space.

6 Conclusion

In this paper, it has been shown that the marginal cost pricing mechanism is informationally the most efficient decentralized mechanism among the class of mechanisms that realize Pareto efficient allocations over a certain class of economies with non-convex production technologies. Since it covers convex production economies as a subset, our result includes the informational efficiency of the Walrasian process as a special case. By examining the upper bound of the message space of the MCPT allocation mechanism, we also investigated the informational requirements for achieving Pareto efficient allocations for the general class of non-convex production economies and showed that Pareto efficient allocations can be realized by the MCPT mechanism that is informationally decentralized and has a finite-dimensional message space. This result is in sharp contrast to the impossibility result of Calsamiglia (1977), which showed that it is impossible to have a non-wasteful informationally decentralized mechanism that has a finite-dimensional message space for the classes of production economies within which at least one firm operates under unbounded increasing returns to scale.

Also, like most of the realization literature, we have assumed that agents follow the rules of the mechanism without regard to self-interest. When the incentive aspect of the Walrasian mechanism is also taken into account, Reichelstein and Reiter (1988) have shown that a Nash implementation typically increases the size of the message space of the mechanism. Williams (1986) and Saijo (1988) have provided general lower bounds of the size of the message space required to Nash implement a social choice correspondence.

Appendix

Proof of Lemma 3. Suppose that there is a message $m \in \mu(e) \cap \mu(\bar{e})$ for $e, \bar{e} \in \bar{E}^{cq}$. It will be proved that $e = \bar{e}$. Since μ is a privacy-preserving correspondence,

$$\mu(e) \cap \mu(\bar{e}) = \mu(\bar{e}_i, e_{-i}) \cap \mu(e_i, \bar{e}_{-i}) \tag{12}$$

for all $i = 1, \ldots, N$ by Remark 7, and hence, in particular,

$$m \in \mu(e) \cap \mu(\bar{e}_i, e_{-i}) \tag{13}$$

for all $i = 1, \ldots, N$. Let $(z, y) = h(m)$. Since the process $\langle M, \mu, h \rangle$ is non-wasteful with respect to $\mathcal{P}$, $(z, y) = h(m)$ and (13) imply that $(z, y) \in \mathcal{P}(e) \cap \mathcal{P}(\bar{e}_i, e_{-i})$. Since (z, y) is an interior point, $(z, y) \in \mathcal{P}(e)$ implies

$$\frac{a_i^l(z_i^1 + \bar{w}_i^1)}{a_i^1(z_i^l + \bar{w}_i^l)} = \frac{1 + b_j^l y_j^l}{b_j^1} \quad l = 2, \ldots, L, i = 1, \ldots, I, j = I + 1, \ldots, N, \tag{14}$$

and

$$b_j^1 y_j^1 = -\sum_{l=2}^{L} \left(y_j^l + \frac{b_l^j}{2} (y_j^l)^2 \right) \quad j = I + 1, \ldots, N. \tag{15}$$

Similarly, $(z, y) \in \mathcal{P}(\bar{e}_i, e_{-i})$ implies

$$\frac{\bar{a}_i^l(z_i^1 + \bar{w}_i^1)}{\bar{a}_i^1(z_i^l + \bar{w}_i^l)} = \frac{1 + b_j^l y_j^l}{b_j^1} \quad l = 2, \ldots, L, i = 1, \ldots, I, j = I + 1, \ldots, N. \tag{16}$$

From equations (14) and (16), we derive

$$\frac{\bar{a}_i^l}{\bar{a}_i^1} = \frac{a_i^l}{a_i^1} \quad l = 2, \ldots, L, i = 1, \ldots, I. \tag{17}$$

As $\sum_{l=1}^{L} a_i^l = 1$ and $\sum_{l=1}^{L} \bar{a}_i^l = 1$, equation (17) implies

$$a_i^l = \bar{a}_i^l \quad l = 1, \ldots, L, i = 1, \ldots, I, \tag{18}$$

and thus $a = \bar{a}$.

As for producers, $(z, y) \in \mathcal{P}(\bar{e}_j, e_{-j})$ implies

$$\frac{a_i^l(z_i^1 + \bar{w}_i^1)}{a_i^1(z_i^l + \bar{w}_i^l)} = \frac{1 + \bar{b}_j^l y_j^l}{\bar{b}_j^1} \quad l = 2, \ldots, L, i = 1, \ldots, I, j = I + 1, \ldots, N, \tag{19}$$

and

$$\bar{b}_j^1 y_j^1 = -\sum_{l=2}^{L} \left(y_j^l + \frac{\bar{b}_j^l}{2} (y_j^l)^2 \right). \tag{20}$$

From equations (14) and (19), we derive

$$\frac{b_j^1}{\bar{b}_j^1} = \frac{1 + b_j^l y_j^l}{1 + \bar{b}_j^l y_j^l} \quad l = 2, \ldots, L, \; j = I + 1, \ldots, N. \tag{21}$$

From equations (15) and (20), we derive

$$\frac{b_j^1}{\bar{b}_j^1} = \frac{\sum_{l=2}^{L}(1+\frac{b_j^l}{2}y_j^l)y_j^l}{\sum_{l=2}^{L}(1+\frac{\bar{b}_j^l}{2}y_j^l)y_j^l} \quad j = I+1,\ldots,N. \tag{22}$$

From equations (21) and (22), we have

$$\frac{1+b_j^l y_j^l}{\sum_{l=2}^{L}(1+\frac{b_j^l}{2}y_j^l)y_j^l} = \frac{1+\bar{b}_j^l y_j^l}{\sum_{l=2}^{L}(1+\frac{\bar{b}_j^l}{2}y_j^l)y_j^l} \quad l = 2,\ldots,L,\ j = I+1,\ldots,N. \tag{23}$$

Multiplying y_j^l on both sides of equation (23) and summing, we have

$$\frac{\sum_{l=2}^{L}(1+b_j^l y_j^l)y_j^l}{\sum_{l=2}^{L}(1+\frac{b_j^l}{2}y_j^l)y_j^l} = \frac{\sum_{l=2}^{L}(1+\bar{b}_j^l y_j^l)y_j^l}{\sum_{l=2}^{L}(1+\frac{\bar{b}_j^l}{2}y_j^l)y_j^l} \quad j = I+1,\ldots,N. \tag{24}$$

Simplifying equation (24), we have

$$\sum_{l=2}^{L} b_j^l (y_j^l)^2 = \sum_{l=2}^{L} \bar{b}_j^l (y_j^l)^2 \quad j = I+1,\ldots,N. \tag{25}$$

Multiplying 1/2 and adding $\sum_{l=2}^{L} y_j^l$ to both sides of equation (25), and then applying equations (15) and (20), we have

$$b_j^1 y_j^1 = \bar{b}_j^1 y_j^1 \quad j = I+1,\ldots,N, \tag{26}$$

which implies

$$b_j^1 = \bar{b}_j^1 \quad j = I+1,\ldots,N. \tag{27}$$

Finally, from equations (21) and (27), we have

$$b_j^l = \bar{b}_j^l \quad l = 2,\ldots,L, j = I+1,\ldots,N. \tag{28}$$

Thus, we have proved

$$b_j = \bar{b}_j \quad j = I+1,\ldots,N, \tag{29}$$

which means $b = \bar{b}$. Thus, equations (18) and (29) mean that $e = \bar{e}$. Consequently, the inverse of the stationary message correspondence, $(\mu)^{-1}$, is a single-valued mapping from $\mu(\bar{E}^{cq})$ to $\bar{E}^{cq}$. Q.E.D.

Proof of Lemma 4. Since a MCP equilibrium is a Walrasian equilibrium for convex production economies, the existence of a MCP equilibrium can be obtained by applying the existence theorem of competitive equilibrium by noting that $\mathcal{Y}_i$ is closed and convex, $0 \in \mathcal{Y}_i$, $(-\mathbb{R}_+^L) \in \mathcal{Y}_i$ and $\mathcal{Y}_i \cap (-\mathcal{Y}_i) \subset \{0\}$. To show the MCP equilibrium is unique, we first need to derive the supply and demand functions of agents.

Produce (for $i = I+1, \ldots, N$) chooses his/her production plan so as to maximize profit within $\mathcal{Y}(b_i)$. Thus, he/she solves the following profit maximizing problem:

$$\max_{y_i} p \cdot y_i$$

subject to

$$b_i^1 y_i^1 + \sum_{l=2}^{L} (y_i^l + \frac{b_i^l}{2}(y_i^l)^2) = 0 \tag{30}$$

and

$$-\frac{1}{b_i^l} \leqq y_i^l \leqq 0 \text{ for all } l \neq 1.$$

An interior solution y must satisfy the following first-order conditions:

$$\begin{aligned} p^1 &= \lambda_i b_i^1 && (31) \\ p^l &= \lambda_i (1 + b_i^l y_i^l), \quad l = 2, \ldots, L, && (32) \end{aligned}$$

where λ_i is a Lagrange multiplier. From (31) and (32) and the assumption $y_i^l \leqq 0$, we can obtain the supply functions

$$y_i^l(p) = \begin{cases} \frac{b_i^1 p^l}{b_i^l p^1} - \frac{1}{b_i^l} & \text{if } \frac{p^l}{p^1} < \frac{1}{b_i^1} \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

for $l = 2, \ldots, L$, and thus, by (30),

$$y_i^1(p) = -\frac{1}{b_i^1} \sum_{l=2}^{L} [y_i^l(p) + \frac{b_i^l}{2}(y_i^l(p))^2]. \tag{34}$$

It may be remarked that $y_i^1(p) \geqq 0$ for all $p \in \Delta^{L-1}$. Indeed, for $l = 2, \ldots, L$, if $\frac{p^l}{p^1} < \frac{1}{b_i^1}$, then $y_i^l(p) = \frac{b_i^1 p^l}{b_i^l p^1} - \frac{1}{b_i^l}$ and thus

$$\begin{aligned} y_i^l(p) + \frac{b_i^l}{2}(y_i^l(p))^2 &= \left[1 + \frac{b_i^l}{2} y_i^l(p)\right] y_i^l(p) \\ &= \left[1 + \frac{b_i^l}{2}\left(\frac{b_i^1 p^l}{b_i^l p^1} - \frac{1}{b_i^l}\right)\right]\left[\frac{b_i^1 p^l}{b_i^l p^1} - \frac{1}{b_i^l}\right] \\ &= \frac{1}{2b_i^l}\left(1 + \frac{b_i^1 p^l}{p^1}\right)\left(\frac{b_i^1 p^l}{p^1} - 1\right) \\ &= \frac{1}{2b_i^l}\left[\left(\frac{b_i^1 p^l}{p^1}\right)^2 - 1\right] < 0. && (35) \end{aligned}$$

If $\frac{p^l}{p^1} \geqq \frac{1}{b_i^1}$, then $y_i^l(p) = 0$ by (33). Thus $y_i^l(p) + \frac{b_i^l}{2}(y_i^l(p))^2 \leqq 0$ for all $p \in \Delta^{L-1}$, and therefore by (34), we have $y_i^1(p) \geqq 0$ for all $p \in \Delta^{L-1}$.

Consumer (for $i = 1, \ldots, I$) chooses his/her consumption so as to maximize his/her utility subject to his/her budget constraint. Since all utility functions are Cobb-Douglas, it is well known that the net demand functions are given by

$$z_i^l(p) = \frac{a_i^l}{p_i^l}[p \cdot \bar{w}_i + \sum_{j=I+1}^{N} \theta_{ij} p \cdot y_j(p)] - \bar{w}_i^l. \tag{36}$$

Define the aggregate net excess demand function by

$$\hat{z} = \sum_{i=1}^{I} z_i(p) - \sum_{i=I+1}^{N} y_i(p). \tag{37}$$

Notice that, since every consumer's budget constraint holds with equality, and the demand and supply functions are clearly continuous, the aggregate excess demand function $\hat{z}(p)$ is continuous and satisfies Walras' Law, i.e., $p \cdot \hat{z}(p) = 0$ for all $p \in \Delta^{L-1}$. Thus, the existence of a MCP/comeptitive equilibrium can be guaranteed by applying an existence theorem in Varian (1992)), i.e., there exists some $p \in \Delta^{L-1}$ such that $\hat{z}(p) \leqq 0$, which means an equilibrium exists for every economy $e \in \bar{E}^{cq}$.

Now we show that every economy $e \in \bar{E}^{cq}$ has a unique equilibrium, and for this, it suffices to show that all goods are gross substitutes at any price $p \in \Delta^{L-1}$, i.e., an increase in price, k, brings about an increase in the excess demand for good l. When $\hat{z}$ is differentiable, the gross substitutes condition becomes $\frac{\partial \hat{z}^l(p)}{\partial p^k} > 0$ for $l \neq k$.

For each $i = I+1, \ldots, N$, from (33), if $\frac{p^l}{p^1} < \frac{1}{b_i^1}$ we have

$$\frac{\partial y_i^l(p)}{\partial p^l} = \frac{b_i^1}{b_i^l p^1} > 0 \quad l = 2, \ldots, L, \tag{38}$$

$$\frac{\partial y_i^l(p)}{\partial p^k} = 0 \quad k \neq l, k \neq 1, l \neq 1, \tag{39}$$

$$\frac{\partial y_i^l(p)}{\partial p^1} = -\frac{b_i^1 p^l}{b_i^l (p^1)^2} < 0 \quad l = 2, \ldots, L, \tag{40}$$

and from (34), we have

$$\frac{\partial y_i^1(p)}{\partial p^1} = -\frac{1}{b_i^1} \sum_{l=2}^{L} [1 + b_i^l y_i^l] \frac{\partial y_i^l}{\partial p^1} > 0, \tag{41}$$

and

$$\frac{\partial y_i^1(p)}{\partial p^l} = -\frac{1}{b_i^1}[1 + b_i^l y_i^l] \frac{\partial y_i^l}{\partial p^l} < 0. \tag{42}$$

When $\frac{p^l}{p^1} \geqq \frac{1}{b_i^1}$, $y_i^l(p)$ are constant functions for $l = 2, \ldots, L$. Thus, $y_i^l(p)$ is a nonincreasing function in p^k for any $l \neq k$ and any $p \in \Delta^{L-1}$.

Note that, by Hotelling's Lemma (cf. Varian (1992, p. 43), $\frac{\partial p \cdot y_j(p)}{\partial p^k} = y_j^k(p)$, and $-\frac{1}{b_i^k} > -\frac{\bar{w}_i^k}{J}$ by the assumption that $b_i^k > \frac{J}{\bar{w}_i^k}$. Then, for each $i = 1, \ldots, I$, from (36), we have

$$\begin{aligned} \frac{\partial z_i^l(p)}{\partial p^k} &= \frac{a_i^l}{p^l}\left[\bar{w}_i^k + \sum_{j=I+1}^{N} \theta_{ij} y_j^k(p)\right] \\ &\geqq \frac{a_i^l}{p^l}\left[\bar{w}_i^k + \sum_{j=I+1}^{N} \theta_{ij}\left(-\frac{1}{b_i^k}\right)\right] \\ &> \frac{a_i^l}{p^l}\left[\bar{w}_i^k + \sum_{j=I+1}^{N}\left(-\frac{\bar{w}_i^k}{J}\right)\right] = 0 \end{aligned} \tag{43}$$

for $l \neq k, k \neq 1$, and

$$\frac{\partial z_i^l(p)}{\partial p^1} = \frac{a_i^l}{p^l}\left[\bar{w}_i^1 + \sum_{j=I+1}^{N} \theta_{ij} y_j^1(p)\right] > 0 \tag{44}$$

by noting that $y_j^1(p) \geqq 0$. Thus, the net demand function $z_i^l(p)$ is an increasing function and the supply function $y_i^l(p)$ is a non-increasing function in price $k \neq l$ for every $p \in \Delta^{L-1}$. Therefore, an increase in price, k, brings about an increase in the excess demand for good l, and thus all goods are gross substitutes. Hence, the equilibrium must be unique (cf. Varian (1992)). Q.E.D.

Proof of Lemma 5. By Lemma 4, we know $\mu_{cq} = (p, z, y)$ is a (single-valued) function. Also, from (33), (34) and (36), we know that the demand function $z(p; c)$ and supply function $y(p; c)$ are continuous in p and $c := (a, b)$. So we only need to show the price vector p is a continuous function on $\bar{E}^{cq}$.

Let $\{e(k)\}$ be a sequence in $\bar{E}^{cq}$ and $e(k) \to e \in \bar{E}^{cq}$. Since any economy in $\bar{E}^{cq}$ is fully specified by the parameter vector c, $e(k) \to e$ implies $c(k) \to c$.

Let $\mu_{cq} = (p, z(p; c), y(p; c))$ and $\mu_{cq}(k) = (p(k), z(p(k); c(k)), y(p(k); c(k)))$. Then we have, $\hat{z}(p; c) = 0$ and $\hat{z}(p(k); c(k)) = 0$, i.e.,

$$\sum_{i=1}^{I} z_i(p; c) = \sum_{j=I+1}^{N} y_j(p; c)$$

and

$$\sum_{i=1}^{I} z_i(p(k); c(k)) = \sum_{j=I+1}^{N} y_j(p(k); c(k)).$$

Since the sequence $\{p(k)\}$ is contained in the compact set $\bar{\Delta}^{L-1} = \{p \in \mathbb{R}_+^L : \sum_{l=1}^L p^l = 1\}$, there exists a subsequence $\{p(k_t)\}$ that converges, say, $\bar{p} \in \bar{\Delta}^{L-1}$ and $\hat{z}(p(k_t); c(k_t)) = 0$. Since $z_i(p(k); c(k))$ and $z_i(p(k); c(k))$ are continuous in c, $\hat{z}(p; c)$ is continuous in c and thus we have $\hat{z}(p(k_t); c(k_t)) \to \hat{z}(\bar{p}, c)$ as $k_t \to \infty$ and $c(k_t) \to c$. However, since every $e \in \bar{E}^{cq}$ has the unique

MCP equilibrium price p that is completely determined by $\hat{z}(p;c) = 0$, we must have $\bar{p} = p$. Q.E.D.

Proof of Lemma 6. By Lemma 4, we know that μ_{cq} is the restriction of μ_{mc} to $\bar{E}^{cq}$. We first prove that the inverse of μ_{cq}, $(\mu_{cq})^{-1}$ is a function.

Let $m \in \mu_{cq}(\bar{E}^{cq})$ and let $e, e' \in (\mu_{cq})^{-1}(m)$. Then $m \in \mu_{cq}(e) \cap \mu_{cq}(e') = \mu_{mc}(e) \cap \mu_{c}(e') = \mu_{mc}(e'_i, e_{-i}) \cap \mu_{mc}(e_i, e'_{-i})$ for all $i = 1, \ldots, N$ by Remark 7. Let $(z, y) = h_{cd} \in W(\bar{E}^{cq})$ be the MCP outcome function. Since u_i is monotonically increasing, we know (z, y) is Pareto efficient by the First Theorem of Welfare Economics. Then, the allocation process $\langle M_c, \mu_{cq}, h_{cq} \rangle$ is privacy-preserving and non-wasteful over $\bar{E}^{cq}$ with respect to $\mathcal{P}$. By Lemma 3, $e = e'$ and thus $(\mu_{cq})^{-1}$ is a function. Also, by Lemma 5, μ_{cq} is a continuous function. Therefore, μ_{cq} is a continuous one-to-one function on $\bar{E}^{cq}$.

Since every e is fully characterized by $(a, b) \in \mathbb{R}^{LI+LJ}_{++}$ with $\sum_{l=1}^{L} a_i^l = 1$ for $i = 1, \ldots, I$, $\bar{E}^{cq}$ is homeomorphic to the finite-dimensional Euclidean space $\mathbb{R}^{(L-1)I+LJ}$. Thus, it must be homeomorphic to any open ball centered on any of its points, and locally compact. It follows that for any $e \in \bar{E}^{cq}$, we can find a neighborhood $N(e)$ of e and a compact set $\bar{N}(e) \subset N(e)$ with a nonempty interior point. Since μ_{cd} is a continuous one-to-one function on $N(e)$, μ_{cd} is a continuous one-to-one function from the compact space $\bar{N}(e)$ onto an Euclidean (and hence Hausdorff topological) space $\mu_{cd}(\bar{N}(e))$. It follows that the restriction of μ_{cd} to $N(e)$ is a homeomorphic imbedding on $\bar{N}(e)$ by Theorem 5.8 in Kelley (1955, p. 141). Choose an open ball $\mathring{N}(e) \subset \bar{N}(e)$. Then $\mathring{N}(e)$ and $\mu_{cd}(\mathring{N}(e))$ are homeomorphic by a homeomorphism $\mu_{cd}|\mathring{N}(e) : \mathring{N}(e) \to \mu_{cd}(\mathring{N}(e))$. This, together with the fact that $\bar{E}^{cq}$ is homeomorphic to its open ball $\mathring{N}(e)$, implies that $\bar{E}^{cq}$ is homeomorphic to $\mu_{cd}(\mathring{N}(e)) \subset M_{mc}$, implying that $\mu_{cd}(E^c)$ can be homeomorphically imbedded in $\mu_{mc}(E^*)$.

Finally, by Remark 2, the MCP message space M_{mc} is contained within an Euclidean space of dimension $(L-1)I + LJ$. This necessarily implies that M_{mc} and thus $\mu_{mc}(\bar{E}^{cq})$ is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$ because his restriction $\mu_{cq}(\bar{E}^{cq})$ is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$, and consequently, $\mu_c(E^c)$ is homeomorphic to $\bar{E}^{cq}$. Q.E.D.

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