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GENERALIZED QUASI-VARIATIONAL-LIKE INEQUALITY PROBLEM

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This paper gives some very general results on the generalized quasi-variational-like inequality problem. Since the problem includes all the existing extensions of the classical variational inequality problem as special cases, our existence theorems extend the previous results in the literature by relaxing both continuity and concavity of the functional. The approach adopted in this paper is based on continuous selection-type arguments and thus is quite different from the Berge Maximum Theorem or Hahn-Banach Theorem approach used in the literature.

1. Introduction. In recent years, various extensions of the classical variational inequality problem have been proposed and studied such as those in Mosco [18], Aubin [3], Chan and Pang [6], Fang and Peterson [12], Parida and Sen [19], Aubin and Ekeland [4], Shih and Tan [21], Zhou and Chen [30], Ding and Tan [10], Yao [29], Chang and Zhang [7], and Tian and Zhou [25, 26]. Motivations for this come from the fact that the variational inequality problem and its various extensions have applications to problems in mathematical programming, partial differential equation theory, game theory, impulsive control, and economics [2, 3, 9, 14, 18, 23, 24]. In these extensions of the variational inequality problem, a functional $(x, y, z) \mapsto \phi(x, y, z)$ is involved. However, in studies of the existence of solutions to the various generalized variational inequality problems, some (e.g., Chan and Pang [6], Fang and Peterson [12], Parida and Sen [19], and Yao [29]) have used the Berge Maximum Theorem to prove the existence of a solution and restricted their discussions to the finite dimensional Euclidean space and continuous functions. But in many cases, functions are not continuous, and topological spaces are not finite. For instance, in the study of free boundary value problems for partial differential equations, the problems reduce to variational inequality problems over infinite dimensional spaces (see, e.g., [14]). Also, to use the Berge Maximum Theorem, one needs to assume that ϕ is continuous. This is clearly a very strong assumption. On the other hand, some (e.g., [3, p. 281], or [4, p. 349], Zhou and Chen [30], Tian and Zhou [25]) have used the Hahn-Banach theorem to prove the existence of solution to quasi-variational inequality problems. This approach does not need to assume that ϕ is continuous and topological spaces are finite dimensional, but, it needs to assume that ϕ is (γ-diagonally) concave in y, while in problems of variational (minimax) inequalities only (0-diagonal) quasi-concavity and lower semi-continuity are needed to prove the existence. Indeed, (0-diagonal) concavity is a crucial assumption in using the Hahn-Banach theorem since it requires that the sum of the functions satisfy the (quasi-) concavity in order to apply the Ky-Fan minimax inequality.

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In this paper we use a quite different approach to show the existence of a solution to the generalized quasi-variational-like inequality problem. The approach we adopt is based on continuous selection-type arguments and is developed in Tian and Zhou [26] to study the quasi-variational inequality problem. This approach enables us to generalize the existing results by relaxing both the (0-diagonal) concavity and continuity conditions. Since the generalized quasi-variational-like inequality problem includes the classical variational inequality, generalized variational inequality, generalized variational-like inequality, quasi-variational inequality, and generalized quasi-variational inequality problems as special cases, our results also prove the existence of solutions to these problems by relaxing both the (0-diagonal) concavity and continuity conditions.

2. Notation and definitions. Let X and Y be two topological spaces, and let 2^Y be the collection of all subsets of Y. A correspondence $F\colon X \to 2^Y$ is said to be *upper semi-continuous* (in short, u.s.c.) if the short $\{x \in X\colon F(x) \subset V\}$ is open in X for every open subset V of Y. A correspondence $F\colon X \to 2^Y$ is said to be *lower semi-continuous* (in short, l.s.c.) if the set $\{x \in X\colon F(x) \cap V \neq \varnothing\}$ is open in X for every open subset V of Y. A correspondence $F\colon X \to 2^Y$ is said to be *continuous* if it is both u.s.c. and l.s.c. A correspondence $F\colon X \to 2^Y$ is said to have *open lower sections* if the set $F^{-1}(y) = \{x \in S\colon y \in F(x)\}$ is open in X for every $y \in Y$. A correspondence $F\colon X \to 2^Y$ is said to have *open upper sections* if for every $x \in X$, $F(x)$ is open in Y. A correspondence $F\colon X \to 2^Y$ is said to be *closed* if the correspondence has a closed graph, i.e., if the set $\{(x, y) \in X \times Y\colon y \in F(x)\}$ is closed in $X \times Y$. A correspondence $F\colon X \to 2^Y$ is said to have an *open graph* if the set $\{(x, y) \in X \times Y\colon y \in F(x)\}$ is open in $X \times Y$. A set X is said to be *contractible* if there is a point $x_0 \in X$ and a continuous function $g\colon X \times [0, 1] \to X$ such that $g(x, 0) = x$ and $g(x, 1) = x_0$ for all $x \in X$. Note that any convex set is contractible. A set X in a topological vector space is said to be *finite dimensional* if the number of linearly independent vectors (points) in the set is finite.

A subset K in a topological space X is said to be *solid* if its interior set int $K \neq \varnothing$. Denote by co B and $\bar{B}$ the convex hull and closure of the set B, respectively.

REMARK 1. It is known that if a correspondence F has an open graph then F has open upper and lower sections, and the converse statement may not be true (cf. [5, pp. 265–266]). Also, Yannelis and Prabhakar [28, p. 237] showed that, if F has open lower sections, then it is l.s.c., and the converse statement may not be true.

REMARK 2. There has been some blurring in the literature of the distinction between closed correspondences and upper semi-continuous correspondences. (Many people use the definition of closed correspondences as the definition of upper semi-continuous correspondences.) In general, a correspondence may be closed without being upper semi-continuous, and vice versa. For instance, define $F\colon \mathbb{R} \to 2^{\mathbb{R}}$ via

$$F(x) = \begin{cases} \left\{\dfrac{1}{x}\right\} & \text{if } x \neq 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

Then F is closed but not upper semi-continuous. Define $G\colon \mathbb{R} \to 2^{\mathbb{R}}$ via $G(x) = (0, 1)$. Then G is upper semi-continuous but not closed. Nevertheless, the following relationships exist under some additional conditions. For a correspondence $F\colon X \to 2^Y$, if Y is compact and F is closed, then F is upper semi-continuous (cf. Aubin and

Ekeland [4, p. 111]). And, if F is upper semi-continuous and closed-valued, then it is closed (cf. Aubin and Ekeland [4, p. 111]).

Let X be a topological space. A function $f\colon X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *lower semi-continuous* (in short, l.s.c.) on X if for each point $x' \in X$, we have

$$\liminf_{x \to x'} f(x) \geqslant f(x'),$$

or equivalently, its epigraph $\operatorname{epi} f \equiv \{(x, a) \in X \times \mathbb{R}\colon f(x) \leqslant a\}$ is a closed subset of $X \times \mathbb{R}$. A function $f\colon X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *upper semi-continuous* (in short, u.s.c.) on X if $-f$ is l.s.c. on X.

Let X be a convex set of a topological vector space E and let $\phi\colon X \times X \to \mathbb{R} \cup \{\pm\infty\}$ be a functional. The functional $(x, y) \mapsto \phi(x, y)$ is said to be *γ-diagonally concave* (in short, γ-DCV) in y (cf. [30]), if for any finite subset $\{y_1, \dots, y_m\} \subset X$ and any $y_\lambda \in \operatorname{co}\{y_1, \dots, y_m\}$ (i.e., $y_\lambda = \sum_{j=1}^m \lambda_j y_j$ for $\lambda_j \geqslant 0$ with $\sum_{j=1}^m \lambda_j = 1$), we have

$$\sum_{j=1}^{m} \lambda_j \phi(y_\lambda, y_j) \leqslant \gamma, \tag{1}$$

where γ is a real number. A functional $(x, y) \mapsto \phi(x, y)$ is said to be *γ-diagonally quasi-concave* (in short, γ-DQCV) in y if for any finite subset $\{y_1, \dots, y_m\} \subset X$ and any $y_\lambda \in \operatorname{co}\{y_1, \dots, y_m\}$,

$$\min_j \phi(y_\lambda, y_j) \leqslant \gamma.$$

A functional $(x, y) \mapsto \phi(x, y)$ is said to be *γ-diagonally (quasi-)convex* (in short, γ-DQCX) in y if $-\phi$ is γ-diagonally (quasi-)concave.

Let X be a set in a Banach space E and let Y be a set in the dual E' of E. We now state the definition of the classical variational inequality problem as well as its various extensions.

Let $f\colon X \to 2^Y$ be a single-valued function. The classical *variational inequality problem* (VIP) is to find a vector $x^* \in X$ such that

$$\langle x - x^*, f(x^*) \rangle \geqslant 0, \qquad \forall x \in X.$$

Let $F\colon X \to 2^Y$ be a correspondence. The *generalized variational inequality problem* (GVIP) (cf. [12]) is to find a vector $x^* \in X$ and a vector $y^* \in F(x^*)$ such that

$$\langle x - x^*, y^* \rangle \geqslant 0, \qquad \forall x \in X.$$

Note that GVIP reduces to VIP by letting F be a single-valued function.

Let $f\colon X \to 2^Y$ be a single-valued function and let $K\colon X \to 2^X$ be a correspondence. The *quasi-variational inequality problem* (QVIP) (cf. [3, 18]) is to find a vector $x^* \in K(x^*)$ such that

$$\langle x - x^*, f(x^*) \rangle \geqslant 0, \qquad \forall x \in K(x^*).$$

Note that QVIP reduces to VIP by letting $K(x) \equiv X$ for all $x \in X$.

Let $K\colon X \to 2^X$ and $F\colon X \to 2^Y$ be two correspondences. The *generalized quasi-variational inequality problem* (GQVIP) (cf. [3, 6, 25]) is to find a vector $x^* \in K(x^*)$

and a vector $y^* \in F(x^*)$ such that

$$\langle x - x^*, y^* \rangle \geqslant 0, \qquad \forall x \in K(x^*).$$

Note that GQVIP reduces to GVIP by letting $K(x) \equiv X$ for all $x \in X$ and reduces to QVIP by letting F be a single-valued function.

Let F: $X \to 2^Y$ be a correspondence. Let θ: $X \times Y \to E$ and π: $X \times X \to E'$ be two single-valued functions. The *generalized variational-like inequality problem* (GVLIP) (cf. [19]) is to find a vector $x^* \in X$ and a vector $y^* \in F(x^*)$ such that

$$\langle \theta(x^*, y^*), \pi(z, x^*) \rangle \geqslant 0, \qquad \forall z \in X.$$

Note that GVLIP reduces to GVIP by letting $\theta(x, y) = y$ and $\pi(x, z) = z - x$.

Let K: $X \to 2^X$ and F: $X \to 2^Y$ be two correspondences. Let θ: $X \times Y \to E$ and π: $X \times X \to E'$ be two single-valued functions. The *generalized quasi-variational-like inequality problem* (GQVLIP) (cf. [29]) is to find a vector $x^* \in K(x^*)$ and a vector $y^* \in F(x^*)$ such that

$$\langle \theta(x^*, y^*), \pi(z, x^*) \rangle \geqslant 0, \qquad \forall z \in K(x^*).$$

Note that GQVLIP which is introduced by Yao [29] contains all the above "variational inequality" problems as special cases.

Before proceeding to the main theorems, we state some technical lemmas which are needed in the discussions below. Lemmas 1 and 2 are due to Yannelis [27, p. 103], Lemma 3 is due to Michael [17, Theorem 3.1‴], and Lemmas 4–6 are due to Yannelis and Prabhakar [28].

LEMMA 1. *Let X and Y be two topological spaces, and let G: $X \to 2^Y$, K: $X \to 2^Y$ be correspondences such that*

(i) *G has an open graph,*

(ii) *K is l.s.c.*

Then the correspondence F: $X \to 2^Y$ defined by $F(x) = G(x) \cap K(x)$ is l.s.c.

LEMMA 2. *Let X be a topological space and Y a convex set of a topological vector space, and let G: $X \to 2^Y$ have an open graph. Then the correspondence F: $X \to 2^Y$ defined by $F(x) = \operatorname{co} G(x)$ has an open graph.*

LEMMA 3. *Let X be a perfectly normal T_1-topological space and Y be a separable Banach space. Let $\mathscr{D}(Y)$ be the set of all nonempty and convex subsets of Y which are either finite-dimensional or closed or solid. Suppose F: $X \to \mathscr{D}(Y)$ is a lower semicontinuous correspondence. Then there exists a continuous function: f: $X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.*

LEMMA 4. *Let X and Y be two topological spaces, and let G: $X \to 2^Y$ and K: $X \to 2^Y$ be correspondences having an open lower sections. Then the correspondence θ: $X \to 2^Y$ defined by, for all $x \in X$, $\theta(x) = G(x) \cap K(x)$, has open lower sections.*

LEMMA 5. *Let X be a topological space and let Y be a convex set of a topological vector space. Suppose a correspondence G: $X \to 2^Y$ has open lower sections. Then the correspondence F: $X \to 2^Y$ defined by $F(x) = \operatorname{co} G(x)$ for all $x \in X$ has open lower sections.*

LEMMA 6. *Let X be a paracompact Hausdorff space and Y be a topological vector space. Suppose F: $X \to 2^Y$ is a correspondence with nonempty convex values and has* *open lower sections. Then there exists a continuous function $f: X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.*

3. Existence theorems for general functions. The following theorems are important in establishing existence results for the generalized quasi-variational-like inequality problem and extend the previous existence results in the literature by relaxing both the continuity and (γ-diagonally) concavity conditions. For instance, these theorems extend the results of Chan and Pang [6, Theorem 3.1], Parida and Sen [19, Theorem 1], and Yao [29, Theorem 3.1] by relaxing the continuity of ϕ and the finite dimensionality of topological spaces and extend the results of Aubin [3, Theorem 9.3.2] and Aubin and Ekeland [4, Corollary 6.4.22], Zhou and Chen [30, Theorem 3.1] by relaxing the (γ-diagonal) concavity condition. These theorems also include the results in Hartman and Stampacchia [14], Saigal [20], and Tian and Zhou [26, Theorem 2] as special cases.

For simplicity, we state the following theorem and some other theorems below with the weak topology even though they hold for any Hausdorff vector space topology τ provided it is weaker than the norm topology.

THEOREM 1. *Let X and Y be two nonempty weakly compact convex subsets in two separable Banach spaces, and let X_w and Y_w denote the same sets X and Y endowed with the weak topology, respectively. Suppose that*

(i) *$K: X_w \to 2^X$ is a continuous correspondence with nonempty closed and convex values such that $K(x)$ is either finite dimensional or solid for each $x \in X_w$;*

(ii) *$F: X_w \to 2^Y$ is an upper semi-continuous correspondence with nonempty closed and convex values;*

(iii) *$\phi: X_w \times Y_w \times X \to \mathbb{R} \cup \{\pm\infty\}$ is l.s.c. and is γ-diagonal quasi-concave in z.*

Then there exist $x^* \in K(x^*)$ and $y^* \in F(x^*)$ such that $\sup_{z \in K(x^*)} \phi(x^*, y^*, z) \leqslant \gamma$.

PROOF. Define a correspondence $P: X_w \times Y_w \to 2^X$ by, for each $(x, y) \in X_w \times Y_w$, $P(x, y) = \{z \in X: \phi(x, y, z) > \gamma\}$. Thus, proving the theorem is equivalent to proving that there exist $x^* \in K(x^*)$ and $y^* \in F(x^*)$ such that $K(x^*) \cap P(x^*, y^*) = \varnothing$.

Since ϕ is l.s.c. in $X_w \times Y_w \times X$, the set $\{(x, y, z) \in X_w \times Y_w \times Z: z \in P(x, y)\} = \{(x, y, z) \in X_w \times Y_w \times Z: \phi(x, y, z) > \gamma\}$ is open and thus P has an open graph in $X_w \times Y_w$. Also, by the γ-diagonal quasi-concavity, $x \notin \operatorname{co} P(x, y)$ for all $x \in X$ and $y \in Y$. To see this, suppose, by way of contradiction, that there exist $x_\lambda \in X$ and $y \in Y$ such that $x_\lambda \in \operatorname{co} P(x_\lambda, y)$. Then there exist finite points, $x_1, \dots, x_m$ in X, and $\lambda_j \geqslant 0$ and $\sum_{j=1}^m \lambda_j = 1$ such that $x_\lambda = \sum_{j=1}^m \lambda_j x_j$ and $x_i \in P(x_\lambda, y)$ for all $i = 1, \dots, m$. That is, $\phi(x_\lambda, y, x_i) > \gamma$ for all i, which contradicts the hypothesis that ϕ is γ-DQCV in z for each y.

Define another correspondence $G: X_w \times Y_w \to 2^X$ by $G(x, y) = K(x) \cap \operatorname{co} P(x, y)$. Let $U_w = \{(x, y) \in X_w \times Y_w: G(x, y) \neq \varnothing\}$. If $U_w = \varnothing$, this implies $K(x) \cap P(x, y) = \varnothing$ for every $x \times y \in X_w \times Y_w$, and so to get the result, we need only to show that $(K(x), F(x))$ has a fixed point (x^*, y^*). But this is guaranteed by the Takutani-Himmelberg fixed point theorem (cf. [15]) and by noting the fact that a correspondence is u.s.c from the weak topology to the weak topology if it is u.s.c. from the weak topology to the norm topology (this is because every weak open set is a norm open set). Now assume $U_w \neq \varnothing$. Since P has an open graph in $X_w \times Y_w$, by Lemma 2, $\operatorname{co} P$ has an open graph in $X_w \times Y_w$. Then, by Lemma 1, G is l.s.c. in $X_w \times Y_w$ and thus the correspondence $G|U_w: U_w \to 2^X$ is l.s.c. in U_w. Also, for all $(x, y) \in U_w$, $G(x, y)$ is nonempty and convex. Now we claim $G(x, y)$ is either solid or finite dimensional. This is clearly true if $K(x)$ is finite dimensional. So we only need to show that $G(x, y)$ has an interior point if $K(x)$ contains an interior point z_0. To see this, let $(x, y) \in U_w$ and $z \in G(x, y) = K(x) \cap \operatorname{co} P(x, y)$. Since $K(x)$ is convex, $z_\lambda = z + \lambda(z_0 - z)$ is an interior point for any $0 < \lambda < 1$. Thus any neighborhood $\mathscr{N}_1(z)$ of z contains an interior point of $K(x)$. Since $\operatorname{co} P(x, y)$ has a relative open graph, $\operatorname{co} P(x, y)$ is open relative to X that contains $K(x)$ and $P(x, y)$. There should be a neighborhood $\mathscr{N}_2(z)$ such that $\mathscr{N}_2(z) \cap X \subset \operatorname{co} P(x, y)$. So $\mathscr{N}_2(z)$ contains an interior point of $G(x, y) = K(x) \cap \operatorname{co} P(x, y)$, i.e., $G(x, y)$ is solid.

Next we show that Z_w is a perfectly normal T_1-topological vector space if it is a weakly compact subset in a separable Banach space E. It is clear that Z_w is a normal T_1-topological space, since the dual E^* of a Banach space E separates points in E, and Z_w is weakly compact. To show that Z_w is perfectly normal, we have to show that any closed set C of Z_w can be written as an intersection of countable open sets. By the assumption that E is separable, E_w is also separable, since the norm-convergence implies the weak-convergence. Let Ω be a countable dense set in E_w. For each closed set C in Z_w, the set $\Omega \setminus C$ is also countable and dense in $E_w \setminus C$. For each $x \in (\Omega \setminus C)$, there are neighborhoods $\mathscr{N}(x)$ and $\mathscr{N}_x(C)$ such that $\mathscr{N}(x) \cap \mathscr{N}_x(C) = \varnothing$. It is clear that $C = \bigcap_{x \in (\Omega \setminus C)} \mathscr{N}_x(C)$, since$(\bigcap_{x \in (\Omega \setminus C)} \mathscr{N}_x(C)) \cap (\bigcup_{x \in (\Omega \setminus C)} \mathscr{N}(x)) = \varnothing$ and $(X_w \setminus C) \subset \bigcup_{x \in (\Omega \setminus C)} \mathscr{N}(x)$.

Hence, we can apply Lemma 3 to assure that there exists a continuous function $g: U_w \to X$ such that $g(x, y) \in G(x, y)$ for all $(x, y) \in U_w$. Note that U_w is relatively open since G is l.s.c. Define the correspondence $M: X_w \times Y_w \to 2^{X \times Y}$ by

$$M(x, y) = \begin{cases} (g(x, y), F(x)) & \text{if } (x, y) \in U_w, \\ (K(x), F(x)) & \text{otherwise}. \end{cases} \tag{2}$$

Then $M: X_w \times Y_w \to 2^{X \times Y}$ is u.s.c from the weak topology to the norm topology and thus $M: X_w \times Y_w \to 2^{X_w \times Y_w}$ is u.s.c. from the weak topology to the weak topology. And, for all $(x, y) \in X \times Y$, $M(x, y)$ is nonempty, closed, and convex. Hence, by the Takutani-Himmelberg fixed point theorem, there exists a point $(x^*, y^*) \in X \times Y$ such that $(x^*, y^*) \in M(x^*, y^*)$. Note that, if $(x^*, y^*) \in U_w$, then $x^* = g(x^*, y^*) \in G(x^*, y^*) \subset \operatorname{co} P(x^*, y^*)$, a contradiction to $x^* \notin \operatorname{co} P(x^*, y^*)$. Hence, $(x^*, y^*) \notin U_w$ and thus $x^* \in K(x^*)$, $y^* \in F(x^*)$, and $K(x^*) \cap \operatorname{co} P(x^*, y^*) = \varnothing$ which implies $K(x^*) \cap P(x^*, y^*) = \varnothing$. □

In Theorem 1, we need to assume that X is a subset of a separable Banach space and need to use a topology which is weaker than the norm topology. The following theorem relaxes these assumptions. Note that in Theorem 2 below the conditions on ϕ are weaker than those in Theorem 1, but we need to strengthen K to have open lower sections. However, when K is a constant correspondence so that $K(x) \equiv X$ for all $x \in X$, the conditions in Theorem 2 are strictly weaker than those in Theorem 1.

THEOREM 2. *Let X and Y be two nonempty, compact, convex, and metrizable sets in two locally convex Hausdorff topological vector spaces, respectively. Suppose that*

(i) $K: X \to 2^X$ *is a nonempty closed convex valued upper semi-continuous correspondence which has open lower sections;*

(ii) $F: X \to 2^Y$ *is a nonempty closed contractible valued upper semi-continuous correspondence;*

(iii) $\phi: X \times Y \times X \to \mathbb{R} \cup \{\pm\infty\}$ *is l.s.c. in x and y and is γ-diagonally quasi-concave in z.*

Then there exists $x^* \in K(x^*)$ and $y^* \in F(x^*)$ such that $\sup_{z \in K(x^*)} \phi(x^*, y^*, z) \leqslant \gamma$.

PROOF. The proof of this theorem is very similar to that of Theorem 1. Define a correspondence $P: X \times Y \to 2^X$ as before. Again we only need to show that there exist $x^* \in K(x^*)$ and $y^* \in F(x^*)$ such that $K(x^*) \cap P(x^*, y^*) = \varnothing$. Since ϕ is l.s.c. in x and y, then for each $(x, y) \in X \times Y$, $P^{-1}(z) = \{(x, y) \in X \times Y: \phi(x, y, z) > \gamma\}$ is open. Thus P has open lower sections. Also, $x \notin \mathrm{co}\, P(x, y)$ for all $x \in X$ by γ-DQCV condition.

Also define the correspondence $G: X \times Y \to 2^X$ and U as before. Since K and P have open lower sections in $X \times Y$, so they have open lower sections in U. Then, by Lemma 5, co P has open lower sections in U. Hence, by Lemma 4, the correspondence $G|U: U \to 2^X$ has open lower sections in U and for all $(x, y) \in U$, $G(x, y)$ is nonempty and convex. Also, since X is a metrizable space, it is paracompact (cf. Michael [17, p. 831]). Hence, by Lemma 6, there exists a continuous function $g: U \to X$ such that $g(x, y) \in G(x, y)$ for all $(x, y) \in U$. Note that, since G has open lower sections and thus is l.s.c. (cf. Remark 1), U is open. Define the correspondence $M: X \times Y \to 2^{X \times Y}$ by

$$M(x, y) = \begin{cases} (g(x, y), F(x)) & \text{if } (x, y) \in U, \\ (K(x), F(x)) & \text{otherwise}. \end{cases} \tag{3}$$

The remaining arguments are as in the proof of Theorem 1 except for using the Eilenberg-Montgomery fixed point theorem [11] instead of Takutani-Himmelberg fixed point theorem. □

REMARK 3. When $K(x) \equiv X$ for all $x \in X$, Theorem 2 (and thus Theorem 1) can be used to establish the existence of a solution to the GVLIP which significantly generalizes Theorem 1 of Parida and Sen [19] in three directions: (1) Their theorem assumes that $\phi(x, y, z)$ is continuous while our theorem only needs to assume that $\phi(x, y, z)$ is lower semi-continuous in x and y. (2) Their theorem assumes that $\phi(x, y, z) \geqslant 0$ for all $x \in X$ and ϕ is quasi-concave while our theorem only needs to assume that ϕ is 0-diagonally quasi-concave in z. (3) Their theorem assumes that the topological spaces are finite dimensional while our theorem allows the topological spaces to be infinite dimensional.

REMARK 4. If F is convex-valued, it is sufficient to use the Takutani-Himmelberg fixed point theorem instead of the more general Eilenberg-Montgomery fixed point theorem in the proofs of Theorem 2.

REMARK 5. In the case where Y is finite dimensional, the compactness of Y can be relaxed. Indeed, since $F(X)$ is compact, $H \equiv \mathrm{co}\, F(X)$ is also compact if Y is finite dimensional.

REMARK 6. The lower semi-continuity of ϕ and openness of lower sections of K in Theorem 2 can be further weakened using the transfer continuity method which is introduced in Tian [24] and Zhou and Tian [31].

A slight generalization of Theorem 2 can be obtained by relaxing the closedness of K.

PROPOSITION 1. *Let X and Y be two nonempty, compact, convex, and metrizable sets in two locally convex Hausdorff topological vector spaces, respectively. Suppose that*

(i) *$K: X \to 2^X$ is a correspondence with nonempty convex values and has open lower sections such that $\bar{K}: X \to 2^X$ is u.s.c.;*

(ii) *$F: X \to 2^Y$ is a nonempty closed contractible valued upper semi-continuous correspondence;*

(iii) *$\phi: X \times Y \times X \to \mathbb{R} \cup \{\pm\infty\}$ is l.s.c. in x and y and is γ-diagonally quasi-concave in z.*

Then there exist $x^* \in \bar{K}(x^*)$ and $y^* \in F(x^*)$ such that $\sup_{z \in K(x^*)} \phi(x^*, y^*, z) \leqslant \gamma$.

The proof of this proposition is the same as the one of Theorem 2 except replacing $K(x)$ in (3) by $\bar{K}(x)$.

The compactness of X and Y in Theorems 1 and 2 can be relaxed if we make some additional assumptions. In the following theorem and other theorems below, when conditions are connected with Theorem 1, compactness or closedness means weakly compactness or weakly closedness.

THEOREM 3. *Suppose all the conditions in Theorem 1 or Theorem 2 are satisfied except for the compactness of X and Y. If there exist nonempty compact convex sets $Z \subset X$ and $D \subset Y$, and a nonempty subset $C \subset Z$ such that*

(a) $K(C) \subset Z$;

(b) $(K(x) \cap Z, F(x) \cap D) \neq \emptyset$ *for all* $x \in Z$;

(c) *for each* $x \in Z \setminus C$ *there exists* $z \in K(x) \cap Z$ *with* $\phi(x, y, z) > \gamma$ *for all* $y \in F(x)$,

then there exist $x^* \in K(x^*)$ *and* $y^* \in F(x^*)$ *such that* $\sup_{z \in K(x^*)} \phi(x^*, y^*, z) \leqslant \gamma$.

PROOF. Define a correspondence $G: Z \to 2^Z$ by, for each $x \in Z$,

$$G(x) = K(x) \cap Z. \tag{4}$$

Then $G(x)$ is nonempty and convex for all $x \in Z$. Since Z is compact and K is closed by Proposition 3.7 in Aubin and Ekeland [4, p. 111], G is closed and therefore is u.s.c. on Z by Theorem 3.8 in Aubin and Ekeland [4, p. 111]. Similarly, we can show that the correspondence $M: Z \to 2^D$, defined via $M(x) = F(x) \cap D$, is a nonempty closed contractible valued upper semi-continuous correspondence. Also, note that

$$G(x) = \begin{cases} K(x) & \text{if } x \in C, \\ K(x) \cap Z & \text{otherwise}. \end{cases} \tag{5}$$

Then, by Theorem 1 or Theorem 2, there is a vector $x^* \in Z$ and a vector $y^* \in F(x^*) \cap D \subset F(x^*)$ such that $\sup_{z \in G(x^*)} \phi(x^*, y^*, z) \leqslant \gamma$. Now $x^* \in C$, for otherwise Hypothesis (c) would be violated; and hence $G(x^*) = K(x^*)$. Therefore, we have $x^* \in K(x^*)$, $y^* \in F(x^*)$, and $\sup_{z \in K(x^*)} \phi(x^*, y^*, z) \leqslant \gamma$. □

Observe that in the case where X and Y are compact, Assumptions (a)–(b) in Theorem 3 are satisfied by taking $C = Z = X$ and $D = Y$ and thus Theorem 3 reduces to Theorem 1 or to Theorem 2. The assumption that $K(x) \cap Z \neq \emptyset$ for all $x \in Z$ is the necessary and sufficient condition for the correspondence K to have a fixed point when X is not compact (cf. Tian [22]). Assumption (c) is similar to the condition imposed by Allen [1] for variational inequalities with noncompact sets.

REMARK 7. When X is finite dimensional, the conditions in Theorem 3 can be replaced by the following conditions: There exists a nonempty compact set $C \subset X$ and a nonempty compact contractible set $D \subset Y$ such that

(a) $(K(x) \cap Z, F(x) \cap D) \neq \emptyset$ for all $x \in Z$, where $Z = \mathrm{co}\{K(C) \cup C\}$;

(b) for each $x \in Z \setminus C$ there exists $y \in K(x) \cap Z$ such that $\phi(x, y) > \gamma$.

REMARK 8. We can also similarly use the conditions imposed in Chan and Pang [6], Parida and Sen [19], and Yao [29] to relax the compactness of X in Theorems 1 and 2. However, these types of generalizations are needed to strengthen ϕ from (γ-diagonally) quasi-concavity to concavity.

4. Existence results for GQVLIP. By applying Theorems 1 and 2, we immediately have the following existence results for the GQVLIP.

THEOREM 4. *Let X and Y be two nonempty weakly compact convex subsets in two separable Banach spaces, and let X_w and Y_w denote the same sets X and Y endowed with the weak topology, respectively. Suppose that*

(i) K: $X_w \to 2^X$ *is a continuous correspondence with nonempty closed and convex values such that $K(x)$ is either finite dimensional or solid for each $x \in X$;*

(ii) F: $X_w \to 2^Y$ *is an upper semi-continuous correspondence with nonempty closed and convex values;*

(iii) θ: $X \times Y \to E$ *and* π: $X \times X \to E'$ *are two single-valued functions such that* ψ: $X_w \times Y_w \times X \to \mathbb{R}$, *defined by* $\psi(x, y, z) = \langle \theta(x, y), \pi(z, x) \rangle$, *is u.s.c. and is 0-diagonally quasi-convex in z.*

Then there exists a solution to the GQVLIP. That is, there exist $x^* \in K(x^*)$ and $y^* \in F(x^*)$ such that $\langle \theta(x^*, y^*), \pi(z, x^*) \rangle \geqslant 0$ for all $z \in K(x^*)$.

THEOREM 5. *Let X and Y be two nonempty, compact, convex, and metrizable sets in two locally convex Hausdorff topological vector spaces. Suppose that*

(i) K: $X \to 2^X$ *is a nonempty closed convex valued upper semi-continuous correspondence which has open lower sections;*

(ii) F: $X \to 2^Y$ *is a nonempty closed contractible valued upper semi-continuous correspondence;*

(iii) θ: $X \times Y \to E$ *and* π: $X \times X \to E'$ *are two single-valued functions such that* ψ: $X \times Y \times X \to \mathbb{R}$, *defined by* $\psi(x, y, z) = \langle \theta(x, y), \pi(z, x) \rangle$, *is u.s.c. in x and y and is 0-diagonally quasi-convex in z.*

Then there exists a solution to the GQVLIP.

PROOFS OF THEOREMS 4 AND 5. Define $\phi(x, y, z) = -\psi(x, y, z) = -\langle \theta(x, y), \pi(z, x) \rangle$, and apply Theorems 1 and 2, respectively. □

Similarly, by applying Theorem 3, the compactness of X and Y in Theorems 4 and 5 can be relaxed.

THEOREM 6. *Suppose all the conditions in Theorem 4 or Theorem 5 are satisfied except for the compactness of X and Y. If there exist nonempty compact convex sets $Z \subset X$ and $D \subset Y$, and a nonempty subset $C \subset Z$ such that*

(a) *$K(C) \subset Z$;*

(b) *$(K(x) \cap Z, F(x) \cap D) \neq \varnothing$ for all $x \in Z$;*

(c) *for each $x \in Z \setminus C$ there exists $z \in K(x) \cap Z$ with $\langle \theta(x, y), \pi(z, x) \rangle > 0$ for all $y \in F(x)$.*

Then there exists a solution to the GQVLIP.

If $K(x) \equiv X$ for all $x \in X$, the GQVLIP reduces to the GVLIP. Thus, by applying Theorem 5, we have the following corollary which generalizes Theorem 2 of Parida and Sen [19].

COROLLARY 1. *Let X and Y be two nonempty, compact, convex, and metrizable sets in two locally convex Hausdorff topological vector spaces. Suppose that*

(i) F: $X \to 2^Y$ *is a nonempty closed contractible valued upper semi-continuous correspondence;*

(ii) θ: $X \times Y \to E$ *and* π: $X \times X \to E'$ *are two single-valued functions such that* ψ: $X \times Y \times X \to \mathbb{R}$, *defined by* $\psi(x, y, z) = \langle \theta(x, y), \pi(z, x) \rangle$, *in u.s.c. in x and y and is 0-diagonally quasi-convex in z.*

Then there exists a solution to the GVLIP, i.e., there exists a vector $x^* \in X$ and a vector $y^* \in F(x^*)$ such that $\langle \theta(x^*, y^*), \pi(z, x^*) \rangle \geqslant 0$ for all $z \in X$.

5. Applications. In this section we apply our general results to some problems in mathematical programming and equilibrium analysis which are two major areas of the applications. The applications are generalized saddle point problems and equilibrium problems of generalized abstract economies.

5.1. *Generalized saddle point problems.* The saddle point problem is a basic problem in optimization theory. It states that under some conditions a saddle point of the Lagrangian function is equivalent to an optimum of the associated programming problem satisfying a constraint qualification (cf. Mangasarian and Ponstein [16], Yao [29]). In this subsection we give a general result on the existence of solutions to the (generalized) saddle point problems. We first give some notation and definitions.

Let X and Y be two topological spaces, let $K: X \to 2^X$ and $F: X \to 2^Y$ be two correspondences, and let φ be a real function on $X \times Y$.

The *generalized saddle point problem* (GSPP) (cf. [29]) is to find $x^* \in K(x^*)$ and $y^* \in F(x^*)$ such that

$$\varphi(x^*, y) \leqslant \varphi(x^*, y^*) \leqslant \varphi(x, y^*)$$

for all $x \in K(x^*)$ and all $y \in F(x^*)$. Note that GSPP reduces to the conventional SPP by letting $K(x) = X$ and $F(x) = Y$ for all $x \in X$ introduced in [16].

Let X be a set in $\mathbb{R}^n$. A differentiable function f defined on X is said to be *invex* (cf. [8, 13, 29]) if there exists a vector function $\pi: X \times X \to \mathbb{R}^n$ such that

$$f(x) - f(y) \geqslant \langle \nabla f(y), \pi(x, y) \rangle, \qquad \forall x, y \in X.$$

It is clear that every differentiable convex function is invex but the converse statement may not be true.

By using Theorem 4, we have the following existence result for (GSPP).

THEOREM 7. *Let X and Y be a nonempty compact convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $K: X \to 2^X$ and $F: X \to 2^Y$ be two nonempty closed convex valued continuous correspondences. Let $\varphi: X \times Y \to \mathbb{R}$ be a continuous function and $\pi: X \times X \to \mathbb{R}^n$ be a function such that*

(i) *$\psi(x, y, z) \equiv \langle \nabla_x \varphi(x, y), \pi(z, x) \rangle$ is u.s.c. and is 0-diagonally quasi-convex in z for each y;*

(ii) *$\varphi(x, y)$ is invex with respect to π in x for each fixed $y \in Y$, and concave in y for each fixed $x \in X$.*

Then there is a solution to the GSPP. That is, there exist $x^* \in K(x^*)$ and $y^* \in F(x^*)$ such that

$$\varphi(x^*, y) \leqslant \varphi(x^*, y^*) \leqslant \varphi(x, y^*)$$

for all $x \in K(x^*)$ and all $y \in F(x^*)$.

PROOF. Define a correspondence $M: X \to 2^Y$ by, for each $x \in X$, $M(x) = \{y \in F(x): \varphi(x, y) \geqslant \varphi(x, u)\ \forall u \in F(x)\}$. Since φ and F are both continuous, by the Berge Maximum Theorem, M is a nonempty compact valued upper semi-continuous correspondence. Also, since φ is concave in y, M is convex valued. All the conditions of Theorem 4 are satisfied and thus there exist $x^* \in K(x^*)$ and $y^* \in M(x^*)$ such that $\langle \nabla_x \varphi(x^*, y^*), \pi(x, x^*) \rangle \geqslant 0$ for all $x \in K(x^*)$. Then, by the invexity of φ, we have for any $x \in K(x^*)$,

$$\varphi(x, y^*) - \varphi(x^*, y^*) \geqslant \langle \nabla_x \varphi(x^*, y^*), \pi(x, x^*) \rangle \geqslant 0.$$

On the other hand, since $y^* \in M(x^*)$, we have for all $y \in F(x^*)$,

$$\varphi(x^*, y) \leqslant \varphi(x^* y^*).$$

Hence (x^*, y^*) solves the generalized saddle point problem. □

5.2. *Generalized abstract economies.* Another application of our general results is to establish the existence of equilibria in generalized abstract economies. The notion of generalized abstract economies is very general and include the conventional games, abstract economies (the so-called generalized games) introduced by Debreu [9], and the competitive market economic mechanism as special cases. We first introduce the notion of generalized abstract economies.

Let I be the set of agents which is any (finite or infinite) countable set. Each agent i chooses an action (x_i, y_i) from his strategy set $X_i \times Y_i$. Let S_i: $X \to 2^{X_i}$ and F_i: $X \to 2^{Y_i}$ be two feasible constraints, and let u_i: $X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be the payoff function of agent i. Hence $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. Denote $X_{-i} = \prod_{j \in I \setminus \{i\}} X_j$, $S = \prod_{i \in I} S_j$, and $F = \prod_{i \in I} F_j$. Denote by x and x_{-i} an element of X and an element of X_i, respectively.

A *generalized abstract economy* $\Gamma = (X_i \times Y_i, S_i \times F_i, u_i)_{i \in I}$ is defined as a family of ordered triples $(X_i \times Y_i, S_i \times F_i, u_i)$. An *equilibrium* for Γ is a pair $(x^*, y^*) \in X \times Y$ such that $x^* \in S(x^*)$, $y^* \in F(x^*)$, and $u_i(x^*, y^*) \geqslant u(x^*_{-i}, x_i, y^*)$ for all $x_i \in S(x^*_{-i})$ and all $i \in I$.

Note that if F_i is a constant single-valued mapping for all $i \in I$, the generalized abstract economy reduces to the conventional abstract economy $\Gamma = (X_i, S_i, u_i)$. Further, if $S_i(x) = X_i$ and F_i is a constant single-valued mapping for all $i \in I$, the generalized abstract economy reduces to the conventional game $\Gamma = (X_i, u_i)$ and the equilibrium is called a *Nash equilibrium*.

Accordingly, we introduce an aggregate payoff function U: $X \times Y \times X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$U(x, y, z) = \sum_{i \in I} \frac{1}{2^i} \left[u_i(x, y) - u_i(x_{-i}, z_i, y) \right]. \tag{6}$$

We then have the following theorem which generalize the results in [4, 6, 25, 26, 29].

THEOREM 8. *Let X and Y be two nonempty weakly compact convex subsets in two separable Banach spaces, and let X_w and Y_w denote the same sets X and Y endowed with the weak topology, respectively. Suppose that*

(i) *S: $X_w \to 2^X$ is a continuous correspondence with nonempty closed and convex values such that $S(x)$ is either finite dimensional or solid for each $x \in X$;*

(ii) *F: $X_w \to 2^Y$ is an upper semi-continuous correspondence with nonempty closed and convex values;*

(iii) *U: $X_w \times Y_w \times X \to \mathbb{R} \cup \{\pm\infty\}$ is u.s.c. and is 0-diagonally quasi-convex in z.*

Then Γ has an equilibrium.

PROOF. Let $\phi(x, y, z) = -U(x, y, z)$. Then all the conditions of Theorem 1 are satisfied and thus we know there is $x^* \in S(x^*)$ and $y^* \in F(x^*)$ such that $\phi(x^*, y^*, z) \leqslant 0$ for all $z \in S(x^*)$. Thus, $U(x^*, y^*, z) \geqslant 0$ for all $z \in S(x^*)$. Now let $z = (x^*_{-i}, z_i)$. We then have

$$\frac{1}{2^i}\left[u_i(x^*, y^*) - u_i(x^*_{-i}, z_i, y^*)\right] \geqslant 0$$

for any $z_i \in S_i(x^*)$ and all $i \in I$. Hence (x^*, y^*) is an equilibrium of the generalized abstract economy. □

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