Existence of Equilibria in Games with Arbitrary Strategy Spaces and Preferences: A Full Characterization*

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Abstract

This paper provides a complete solution to the existence of equilibria in games with any number of players that may be finite, infinite, or even uncountable; arbitrary strategy spaces that may be discrete, continuum, non-compact or non-convex; payoffs (resp. preferences) that may be discontinuous or do not have any form of quasi-concavity (resp. nontotal, nontransitive, discontinuous, nonconvex, or nonmonotonic). We establish a single condition, *recursive diagonal transfer continuity* for aggregate payoffs and "upsetting relation" or *recursive weak transfer quasi-continuity* for individuals' preferences, which is necessary and sufficient for the existence of Nash equilibria in general games. The results not only provide a way of fully understanding the intrinsic nature of equilibrium, but also allow us to develop new sets of sufficient conditions for the existence of equilibrium and to ascertain the existence of solution for other important optimization problems. As illustrations, we show how they can be employed to fully characterize the existence of competitive equilibrium in economies with excess demand functions, stable matchings, and greatest and maximal elements of weak and strict preferences. The method of proof adopted to obtain our main results is also new and elementary — a non-fixed-point-theorem approach.

Keywords: full characterization, Nash equilibrium; discontinuous games, non-ordered preferences, finite or uncountable players, arbitrary strategy space, recursive transfer continuity.

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1 Introduction

The notion of Nash equilibrium is probably one of the most important solution concepts in economics in general and game theory in particular, which has wide applications in almost all areas of economics and in business and other social sciences. The classical existence theorems on Nash equilibrium (e.g. in Nash (1950, 1951), Debreu (1952), Glicksberg (1952), Fan (1953), Nikaido and Isoda (1955)) typically assume *continuity* and *(quasi)concavity* for the payoff functions, in addition to convexity and compactness of strategy spaces. However, in many important economic models, such as those in Bertrand (1883), Hotelling (1929), Milgrom and Weber (1982), Dasgupta and Maskin (1986), and Jackson (2005), etc., payoffs are discontinuous and/or non-quasiconcave, and strategy spaces are nonconvex and/or noncompact.

Accordingly, economists continually strive to seek weaker conditions that can guarantee the existence of equilibrium. Some seek to weaken the quasiconcavity of payoffs or substitute it with some forms of transitivity/monotonicity of payoffs (cf. McManus (1964), Roberts and Sonnenschein (1976), Nishimura and Friedman (1981), Topkis (1979), Vives (1990), and Milgrom and Roberts (1990)), some seek to weaken the continuity of payoff functions (cf. Dasgupta and Maskin (1986), Simon (1987), Simon and Zame (1990), Tian (1992a, 1992b, 1992c, 1994), Reny (1999), Bagh and Jofre (2006), Reny (2009), while others seek to weaken both quasiconcavity and continuity (cf. Baye, Tian, and Zhou (1993), Nessah and Tian (2008), and Barelli and Soza (2009), McLennan, Monteiro, and Tourky (2009)).

However, all the existing results only provide sufficient conditions for the existence of equilibrium. In order to apply a fixed-point theorem (say, Brouwer, Kakutani, Tarski's fixed point theorem, or KKM lemma, etc.), they all need to assume some forms of quasiconcavity[1] (or transitivity/monotonicity) and continuity of payoffs, in addition to compactness and convexity of strategy spaces.[2] As such, the intrinsic nature of equilibrium has not been fully understood yet. Why does or does not a game have an equilibrium? What kind of games can or cannot have equilibria? How can we know a game has or does not have an equilibrium? Are continuity and quasiconcavity both essential to the existence of equilibrium? If so, can continuity and quasiconcavity be combined into one single condition? If not, which one can be dropped?

No complete answers to these questions have been given so far. Since the existing results only give sufficient conditions for the existence of equilibrium, we can easily find many simple examples of economic games that have or do not have an equilibrium, but none of them can be used to reveal the existence/non-existence of equilibria in these games. Also, neither a unified condition nor full characterization on Nash equilibrium has been given yet for general games.[3]

[1] For mixed strategy Nash equilibrium, quasiconcavity is automatically satisfied since the mixed extension has linear payoff functions. Thus only some form of continuity matters for the existence of mixed strategy Nash equilibrium.

[2] Thus, convexity assumption excludes the possibility of considering discrete games.

[3] In the mechanism design literature, a lot of studies on full characterizations of Nash implementation of a social

In this paper we will answer these questions and provide a complete solution to the question of the existence of equilibria in general games. We fully characterize the existence of equilibria in games with any number of players that may be finite, infinite, or even uncountable; arbitrary strategy spaces that may be discrete, continuum, non-compact or non-convex; payoffs (resp. preferences) that may be discontinuous or do not have any form of quasi-concavity (resp. nontotal, nontransitive, discontinuous, nonconvex, or nonmonotonic). We introduce the notions of recursive transfer continuities, specifically *recursive diagonal transfer continuity* for aggregate payoffs and *recursive weak transfer quasi-continuity* for individuals' preferences, respectively.

It is shown that the single condition, recursive diagonal transfer continuity (or recursive weak transfer quasi-continuity) is necessary, and further, under compactness, sufficient for the existence of pure strategy Nash equilibrium in games with general topological strategy spaces and payoffs (or preferences). We also provide a complete solution for the case of any arbitrary strategy space that may be noncompact. We show that recursive diagonal transfer continuity (or recursive weak transfer quasi-continuity) with respect to a compact set B is necessary and sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary (possibly noncompact or open) strategy spaces and general preferences. As such, it strictly generalizes all the existing theorems on the existence of pure strategy Nash equilibrium. Recursive diagonal transfer continuity (or recursive weak transfer quasi-continuity) defined on respective spaces also permits full characterization of symmetric pure strategy, mixed strategy Nash, and Bayesian Nash equilibria in games with general strategy spaces and payoffs.

The approach and main results obtained in the paper can also allow us to develop new sufficient conditions for the existence of equilibrium and to ascertain the existence of equilibria in important classes of economic games and other optimization problems. As illustrations, we provide some new classes of sufficient conditions for the existence of equilibrium without imposing any form of quasiconcavity and also show how they can be employed to fully characterize the existence of competitive equilibrium for economies with excess demand functions, stable matchings, and greatest and maximal elements of weak and strict preferences. The results generalize all the existing results in the respective fields. For instance, we generalize the existing results on the existence of stale matchings to allow any number of agents that could be finite, countable or even uncountable. This also gives a way to characterize the essence of equilibrium with a finite choice set. Our results not only help us to understand what kind of games can have or cannot have equilibria, but also introduce new techniques and methods for possibly studying other optimization problems and extending some basic mathematics results such as fixed point theorems, variational inequalities, etc. The method of proof employed to obtain our main results is also new. Moreover, a remarkable advantage of the proof is that it is simple and elementary without using advanced

choice correspondence have been given such as those in Maskin (1999), Moore and Repullo (1990), Dutta and Sen (1991), etc. Also, Rahman (2008) recently provides a full characterization of correlated equilibrium.

math.

The notions of recursive transfer continuities extend the notions of transfer continuities, which cannot sufficiently guarantee the existence of equilibrium without imposing additional conditions, from direct transfers to allow indirect (called recursive or sequential) transfers so that they turn out to be necessary and sufficient for the existence of equilibrium without imposing any additional conditions. Roughly speaking, an upsetting relation $\succ$ is recursively diagonal transfer continuous if, whenever x is not an equilibrium, there exists a starting transfer strategy profile y^0 and a neighborhood of x, all of which are upset by any z that recursively upsets y^0.[4] Note that, unlike all the existing notions of transfer continuity introduced in the literature, we may choose the starting transfer strategy profile y^0 as any point in the strategy space, which enables us to show that recursive diagonal transfer continuity is a necessary condition for the existence of equilibrium in any game by choosing any Nash equilibrium as y^0. As a result, quasiconcavity/ monotonicity of payoffs (or convexity of preferences) is unnecessary for characterizing the existence of Nash equilibria. When the number of such securing strategy profiles is m, $\succ$ is then called m-recursive diagonal transfer continuity. Then diagonal transfer continuity, introduced by Baye, Tian, and Zhou (1993) implies 1-recursive diagonal transfer continuity, and weak transfer quasi-continuity introduced by Nessah and Tian (2008) implies 1-recursive weak transfer quasi-continuity (by letting $y^0 = x$), respectively. Since they are in the notion of direct transfer continuity, diagonal transfer continuity or weak transfer quasi-continuity is neither necessary nor sufficient, and thus some form of quasiconcavity such as (strong) diagonal transfer quasiconcavity is needed for the existence of equilibrium as studied in Baye, Tian, and Zhou (1993) and Nessah and Tian (2008).

It may be worth pointing out that the relation of recursive transfer continuities and direct transfer continuities is somewhat like that of the weak axiom of revealed preference (WARP) and strong axiom of revealed preference (SARP). Directly revealing a preference by WARP is not enough to fully reveal individuals' preferences, and then one may resort to indirectly revealing a preference by SARP to fully reveal an individual rational behavior. Similarly, diagonal transfer continuity or better-reply security alone is not enough to guarantee the existence of Nash equilibrium, one then may need to use a notion of recursive transfer continuity to fully characterize the existence of equilibrium.

We now describe the basic idea why recursive diagonal transfer continuity of an upsetting relation $\succ$ ensures the existence of pure strategy Nash equilibrium for a compact game. When a game fails to have a pure strategy Nash equilibrium on a compact strategy space X, by recursive diagonal transfer continuity, for every x, there is a starting transfer strategy profile y^0 such that all points in a neighborhood of x will be upset by any z that recursively upsets y^0. Then there are finite strategy profiles $\{x^1, x^2, \ldots, x^n\}$ whose neighborhoods cover X. Thus, all of the points in

[4] We say a strategy profile z recursively upsets a strategy profile y^0 if there exists a finite set of strategy profiles $\{y^1, \ldots, m^{m-1}, z\}$ such that z upsets y^{m-1}, y^{m-1} upsets y^{m-1}, and so on till finally y^1 upsets y^0.

a neighborhood, say $\mathcal{V}_{x^1}$, will be upset by a corresponding deviation profile z^1, which means z^1 cannot be an element in $\mathcal{V}_{x^1}$. If it is in some other neighborhood, say, $\mathcal{V}_{x^2}$, then it can be shown that z^2 will upset all strategy profiles in the union of $\mathcal{V}_{x^1}$ and $\mathcal{V}_{x^2}$ so that z^2 is not in the union of $\mathcal{V}_{x^1}$ and $\mathcal{V}_{x^2}$. Suppose $z^2 \in \mathcal{V}_{x^3}$. Then we can similarly show that z^3 is not in the union of $\mathcal{V}_{x^1}$, $\mathcal{V}_{x^2}$ and $\mathcal{V}_{x^3}$. With this process going on, we can finally show that $z^n \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \ldots, \cup \mathcal{V}_{x^n}$, which means z^n will not be in the strategy space X, reaching a contradiction.

The basic transfer method is systematically developed in Tian (1992, 1993), Tian and Zhou (1992, 1995), Zhou and Tian (1992), and Baye, Tian, and Zhou (1993) for studying the maximization of binary relations that may be nontotal or nontransitive and the existence of equilibrium in games that may have discontinuous or nonquasiconcave payoffs. These papers, especially Zhou and Tian (1992), have developed three types of transfers: transfer continuities, transfer convexities, and transfer transitivities to study the maximization of binary relations and the existence of equilibrium in games with discontinuous and/or nonquasiconcave payoffs. Various notions of transfer continuities, transfer convexities and transfer transitivities provide complete solutions to the question of the existence of maximal elements for complete preorders and interval orders (cf. Tian (1993) and Tian and Zhou (1995)). Incorporating recursive transfers into various transfer continuities will likely allow us to obtain full characterization results for many other solution problems as shown in the application section.

The remainder of the paper is organized as follows. Section 2 provides basic notation and definitions, and analyze the intrinsic nature of Nash equilibrium and why all the existing results provide only sufficient but not necessary conditions. Section 3 fully characterizes the existence of pure strategy Nash equilibrium by using aggregate payoffs and individuals' payoffs or preferences, respectively. We also provide sufficient conditions for our recursive transfer continuities. Section 4 extends the full characterization results to symmetric pure strategy Nash equilibrium. Section 5 shows how our main results can be employed to fully characterize the existence of competitive equilibrium for economies with excess demand functions, stable matchings, and greatest and maximal elements of weak and strict preferences. Concluding remarks are offered in Section 6. Full characterizations on the existence of mixed strategy and Bayesian Nash equilibrium are provided in Appendix.

2 Preliminaries: Nash Equilibrium and Its Intrinsic Nature

2.1 Notion and Definitions

Let I be the set of players that can be finite, infinite or even uncountable. Suppose that each player i's strategy set X_i is a nonempty subset of a topological space E_i. Denote by $X = \prod_{i \in I} X_i$ the set of strategy profiles. For each player $i \in I$, denote by $-i$ all other players rather than player i. Also

denote by $X_{-i} = \prod_{j \neq i} X_j$ the Cartesian product of the sets of strategies of players $-i$. Without loss of generality, suppose player i's preference relation is given by the weak preference $\succcurlyeq_i$ defined on X, which may be nontotal or nontransitive.[5] Let $\succ_i$ denote the asymmetric part of $\succcurlyeq_i$, i.e., $y \succ_i x$ if and only if $y \succcurlyeq_i x$ but not $x \succcurlyeq_i y$.

A *game* $G = (X_i, \succcurlyeq_i)_{i \in I}$ is simply a family of ordered tuples $(X_i, \succcurlyeq_i)$.

When $\succcurlyeq_i$ can be represented by a payoff function $u_i : X \longrightarrow \mathbb{R}$, the game $G = (X_i, u_i)_{i \in I}$ is a special case of $G = (X_i, \succcurlyeq_i)_{i \in I}$.

A strategy profile $x^* \in X$ is a *pure strategy Nash equilibrium* of a game G if,

$$x^* \succcurlyeq_i (y_i, x^*_{-i}) \; \forall i \in I, \; \forall y_i \in X_i.$$

A game $G = (X_i, \succcurlyeq_i)_{i \in I}$ is compact, convex, and upper continuous if, for all $i \in I$, X_i is compact, convex, and the weakly upper contour set $U_w(x) = \{y \in X : y \succcurlyeq x\}$ of $\succcurlyeq$ is closed for all $x \in X$, respectively. A game $G = (X_i, \; u_i)_{i \in I}$ is quasiconcave if, for every $i \in I$, X_i is convex and the payoff function u_i is quasiconcave in x_i.

2.2 The Essence of Equilibrium and Why the Existing Results are Only Sufficient

Before proceeding to the details about the notions of recursive transfer continuities and get deep insights on why they can be powerfully used to characterize the existence of equilibrium, we first analyze the intrinsic nature of Nash equilibrium, and why the conventional continuity is unnecessarily strong and all the existing results provide only sufficient but not necessary conditions.

In doing so, we define an "upsetting" binary relation, denoted by $\succ$ as follows:[6]

$$y \succ x \;\text{ iff }\; \exists i \in I \; s.t. \; u_i(y_i, x_{-i}) > u_i(x). \tag{1}$$

In this case, we say strategy profile y *upsets* strategy profile x. When the number of players is finite,[7] define the aggregator function, $U : X \times X \to \Re$ by

$$U(y, x) = \sum_{i \in I} u_i(y_i, x_{-i}), \;\; \forall (x, y) \in X \times X, \tag{2}$$

which refers to the aggregate payoff across individuals where for every player i assuming she or he deviates to y_i given that all other players follow the strategy profile x. The "upsetting" relation

[5]The results obtained for weak preferences $\succcurlyeq_i$ can be also used to get the results for strict preferences $\succ_i$. Indeed, from $\succ_i$, we can define a weak preferences $\succcurlyeq_i$ defined on $X \times X$ as follows: $y \succcurlyeq_i x$ if and only if $\neg x \succ_i y$. The preference $\succcurlyeq_i$ defined in such a way is called the completion of $\succ_i$. A preference $\succcurlyeq_i$ is said to be complete if, for any $x, y \in X$, either $x \succcurlyeq_i y$ or $y \succcurlyeq x$. A preference $\succcurlyeq_i$ is said to be total if, for any $x, y \in X$, $x \neq y$ implies $x \succcurlyeq_i y_i$ or $y \succcurlyeq_i x$.

[6]We can also define this upsetting relation by using individuals' preferences $\succcurlyeq_i$ when they may not be represented by numerical payoff functions u_i: $y \succ x$ iff $\exists i \in I \; s.t. (y_i, x_{-i}) \succ_i x$.

[7] I can be a countably infinite set. In this case, one may define U according to $U(y, x) = \sum_{i \in I} \frac{1}{2^i} u_i(y_i, x_{-i})$. This is a more general formulation.

$\succ$ is then defined as:

$$y \succ x \text{ iff } U(y,x) > U(x,x). \tag{3}$$

Then, one can easily see that a strategy profile $x^* \in X$ is a pure strategy Nash equilibrium if and only if there does not exist any strategy y in X that upsets x^*.

When $x \in X$ is not a pure strategy Nash equilibrium, then there exists a strategy profile $y \in X$ such that $y \succ x$. To ensure the existence of an equilibrium, it is necessary to require *all* strategies in a neighborhood $\mathcal{V}_x$ of x be upset by some strategy profile $z \in X$, denoted by $z \succ \mathcal{V}_x$. The topological structure of the conventional continuity surely secures this upsetting relation locally at x by using the strategy profile y, i.e., there always exists a neighborhood $\mathcal{V}_x$ of x such that $y \succ \mathcal{V}_x$. As such, no transfers (say, from y to z) or switchings (from player i to j) are needed for securing this upsetting relation locally at x. However, when u_i is not continuous, such a topological relation between the upsetting point y and the neighborhood $\mathcal{V}_x$ may no longer be true, i.e., we may not have $y \succ \mathcal{V}_x$. But, if y can be transferred to z so that $z \succ \mathcal{V}_x$, then the upsetting relation $\succ$ can be secured locally at x. This naturally leads to the following notion of transfer continuity, which is a weak notion of continuity.

DEFINITION 2.1 The upsetting relation $\succ$ is diagonally transfer continuous if, whenever $y \succ x$ for $x, y \in X$, there exists an other deviation strategy profile $z \in X$ and a neighborhood $\mathcal{V}_x \subset X$ of x such that $z \succ x'$ for all $x' \in \mathcal{V}_x$.

Note that "$y \succ x$ for $x, y \in X$" means "$x \in X$ is not an equilibrium". We will use these terms interchangeably. The above definition in turn immediately reduces to the notion of diagonal transfer continuity introduced by Baye, Tian, and Zhou (1993) for aggregator function.

DEFINITION 2.2 A game $G = (X_i,\, u_i)_{i \in I}$ is diagonally transfer continuous if, whenever $U(y,x) > U(x,x)$ for $x, y \in X$, there exists another deviation strategy profile $z \in X$ and a neighborhood $\mathcal{V}_x \subset X$ of x such that $U(z,x') > U(x',x')$ for all $x' \in \mathcal{V}_x$.

Tian and Zhou (1992), Baye, Tian, and Zhou (1993) and Tian and Zhou (1995) were the first ones to use such weak notions of continuity, such as diagonal transfer continuity, to study the existence of Nash and dominant strategy equilibria. Better-reply security introduced by Reny (1999) and its extensions by many others actually also fall in forms of transfer continuity. Nessah and Tian (2008) introduce various notions of transfer continuities to study the existence of equilibrium in discontinuous games.

Also, note that, to secure "upsetting" relation locally at x by z: $z \succ \mathcal{V}_x$, it is unnecessary to have $u_i(z_i, \mathcal{V}_x) > u_i(\mathcal{V}_x)$ for all players i, but is enough for just one player. Diagonal transfer continuity in Baye, Tian, and Zhou (1993), better-reply security in Reny (1999), weak transfer continuity in Nessah and Tian (2008), for instance, weaken the conventional continuity along this line.

Moreover, to secure "upsetting" relation locally at x by z: $z \succ \mathcal{V}_x$, it is unnecessary to just fix one player so that $u_i(z_i, \mathcal{V}_x) > u_i(\mathcal{V}_x)$, but players can be switched to secure this upsetting relation locally at x. If for every $x' \in \mathcal{V}_x$, there exists a player i such that $u_i(z_i, x'_{-i}) > u_i(x')$, all is done here. In other words, we can secure this "upsetting" relation locally by possibly switching players for every strategy in a neighborhood. This exactly comes up with the notion of weak transfer quasi-continuity introduced by Nessah and Tian (2008).

DEFINITION 2.3 A game $G = (X_i, u_i)_{i \in I}$ is said to be *weakly transfer quasi-continuous* if, whenever $x \in X$ is not an equilibrium, there exists a strategy profile $y \in X$ and a neighborhood $\mathcal{V}(x)$ of x so that for every $x' \in \mathcal{V}(x)$, there exists a player i such that $u_i(y_i, x'_{-i}) > u_i(x')$.

Note that, while the weak transfer quasi-continuity, lower single-deviation property in Reny (2009), and Condition B in Barelli and Soza (2009), which are weaker than the conventional continuity, better-reply security, weaker transfer continuity, explicitly exhibit such switchings, the notion of diagonal transfer continuity for aggregator function U internalizes (implicitly allows) the switchings. Such implicit switchings have an advantage that it may become easier to check "upsetting" relations, especially for "complementary discontinuities", by which a downward jump in one player's payoff can always be accompanied by an upward jump in another player's payoff (cf. Maskin and Dasgupta (1986) and Simon (1987)). [8]

Thus, to have the existence of an equilibrium, it is necessary to secure "upsetting" relation locally for every non-equilibrium. However, it may not be sufficient for the existence of equilibrium unless imposing some forms of quasiconcavity, transitivity, or monotonicity. This is why, to make this upsetting relation sufficient for the existence of equilibrium, all the existing results impose additional conditions such as convexity of strategy spaces and (weak forms of) quasiconcavity of payoffs or transitivity/monotonicity of payoffs in order to use a fixed point theorem. But, such kind of combined conditions are only sufficient but not necessary. As mentioned in the introduction, to apply a fixed-point theorem to prove the existence of equilibrium, they all need to assume some forms of quasiconcavity (or transitivity/monotonicity) and continuity of payoffs, in addition to compactness and convexity of strategy spaces.

As such, a direct upsetting transfer along may not be enough to guarantee the existence of an equilibrium, a recursive (sequential) indirect upsetting transfer starting from a strategy profile y^0 may be needed for guaranteeing the existence of equilibrium without imposing quasiconcavity or monotonicity condition. This is what we study in this paper. With such recursively upsetting relations, we are able to allow not only sequential transfers, but also the switchings of players

[8] While diagonal transfer continuity defined by aggregate payoffs is stronger than weak transfer quasi-continuity defined by individuals' preferences as shown in Nessah and Tian (2009), they are equivalent when diagonal transfer continuity is defined by the upsetting relation $\succ$ according to individuals' payoffs (see Lemma 3.1 below). An "upsetting" relation $\succ$ is said to be diagonally transfer continuous if, whenever $x \in X$ is not an equilibrium, there exists a strategy profile $y \in X$ and a neighborhood $\mathcal{V}_x$ of x such that $y \succ \mathcal{V}_x$.

in any stage of upsetting transfers and for different strategy profiles in a neighborhood. As a result, such a weak notion of recursive continuity may become both necessary and sufficient for the existence of Nash equilibria. Indeed, we will show that these intuition and insights turn to be correct. It is the two requirements—securing upsetting relation and recursive transfers that characterize the existence or nonexistence of an equilibrium.

In summary, the reasons why the notion of recursive transfer continuity is necessary and sufficient for the existence of equilibrium in general games come down to its four features: (1) it allows us to transfer to a strategy z that secures upsetting relations locally; (2) it allows for recursive upsetting relations; (3) it allows to make the switchings (transfers) among players for any upsettings of a particular transfer; (4) the starting point for making a recursive upsetting chain can be any point, say, it can be an equilibrium point to the necessity of the condition or the nonexistence of such a starting point if a game does not possess an equilibrium.

Now we are ready to discuss our full characterization results.

3 Full Characterization of Pure Strategy Nash Equilibria

In this section we provide a complete solution to the question of the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs/preferences by providing necessary and sufficient conditions for the existence of pure strategy Nash equilibria. We will provide two classes of necessary and sufficient conditions. One is based on aggregate payoffs and the other is based on individuals' payoffs/prerences. These results provide not only a way of fully understanding the essence of equilibrium, but also a way of developing new sufficient conditions for the existence of equilibrium and checking the existence or nonexistence of pure strategy Nash equilibrium in games with noncompact or nonconvex strategy spaces and nontransitive or nonconvex discontinuous preferences. We provide a number of simple examples of economic games that cannot use the existing theorems to show the existence/non-existence of equilibria, but our results can do.

3.1 Full Characterization By the Aggregate Payoffs

In this subsection we assume that individuals' preferences can be represented by payoff functions. We consider a mapping of individual payoffs into the aggregator function $UX \times X \rightarrow \Re$, and then provide a necessary and sufficient condition on the aggregator function for the existence of pure strategy Nash equilibrium. The aggregator function approach is pioneered by Nikaido and Isoda (1955), and is also used by Baye, Tian, and Zhou (1993). Dasgupta and Maskin (1986) also use a similar approach to prove the existence of mixed strategy Nash equilibrium in games with discontinuous payoff functions. An advantege of this approach is that it internalizes the switchings

among players so that checking "upsetting" relations are easier due to the complementarity that a game generally has.

DEFINITION 3.1 (Recursive Upsetting) A strategy profile $y^0 \in X$ is said to be *recursively upset by* $z \in X$ if there exists a finite set of deviation strategy profiles $\{y^1, y^2, \ldots, y^{m-1}, z\}$ such that $U(y^1, y^0) > U(y^0, y^0)$, $U(y^2, y^1) > U(y^1, y^1)$, ..., $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$.

Upsetting relations are critical important for studying and characterizing the existence of equilibrium. We say that a strategy profile $y^0 \in X$ is *m-recursively upset by* $z \in X$ if the number of such deviation strategy profiles is m. For convenience, we say y^0 is *directly upset by* z when $m = 1$, and *indirectly upset by* z when $m > 1$. Recursive upsetting says that a strategy profile y^0 can be directly or indirectly upset by a strategy profile z through sequential deviation strategy profiles $\{y^1, y^2, \ldots, y^{m-1}\}$ in a recursive way that y^0 is upset by y^1, y^1 is upset by y^2, ..., and y^{m-1} is upset by z.

DEFINITION 3.2 (Recursive Diagonal Transfer Continuity) A game $G = (X_i,\ u_i)_{i \in I}$ is said to be *recursively diagonal transfer continuous* if, whenever $U(y, x) > U(x, x)$ for $x, y \in X$, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ for any z that recursively upsets y^0.

In the definition of recursive diagonal transfer continuity, x is transferred to y^0 that could be any point in X. Recursive diagonal transfer continuity merely requires that, whenever x is not an equilibrium, there exists a starting point y^0 such that any recursive upsetting chain $\{y^0, y^1, y^2, \ldots, y^m\}$ disproves the possibility of an equilibrium in a sufficiently small neighborhood of x, i.e., all points in the neighborhood are upset by all securing strategy profiles that directly or indirectly upset y^0. This implies that, if equilibrium fails to exist, then there is a nonequilibrium strategy profile x such that for every $y^0 \in X$ and every neighborhood $\mathcal{V}_x$ of x, some deviation strategy profiles in the neighborhood cannot be upset by a securing strategy profile z that directly or indirectly upsets y^0.

REMARK 3.1 Under recursive diagonal transfer continuity, when $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$, $U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2})$, ..., $U(y^1, y^0) > U(y^0, y^0)$, we have not only $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$, but also $U(y^{m-1}, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$, ..., $U(y^1, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$. That is, any chain of securing strategy profiles $\{y^1, y^2, \ldots, y^{m-j}\}$ obtained by truncating a recursive upsetting chain $\{y^1, y^2, \ldots, y^{m-1}, z\}$ is also a recursive upsetting chain, including y^1. Also, when dealing with an upsetting relation, we may be in a situation where not only one point, say, y^k, is transferred to another point y^{k+1}, but also players may be switched in order to keep the upsetting relation.

Similarly, we can define m-recursive diagonal transfer continuity. A game $G = (X_i,\ u_i)_{i\in I}$ is *m-recursively diagonal transfer continuous* if the phrase "for any z that recursively upsets y^0" in the above definition is replaced by "for any z that m-recursively upsets y^0". Thus, a game $G = (X_i,\ u_i)_{i\in I}$ is recursively diagonal transfer continuous if it is m-recursively diagonal transfer continuous on X for all $m = 1, 2 \ldots$.

REMARK 3.2 It is clear that diagonal transfer continuity implies 1-recursive diagonal transfer continuity by letting $y^0 = x$, but the converse may not be true since x possibly cannot be selected as y^0. Thus, diagonal transfer continuity (thus continuity) is in general stronger than 1-recursive diagonal transfer continuity. Also, recursive diagonal transfer continuity neither implies nor is implied by continuity for games with two or more players.[9] This point becomes clear when one sees recursive diagonal transfer continuity is a necessary and sufficient condition for the existence of pure strategy Nash equilibrium while continuity of the aggregate payoff function is neither a necessary nor sufficient condition for the existence of pure strategy Nash equilibrium.

Now we are ready to state our main results that strictly generalize all the existing results on the existence of pure strategy Nash equilibrium in games. We first show that recursive diagonal transfer continuity is a necessary condition for any game to possess a pure strategy Nash equilibrium.

THEOREM 3.1 (NECESSITY THEOREM) *If a game $G = (X_i, u_i)_{i\in I}$ possesses a pure strategy Nash equilibrium, it must be recursively diagonal transfer continuous on X.*

PROOF. First, note that, if $x^*\in X$ is a pure strategy Nash equilibrium of a game G, we must have $U(y, x^*) \leq U(x^*, x^*)$ for all $y \in X$, which is obtained by summing up

$$u_i(y_i, x^*_{-i}) \leq u_i(x^*) \quad \forall\ y_i \in X_i, \tag{4}$$

for all players.

Let x^* be a pure strategy Nash equilibrium and $U(y, x) > U(x, x)$ for $x, y \in X$. Let $y^0 = x^*$ and $\mathcal{V}_x$ be a neighborhood of x. Since $U(y, x^*) \leq U(x^*, x^*)$ for all $y \in X$, it is impossible to find any securing strategy profile y^1 such that $U(y^1, y^0) > U(y^0, y^0)$ (then of course it is impossible to find any sequence of strategy profiles $\{y^1, y^2, \ldots, y^m\}$ such that $U(y^1, y^0) > U(y^0, y^0)$, $U(y^2, y^1) > U(y^1, y^1), \ldots, U(y^m, y^{m-1}) > U(y^{m-1}, y^{m-1})$). Hence, the recursive diagonal transfer continuity holds trivially. ■

The Necessity Theorem, Theorem 3.1, is especially useful to check the nonexistence of equilibrium of economic games. No such a result is available in the literature. Examples 3.1 - 3.2

[9] In one-player games recursive diagonal transfer continuity is equivalent to the player's utility function possessing a maximum on a compact set, and consequently it implies transfer weak upper continuity introduced in Tian and Zhou (1995), which is weaker than continuity.

below show that no the existing results can be used to argue the nonexistence of pure Nash equilibrium, but, by Theorem 3.1, the games do not have any pure strategy Nash equilibrium since they fail to satisfy the recursive diagonal transfer continuity condition.

EXAMPLE 3.1 (DASGUPTA AND MASKIN) Consider the following game studied by Dasgupta and Maskin (1986): $n = 2$, $X_1 = X_2 = [0, 1]$, and the payoff functions are defined by

$$u_i(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 = 1 \\ x_i & \text{otherwise} \end{cases} \qquad i = 1, 2.$$

It is clear that this simple game does not possess a Nash equilibrium, but no the existing theorem can tell this. However, we can know this by showing that it is not recursively diagonal transfer continuous on X. To see this, for $x = (1, 1)$ and $y \in (0, 1) \times (0, 1)$, we have $U(y, x) > U(x, x)$. We then cannot find any $y^0 \in X$ and neighborhood $\mathcal{V}_x$ of x such that $U(z, x') > U(x', x')$ for every deviation profile z that is upset directly or indirectly by y^0 for all $x' \in \mathcal{V}_x$. We show this by considering two cases.

Case 1. $y^0 \neq (1, 1)$. Then, for any neighborhood $\mathcal{V}_{(1,1)}$ of $(1, 1)$, choosing strategy profiles $z \in X$ and $x' \in \mathcal{V}_{(1,1)}$ such that $y_1^0 + y_2^0 < z_1 + z_2 < x_1' + x_2'$, we then have $U(z, y^0) > U(y^0, y^0)$ but $U(z, x') < U(x', x')$.

Case 2. $y^0 = (1, 1)$. Then, for any neighborhood $\mathcal{V}_{(1,1)}$ of $(1, 1)$, choosing strategy profiles $z \in X$ and $x' \in \mathcal{V}_{(1,1)}$ such that $0 < z_1 + z_2 < x_1' + x_2' < y_1^0 + y_2^0$, we then have $U(z, y^0) > U(y^0, y^0)$ but $U(z, x') < U(x', x')$.

Thus, we cannot find any $y^0 \in X$ and any neighborhood $\mathcal{V}_{(1,1)}$ of $(1, 1)$ such that $U(z, x') > U(x', x')$ for every deviation profile z that is upset by y^0 for all $x' \in \mathcal{V}_x$. Hence, the game is not recursively diagonal transfer continuous on X, and therefore, by Theorem 3.1, there is no pure strategy Nash equilibrium on X.

Now we change the strategy space $X_1 = X_2 = (0, 1)$ to be an open unit interval and the payoff functions to be

$$u_i(x_1, x_2) = x_i \quad i = 1, 2.$$

The game does not possess a Nash equilibrium although it is continuous and quasiconcave. Indeed, given $x \in X$, we cannot find any $y^0 \in (0, 1)$ and any neighborhood $\mathcal{V}_x$ of x such that $U(z, x') > U(x', x')$ for every deviation profile z that is upset by y^0 for all $x' \in \mathcal{V}_x$. We can show this by considering two cases.

Case 1. $y^0 < x$. Then, for any neighborhood $\mathcal{V}_{(x)}$ of x, choosing strategy profiles $z \in X$ and $x' \in \mathcal{V}_x$ such that $y_1^0 + y_2^0 < z_1 + z_2 < x_1' + x_2'$, we then have $U(z, y^0) > U(y^0, y^0)$ but $U(z, x') < U(x', x')$.

Case 2. $y^0 \geq x$. Then, for any neighborhood $\mathcal{V}_{(x)}$ of x, choosing strategy profiles $z \in X$ and $x' \in \mathcal{V}_x$ such that $0 < z_1 + z_2 < x_1' + x_2' < y_1^0 + y_2^0$, we then have $U(z, y^0) > U(y^0, y^0)$ but

$U(z, x') < U(x', x')$. Thus, the game is not recursively diagonal transfer continuous on X, and therefore, by Theorem 3.1, there is no pure strategy Nash equilibrium on X.

EXAMPLE 3.2 (KARLIN) Consider games of "timing" or "silent duel", which have been studied by Karlin (1959), Owen (1968), Jones (1980), and Dasgupta and Maskin (1986). These are symmetric two-person zero-sum games on the unit square so that $n = 2$, $X_1 = X_2 = [0, 1]$, and $U(x, x) = 0$ for all $x \in X$. The version called the "silent duel" has player l's payoff function of the form:

$$u_1(x_1, x_2) = \begin{cases} x_1 - x_2 + x_1 x_2, & \text{if } x_1 < x_2 \\ 0, & \text{if } x_1 = x_2 \\ x_1 - x_2 - x_1 x_2, & \text{if } x_1 > x_2 \end{cases}.$$

It is known that there is no Nash equilibrium for this game, but no the existing theorem can tell. We show the game must not be recursively diagonally transfer continuous. To see this, consider $x = (x_1, x_2) = (1, 1)$. We then cannot find any $y^0 \in X$ and neighborhood $\mathcal{V}_x$ of x such that $U(z, x') > U(x', x')$ for every deviation profile z that is upset directly or indirectly by y^0 for all $x' \in \mathcal{V}_x$. To show this, four cases need to be considered.

Case 1. $y_1^0 < 1$ and $y_2^0 < 1$. Then, for any neighborhood $\mathcal{V}_{(1,1)}$ of $(1, 1)$, choose strategy profiles $z \in X$ and $x' \in \mathcal{V}_{(1,1)}$ such that $y_1^0 < z_1 < x_1'$ and $y_2^0 < z_2 < x_2'$. Since $u_1(y_1, y_2)$ and $u_2(y_1, y_2) = -u_1(y_1, y_2)$ are both increasing in y_1 and y_2, respectively, we have $u_1(z_1, y_2^0) - u_1(y_1^0, y_2^0) > 0$ and $u_2(y_1^0, z_2) - u_2(y_1^0, y_2^0) = u_1(y_1^0, y_2^0) - u_1(y_1^0, z_2) > 0$, $u_1(z_1, x_2') - u_1(x_1', x_2') < 0$ and $u_2(x_1', z_2) - u_2(x_1', x_2') = u_1(x_1', x_2') - u_1(x_1', z_2) < 0$. Thus, we have $U(z, y^0) > U(y^0, y^0)$ but $U(z, x') < U(x', x')$.

Case 2. $y_1^0 = 1$ and $y_2^0 < 1$. Then, for any neighborhood $\mathcal{V}_{(1,1)}$ of $(1, 1)$, choose strategy profiles $z \in X$ and $x' \in \mathcal{V}_{(1,1)}$ such that $y_1^0 = z_1 = x_1'$ and $y_2^0 < z_2 < x_2'$. Then, by the monotonicity of $u_1(y_1, y_2)$ and $u_2(y_1, y_2) = -u_1(y_1, y_2)$, we have $U(z, y^0) > U(y^0, y^0)$ but $U(z, x') < U(x', x')$.

Case 3. $y_1^0 < 1$ and $y_2^0 = 1$. Then, for any neighborhood $\mathcal{V}_{(1,1)}$ of $(1, 1)$, choose strategy profiles $z \in X$ and $x' \in \mathcal{V}_{(1,1)}$ such that $y_1^0 < z_1 < x_1'$ and $y_2^0 = z_2 = x_2'$. Then, by similar reasoning, we have $U(z, y^0) > U(y^0, y^0)$ but $U(z, x') < U(x', x')$.

Case 4. $y_1^0 = 1$ and $y_2^0 = 1$. Then, for any neighborhood $\mathcal{V}_{(1,1)}$ of $(1, 1)$, choose strategy profiles $z \in X$ and $x' \in \mathcal{V}_{(1,1)}$ such that $1/2 < z_1 < x_1'$ and $1/2 = z_2 = x_2'$. We then have $u_1(z_1, y_2^0) - u_1(y_1^0, y_2^0) = 2z_1 - 1 > 0$ and $u_2(y_1^0, z_2) - u_2(y_1^0, y_2^0) = u_1(y_1^0, y_2^0) - u_1(y_1^0, z_2) = 2z_2 - 1 > 0$, $u_1(z_1, x_2') - u_1(x_1', x_2') < 0$ and $u_2(x_1', z_2) - u_2(x_1', x_2') = u_1(x_1', x_2') - u_1(x_1', z_2) < 0$, and consequently, $U(z, y^0) > U(y^0, y^0)$ but $U(z, x') < U(x', x')$.

Thus, we cannot find any $y^0 \in X$ and any neighborhood $\mathcal{V}_0$ of $(1, 1)$ such that $U(z, x') > U(x', x')$ for every deviation profile z that is upset by y^0 for all $x' \in \mathcal{V}_x$. Hence, the game is not

recursively diagonal transfer continuous on X, and therefore, by Theorem 3.1, there is no pure strategy Nash equilibrium on X.

Next, we show that, when the strategy space X of a game is compact, recursive diagonal transfer continuity is not only necessary but also sufficient for the existence of pure strategy Nash equilibrium.

THEOREM 3.2 (SUFFICIENCY THEOREM) *Suppose the strategy space X of a game $G = (X_i, u_i)_{i \in I}$ is compact. Then, if the game is recursively diagonal transfer continuous on X, it possesses a pure strategy Nash equilibrium.*

PROOF. First, note that, if $U(y, x^*) \leq U(x^*, x^*)$ for all $y \in X$, $x^* \in X$ must be a pure strategy Nash equilibrium of a game G. Indeed, letting $y = (y_i, x^*_{-i})$, we have $u_i(y_i, x^*_{-i}) \leq u_i(x^*) \quad \forall \; y_i \in X_i$, which means x^* is a Nash equilibrium.

Suppose, by way of contradiction, that there is no pure strategy Nash equilibrium. Then, for each $x \in X$, there exists $y \in X$ such that $U(y, x) > U(x, x)$. By recursive diagonal transfer continuity, for each $x \in X$, there exists y^0 and a neighborhood $\mathcal{V}_x$ such that $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ whenever $y^0 \in X$ is recursively upset by z, i.e., for any sequence of recursive securing strategy profiles $\{y^1, \ldots, y^{m-1}, z\}$ with $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$, $U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2})$, ..., $U(y^1, y^0) > U(y^0, y^0)$ for $m \geq 1$, we have $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$. Since there is no equilibrium by the contrapositive hypothesis, y^0 is not an equilibrium and thus, by recursive diagonal transfer continuity, such a sequence of recursive securing strategy profiles $\{y^1, \ldots, y^{m-1}, z\}$ exists for some $m \geq 1$.

Since X is compact and $X \subseteq \bigcup_{x \in X} \mathcal{V}_x$, there is a finite set $\{x^1, \ldots, x^L\}$ such that $X \subseteq \bigcup_{i=1}^{L} \mathcal{V}_{x^i}$. For each of such x^i, the corresponding initial deviation profile is denoted by y^{0i} so that $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$ whenever y^{0i} is recursively upset by z^i.

Since there is no equilibrium, for each of such y^{0i}, there exists z^i such that $U(z^i, y^{0i}) > U(y^{0i}, y^{0i})$, and then, by 1-recursive diagonal transfer continuity, we have $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$. Now consider the set of securing strategy profiles $\{z^1, \ldots, z^L\}$. Then, $z^i \notin \mathcal{V}_{x^i}$, otherwise, by $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$, we will have $U(z^i, z^i) > U(z^i, z^i)$, a contradiction. So we must have $z^1 \notin \mathcal{V}_{x^1}$.

Without loss of generality, we suppose $z^1 \in \mathcal{V}_{x^2}$. Since $U(z^2, z^1) > U(z^1, z^1)$ by noting that $z^1 \in \mathcal{V}_{x^2}$ and $U(z^1, y^{01}) > U(y^{01}, y^{01})$, then, by 2-recursive diagonal transfer continuity, we have $U(z^2, \mathcal{V}_{x^1}) > U(\mathcal{V}_{x^1}, \mathcal{V}_{x^1})$. Also, $U(z^2, \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^2}, \mathcal{V}_{x^2})$. Thus $U(z^2, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2})$, and consequently $z^2 \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}$.

Again, without loss of generality, we suppose $z^2 \in \mathcal{V}_{x^3}$. Since $U(z^3, z^2) > U(z^2, z^2)$ by noting that $z^2 \in \mathcal{V}_{x^3}$, $U(z^2, z^1) > U(z^1, z^1)$, and $U(z^1, y^{01}) > U(y^{01}, y^{01})$, by 3-recursive diagonal transfer continuity, we have $U(z^3, \mathcal{V}_{x^1}) > U(\mathcal{V}_{x^1}, \mathcal{V}_{x^1})$. Also, since $U(z^3, z^2) > U(z^2, z^2)$

and $U(z^2, y^{02}) > U(y^{02}, y^{02})$, by 2-recursive diagonal transfer continuity, we have $U(z^3, \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^2}, \mathcal{V}_{x^2})$. Thus, we have $U(z^3, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}) > U(\mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3})$, and consequently $z^3 \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}$.

With this process going on, we can show that $z^k \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \ldots, \cup \mathcal{V}_{x^k}$, i.e., z^k is not in the union of $\mathcal{V}_{x^1}, \mathcal{V}_{x^2}, \ldots, \mathcal{V}_{x^k}$ for $k = 1, 2, \ldots, L$. In particular, for $k = L$, we have $z^L \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \ldots \cup \mathcal{V}_{x^L}$ and so $z^L \notin X \subseteq \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \ldots \cup \mathcal{V}_{x^L}$, a contradiction. ■

Examples 3.3 and 3.4 below are games with pure strategy equilibria that are accounted for by Theorems 3.2, but which violate the conditions of existing theorems.

EXAMPLE 3.3 Consider a variation of "timing" or "silent duel" games, in Example 3.2. The version, called the "noisy duel", has player l's payoff function of the form:

$$u_1(x_1, x_2) = \begin{cases} 2x_1 - 1, & \text{if } x_1 < x_2 \\ 0, & \text{if } x_1 = x_2 \\ 1 - 2x_2, & \text{if } x_1 > x_2 \end{cases} .$$

In this game, the payoff function $u_i(x_1, x_2)$ is neither diagonally transfer continuous nor quasiconcave in y_i for $i = 1$. Therefore, theorems in Baye, Tian, and Zhou (1993) and Reny (1999) are not applicable.

However, the game has a pure strategy Nash equilibrium. To see this, suppose $U(y, x) > U(x, x)$ for $x = (x_1, x_2) \in X$ and $y = (y_1, y_2) \in X$. Let $y^0 = (1/2, 1/2)$ and $\mathcal{V}_x$ be a neighborhood of x. Since $U(y, y^0) \leq U(y^0, y^0)$ for all $y \in X$, it is impossible to find any securing strategy profile y^1 such that $U(y^1, y^0) > U(y^0, y^0)$. Hence, the recursive diagonal transfer continuity holds. Therefore, by Theorem 3.2, this game has a pure strategy Nash equilibrium.

EXAMPLE 3.4 Consider the two-player game with the following payoff functions defined on $[0, 1] \times [0, 1]$ studied by Barelli and Soza (2009).

$$u_i(x_i, x_{-i}) = \begin{cases} 0 & \text{if } x_i \in (0, 1) \\ 1 & \text{if } x_i = 0 \text{ and } x_{-i} \in \mathbb{Q} \\ 1 & \text{if } x_i = 1 \text{ and } x_{-i} \notin \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} ,$$

where $\mathbb{Q} = \{x \in [0, 1] : x \text{ is a rational number}\}$.

This game is convex, compact, bounded and quasiconcave, but it is not weakly transfer quasicontinuous, and consequently, it is not diagonally transfer continuous nor better-reply secure either.[10] Thus, there is no existing theorem that can be applied.

[10] A game $G = (X_i, u_i)_{i \in I}$ is better-reply secure if, whenever $(x^*, u^*) \in \bar{\Gamma}$, x^* is not an equilibrium implies that there is some player i, $\overline{x}_i \in X_i$, and an open neighborhood $\mathcal{V}_{x_{-i}}$ of x_{-i} such that $u_i(\overline{x}_i, y_{-i}) > u_i(x^*)$ for all $y_{-i} \in \mathcal{V}_{x_{-i}}$, where $\bar{\Gamma}$ is the closure of the graph $\Gamma = \{(x, u) \in X \times \mathbb{R}^n : u_i(x) = u_i, \ \forall i \in I\}$.

To see the game is not weakly transfer quasi-continuous, consider the nonequilibrium $x = (1,1)$. We then cannot find any $y \in X$ and any neighborhood $\mathcal{V}_{(1,1)}$ of $(1,1)$ such that for every $x' \in \mathcal{V}_x$, there is a player i with $u_i(y_i, x'_{-i}) > u_i(x')$. We show this by considering two cases.

Case 1. $y_2 \neq 0$. Then, for any neighborhood $\mathcal{V}_{(1,1)}$ of $(1,1)$, choosing $x' \in \mathcal{V}_x$ with $x'_1 = 1$ and $x'_2 \notin \mathbb{Q}$, we have $u_1(y_1, x'_2) \leq u_1(x'_1, x'_2) = 1$ and $u_2(x'_1, y_2) = u_2(x'_1, x'_2) = 0$.

Case 2. $y_2 = 0$. When $y_1 \neq 0$, choosing $x' \in \mathcal{V}_x$ with $x'_2 = 1$ and $x'_1 \notin \mathbb{Q}$, we have $u_1(y_1, x'_2) = u_1(x'_1, x'_2) = 0$ and $u_2(x'_1, y_2) \leq u_2(x'_1, x'_2) = 1$. When $y_1 = 0$, choosing $x' \in \mathcal{V}_x$ with $x'_1 \notin \mathbb{Q}$ and $x'_2 \notin \mathbb{Q}$, we have $u_1(y_1, x'_2) = u_1(x'_1, x'_2) = 0$ and $u_2(x'_1, y_2) = u_2(x'_1, x'_2) = 0$.

Thus, the game is not weakly transfer quasi-continuous, and the existence theorems in Nessah and Tian (2008) can not be applied.

However, it is recursively diagonal transfer continuous. Indeed, suppose $U(y, x) > U(x, x)$ for $x = (x_1, x_2) \in X$ and $y = (y_1, y_2) \in X$. Let $y^0 = (0,0)$ and $\mathcal{V}_x$ be a neighborhood of x. Since $U(y, y^0) \leq U(y^0, y^0)$ for all $y \in X$, it is impossible to find any securing strategy profile y^1 such that $U(y^1, y^0) > U(y^0, y^0)$. Hence, the recursive diagonal transfer continuity holds. Therefore, by Theorem 3.2, this game has a pure strategy Nash equilibrium.

The above Sufficiency Theorem assumes that the strategy space of a game is compact. This may still be a restrictive assumption as some important economic games may not have compact strategy spaces. For instance, it is well known that Walrasian mechanism can be regarded as a generalized game. However, when preferences are strictly monotone, excess demand functions are not well defined for zero prices. In this case, we cannot use Theorem 3.2 to fully characterize the existence of competitive equilibrium.

In the following we show that the compactness of strategy space in Theorem 3.2 can also be removed.[11] To do so, we first introduce the following stronger version of recursive diagonal transfer continuity.

DEFINITION 3.3 Let B be a subset of X. A game $G = (X_i,\ u_i)_{i \in I}$ is said to be *recursively diagonal transfer continuous on X with respect to B* if, whenever x is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that (1) whenever y^0 is upset by a strategy profile in $X \setminus B$, it is upset by a strategy profile in B and (2) $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ for any finite subset of securing strategy profiles $\{y^1, \ldots, y^m\} \subset B$ with $y^m = z$ and $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$, $U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2})$, ..., $U(y^1, y^0) > U(y^0, y^0)$ for $m \geq 1$.

Condition (1) in the above definition ensures that if a strategy profile x is not an equilibrium for the game $G = (X_i,\ u_i)_{i \in I}$, it must not be an equilibrium when the strategy space is constrained to be B.

[11] I thank David Rahman for raising this issue to me. Thanks also to Adam Wong for pointing out a misstatement in an earlier version of Theorem 3.3.

Note that, while $\{y^1, \ldots, y^m\}$ are required to be in B, y^0 is not necessarily in B but can be any point in X. Also, when $B = X$, recursive diagonal transfer continuity on X with respect to B reduces to recursive diagonal transfer continuity on X. We then have the following theorem that generalizes Theorem 3.2 by relaxing compactness of games.

THEOREM 3.3 (FULL CHARACTERIZATION THEOREM) *A game $G = (X_i, u_i)_{i \in I}$ possesses a pure strategy Nash equilibrium if and only if there exists a compact set $B \subseteq X$ such that it is recursively diagonal transfer continuous on X with respect to B.*

PROOF. *Sufficiency* ($\Leftarrow$). The proof of sufficiency is essentially the same as that of Theorem 3.2 and we just outline the proof here. To show the existence of a pure strategy Nash equilibrium on X, it suffices to show that the game possesses a pure strategy Nash equilibrium x^* in B if it is recursively diagonal transfer continuous on X with respect to B. Suppose, by way of contradiction, that there is no pure strategy Nash equilibrium in B. Then, since the game G is recursively diagonal transfer continuous on X with respect to B, for each $x \in B$, there exists y^0 and a neighborhood $\mathcal{V}_x$ such that (1) whenever y^0 is upset by a strategy profile in $X \setminus B$, it is upset by a strategy profile in B and (2) $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ for any finite subset of securing strategy profiles $\{y^1, \ldots, y^m\} \subset B$ with $y^m = z$ and $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$, $U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2})$, …, $U(y^1, y^0) > U(y^0, y^0)$ for $m \geq 1$. Since there is no equilibrium by the contrapositive hypothesis, y^0 is not an equilibrium and thus, by recursive diagonal transfer continuity on X with respect to B, such a sequence of recursive securing strategy profiles $\{y^1, \ldots, y^{m-1}, z\}$ exists for some $m \geq 1$.

Since B is compact and $B \subseteq \bigcup_{x \in X} \mathcal{V}_x$, there is a finite set $\{x^1, \ldots, x^L\} \subseteq B$ such that $B \subseteq \bigcup_{i=1}^{L} \mathcal{V}_{x^i}$. For each of such x^i, the corresponding initial deviation profile is denoted by y^{0i} so that $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$ whenever y^{0i} is recursively upset by z^i through any finite subset of securing strategy profiles $\{y^{1i}, \ldots, y^{mi}\} \subset B$ with $y^{mi} = z^i$. Then, by the same argument as in the proof of Theorem 3.2, we will obtain that z^k is not in the union of $\mathcal{V}_{x^1}, \mathcal{V}_{x^2}, \ldots, \mathcal{V}_{x^k}$ for $k = 1, 2, \ldots, L$. For $k = L$, we have $z^L \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \ldots \cup \mathcal{V}_{x^L}$ and so $z^L \notin B \subseteq \bigcup_{i=1}^{L} \mathcal{V}_{x^i}$, which contradicts that z^L is a securing strategy profile in B.

Necessity ($\Rightarrow$). Suppose x^* is a pure strategy Nash equilibrium. Let $B = \{x^*\}$. Then, the set B is clearly compact. Now, for any nonequilibrium $x \in X$, let $y^0 = x^*$ and $\mathcal{V}_x$ be a neighborhood of x. Since $U(y, x^*) \leq U(x^*, x^*)$ for all $y \in X$ and $y^0 = x^*$ is a unique element in B, there is no other securing strategy profile y^1 such that $U(y^1, y^0) > U(y^0, y^0)$. Hence, the game is recursively diagonal transfer continuous on X with respect to B. ■

Thus, Theorem 3.3 fully characterizes the existence of equilibrium in games with an arbitrary strategy space that may be discrete, continuum, non-convex or non-compact and an arbitrary payoff function that may be discontinuous or nonquasiconcave, and consequently it strictly generalizes all the existing results on the existence of pure strategy Nash equilibrium.

REMARK 3.3 Another way to deal with the existence of pure strategy Nash equilibrium of a game with noncompact strategy space is to extend (rather than restrict) a noncompact X to another compact set $\bar{X}$ (by compactification of topological space) and extend U to $\bar{U}$ on $\bar{X}$. Then a necessary and sufficient condition for the existence of Nash equilibrium is that $\bar{U}$ is recursively diagonal transfer continuous on $\bar{X}$.[12]

The following example shows that, although the strategy space is an open unit interval, highly discontinuous and nonquasiconcave, we can use Theorem 3.3 to argue the existence of equilibrium.

EXAMPLE 3.5 Consider a game with $n = 2$, $X_1 = X_2 = (0, 1)$ that is an open unit interval set, and the payoff functions are defined by

$$u_i(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in \mathbb{Q} \times \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \qquad i = 1, 2,$$

where $\mathbb{Q} = \{x \in (0, 1) : x \text{ is a rational number}\}$.

Then the game is not compact, nor quasiconcave. It is not weakly transfer quasi-continuous either (so it is not diagonally transfer continuous, better-reply secure, or weakly transfer continuous either). To see this, consider any nonequilibrium x that consists of irrational numbers. Then, for any neighborhood $\mathcal{V}_x$ of x, choosing $x' \in \mathcal{V}_x$ with $x'_1 \in \mathbb{Q}$ and $x'_2 \in \mathbb{Q}$, we have $u_1(y_1, x'_2) \leq u_1(x'_1, x'_2) = 1$ and $u_2(x'_1, y_2) \leq u_2(x'_1, x'_2) = 1$ for any $y \in X$. So the game is not weakly transfer quasi-continuous. Thus, there is no existing theorem that can be applied.

However, it is recursively diagonal transfer continuous on X. Indeed, suppose $U(y, x) > U(x, x)$ for $x = (x_1, x_2) \in X$ and $y = (y_1, y_2) \in X$. Let y^0 be any vector with rational numbers, $B = \{y^0\}$, and $\mathcal{V}_x$ be a neighborhood of x. Since $U(y, y^0) \leq U(y^0, y^0)$ for all $y \in X$, it is impossible to find any securing strategy profile y^1 such that $U(y^1, y^0) > U(y^0, y^0)$. Hence, the game is recursively diagonal transfer continuous on X with respect to B. Therefore, by Theorem 3.3, this game has a pure strategy Nash equilibrium. In fact, the set of pure strategy Nash equilibria consists of all rational numbers on $(0, 1)$.

Our main characterization results help us not only understand what is possible for a game to have or not have a pure strategy Nash equilibrium, but also develop new sufficient conditions for the existence of pure strategy Nash equilibrium. It is well known that the convexity of preferences can be substituted for transitivity of preferences in maximizing preferences of individuals. The following results show that this is true also for economic games.

DEFINITION 3.4 (Deviation Transitivity) $G = (X_i, u_i)_{i \in I}$ is said to be *deviational transitive* if $U(y^2, y^1) > U(y^1, y^1)$ and $U(y^1, y^0) > U(y^0, y^0)$ imply that $U(y^2, y^0) > U(y^0, y^0)$. That is, the upsetting dominance relation is transitive.

[12] Thanks Adam Wong for pointing out this to me.

We then we have the following result without assuming the convexity and convexity of strategy space and imposing any form of quasiconcavity.

PROPOSITION 3.1 *Suppose $G = (X_i, u_i)_{i \in I}$ is compact and deviational transitive. Then, there exists a pure strategy Nash equilibrium point if and only if G is 1-recursively diagonal transfer continuous.*

PROOF. We only need to show that, when G is deviational transitive, 1-recursive diagonal transfer continuity implies m-recursive diagonal transfer continuity for $m \geq 1$. Suppose x is not an equilibrium. Then, by 1-recursive diagonal transfer continuity, there exists a strategy profile $y^0 \in X$ and a neighborhood $\mathcal{V}_x$ of x such that $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ whenever $U(z, y^0) > U(y^0, y^0)$ for any $z \in X$.

Now, for any sequence of deviation profiles $\{y^1, \ldots, y^{m-1}, y^m\}$, if $U(y^m, y^{m-1}) > U(y^{m-1}, y^{m-1})$, $U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2})$, ..., $U(y^1, y^0) > U(y^0, y^0)$, we then have $U(y^m, y^0) > U(y^0, y^0)$ by deviation transitivity of U, and thus by 1-recursive diagonal transfer continuity, $U(y^m, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$. Since m is arbitrary, G is recursively diagonal transfer continuous. ■

Baye, Tian, and Zhou (1993) show that a game possesses a pure strategy Nash equilibrium if it is compact, convex, diagonally transfer continuous, and diagonally transfer quasi-concave.[13] Since diagonal transfer continuity implies 1-recursive diagonal transfer continuity, we immediately have the following corollary.

COROLLARY 3.1 *Suppose $G = (X_i, u_i)_{i \in I}$ is compact, deviational transitive, and diagonally transfer continuous. Then, there exists a pure strategy Nash equilibrium.*

The above corollary is a new result that uses diagonal transfer continuity as a weak notion of continuity condition, and assumes neither the convexity and convexity of strategy space nor any form of quasiconcavity.

3.2 Full Characterization By Individuals' Preferences

The aggregator function approach adopted in the previous subsection captures the basic idea of transferring upsetting relation from one strategy or agent to another strategy or agent so that the upsetting relations can be preserved locally. While this approach has a number of advantages such as, it implicitly allows for, or internalizes, the switchings of players in an upsetting relation so that the proof is relatively simpler, and it is relatively easier to check the upsetting relations due to the complementarity that secures payoffs, the aggregator function approach also has a

[13] A game $G = (X_i,\ u_i)_{i \in I}$ is diagonally transfer quasiconcave in y if, for any finite subset $Y^m = \{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $X^m = \{x^1, ..., x^m\} \subset X$ such that for any subset $\{x^{k^1}, x^{k^2}, ..., x^{k^s}\} \subset X^m$, $1 \leq s \leq m$, and any $x \in co\{x^{k^1}, x^{k^2}, ..., x^{k^s}\}$, we have $\min_{1 \leq l \leq s} U(y^{k^l}, x) \leq U(x, x)$.

number of disadvantages. First, we need to assume that the preferences of each player can be represented by a payoff function. Secondly, we need to assume that the number of players is either finite or countable. Thirdly, it is a cardinal approach, but not an ordinal approach. While monotonic transformations preserve individuals' upsetting relations unchanged, it may not be true after aggregation, i.e., with a mapping by the aggregator function, a deviation strategy profile y may no longer upset a strategy profile x after some monotonic transformation although y upsets x before the monotonic transformation. Fourthly, the aggregator function approach only reveals the total upsetting relations, it is less clear about individuals' strategic interactions, and thus it lacks a more natural game theoretical analysis.

Nevertheless, the method developed in this paper does not necessarily need to define the aggregator function U. What matters is the concept of upsetting. We can also get the full characterization results in terms of individuals' payoffs or preferences. The individual preference approach overcomes all the shortcomings above and has the following advantages: (1) Preferences may not be represented by a payoff function; (2) the set of players can be arbitrary, (3) monotonic transformations preserve individuals' upsetting relations, and (4) the analysis reveals more clear individuals' strategic interactions.

DEFINITION 3.5 A game $G = (X_i, \succcurlyeq_i)_{i \in I}$ is said to be *recursively weakly transfer quasi-continuous* if, whenever $x \in X$ is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that for every $x' \in \mathcal{V}_x$ and every finite set of deviation strategy profiles $\{y^1, y^2, \ldots, y^{m-1}, z\}$ with $(y^1_{i_1}, y^0_{-i_1}) \succ_{i_1} y^0$ for some $i_1 \in I$, $(y^2_{i_2}, y^1_{-i_2}) \succ_{i_2} y^1$ for some $i_2 \in I$, ..., $(z_{i_m}, y^{m-1}_{-i_m}) \succ_{i_m} y^{m-1}$ for some $i_m \in I$, there exists player $i \in I$ such that $(z_i, y^{m-1}_{-i_m}) \succ_i x'$.

Note that, the notion of recursive weak transfer quasi-continuity allows the switchings (transfers) of players for any particular transfer of the recursive upsetting transfers, and for any upsettings in the neighborhood $\mathcal{V}_x$.

Similarly, we can define the notions of m-recursive weak transfer quasi-continuity and recursive weak transfer quasi-continuity on X with respect to B for $B \subset X$. Note that, if a game is weakly transfer quasi-continuous, it is 1-recursively weakly transfer continuous by letting $y^0 = x$. But, the converse may not be true. Thus, weak transfer quasi-continuity is in general stronger than 1-recursive weak transfer quasi-continuity.

Now it may be worth knowing what is the relationship between recursive weak transfer quasi-continuity defined above and the recursive diagonal transfer continuity of a game defined in the previous subsection. This relationship may help us to simplify the proof, as it will be seen below. By using the upsetting binary relation $\succ$ defined in the previous section, we can define recursive diagonal transfer continuity accordingly.

DEFINITION 3.6 The "upsetting" relation $\succ$ is said to be *recursively diagonal transfer continuous* if, whenever $x \in X$ is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that $z \succ \mathcal{V}_x$ for any z that recursively upsets y^0.

LEMMA 3.1 *Let $\succ$ be the upsetting relation defined by (1). We then have*

(1) A game $G = (X_i, \succcurlyeq_i)_{i \in I}$ is recursively weakly transfer quasi-continuous on X if and only if $\succ$ is recursively diagonal transfer continuous on X.

(2) A game $G = (X_i, \succcurlyeq_i)_{i \in I}$ is weakly transfer quasi-continuous on X if and only if the upsetting relation $\succ$ is diagonally transfer continuous on X.

The proof is straightforward, and thus it is omitted here.

By Lemma 3.1, we then have the following result that fully characterizes the existence of pure strategy Nash equilibrium in qualitative games with arbitrary compact strategy spaces and general preferences.

THEOREM 3.4 *Recursive weak transfer quasi-continuity is necessary, and further under compactness of X, sufficient for a game $G = (X_i, \succcurlyeq_i)_{i \in I}$ to have a pure strategy Nash equilibrium on X.*

Recursive transfer continuity is needed for guaranteeing the existence of equilibrium. Recently, Reny (2009) provides an example in which the game is weakly transfer quasi-continuous, and thus it is 1-recursively weakly transfer quasi-continuous,[14] but it does not possess a pure strategy Nash equilibrium.

Similarly, the compactness of strategy space in Theorem 3.4 can also be removed and we have the following theorem.

THEOREM 3.5 (FULL CHARACTERIZATION THEOREM) *A game $G = (X_i, \succcurlyeq_i)_{i \in I}$ possesses a pure strategy Nash equilibrium if and only if there exists a compact set $B \subseteq X$ such that it is recursively weakly transfer quasi-continuous on X with respect to B.*

Thus, Theorem 3.5 fully characterizes the existence of equilibrium in games with an arbitrary strategy space that may be discrete, continuum, non-convex or non-compact and preferences that may not be represented by a payoff function, nontotal/nontransitive, non-convex or discontinuous, and consequently it strictly generalizes all the existing results on the existence of pure strategy Nash equilibrium such as those in Nash (1951), Debreu (1952), Nikaido and Isoda (1955), Nishimura and Friedman (1981), Dasgupta and Maskin (1986), Vives (1990), Tian and Zhou (1992, 1995), Zhou (1992), and Baye, Tian, and Zhou (1993), Reny (1999), Carmona (2005),

[14] It is also convex, compact, bounded, and quasiconcave.

Bagh and Jofre (2006), and Morgan and Scalzo (2007), Nessah and Tian (2008), Barelli and Soza (2009), Reny (2009), McLennan, Monteiro, and Tourky (2009), etc.

Similarly, we can provide some new sufficient conditions by using deviation transitivity.

DEFINITION 3.7 (Deviation Transfer Transitivity) $G = (X_i, \succcurlyeq_i)_{i \in I}$ is said to be *deviational transfer transitive* if for $y^0, y^1, y^2 \in X$, $y^2 \succ_{i_2} y^1$ for some $i_2 \in I$ and $y^1 \succ_{i_1} y^0$ for some $i_1 \in I$ imply that $y^2 \succ_{i'} y^0$ for some $i' \in I$.

Note that, like the notion of recursive weak transfer quasi-continuity, deviation transfer transitivity allows the transfers among players for each transfer in the recursive upsetting transfers.

Nessah and Tian (2008) show that a game possesses a pure strategy Nash equilibrium if it is compact, convex, wealy transfer quasi-continuous, and strongly diagonal transfer quasi-concave.[15] Similar to Proposition 3.1, replacing strong diagonal transfer quasiconcavity by deviation trnasfer transitivity, we have the following result without assuming the convexity and convexity of strategy space and imposing any form of quasiconcavity.

PROPOSITION 3.2 *Suppose $G = (X_i, \succcurlyeq_i)_{i \in I}$ is compact and deviational transfer transitive. Then, there exists a pure strategy Nash equilibrium point if and only if G is 1-recursively weakly transfer quasi-continuous.*

We now provide some sufficient conditions for a game $G = (X_i, u_i)_{i \in I}$ to be deviational transfer transitive and 1-recursively transfer quasi-continuous.

DEFINITION 3.8 $u_i : X_1 \times X_2 \to \mathbb{R}$ is diagonally monotonic if (1) $X_1 = X_2 \subset \mathbb{R}$ and (2) for each $x_{-i} \in X_{-i}$, $u_i(x_i, x_{-i})$ is either non-increasing in x_i for all $x_i < x_{-i}$ and $x_i \geqq x_{-i}$ or nondecreasing in x_i for all $x_i \leqq x_{-i}$ and $x_i > x_{-i}$.

Note that diagonal monotonicity of u_i allows discontinuity at $x_i = x_{-i}$. It is clear that u_i is diagonally monotonic if it is monotonic. A game $G = (X_i,\ u_i)_{i \in I}$ is diagonally monotonic if $u_i : X_1 \times X_2 \to \mathbb{R}$ is diagonally monotonic for $i = 1, 2$.

LEMMA 3.2 *If a game $G = (X_i,\ u_i)_{i \in I}$ is diagonally monotonic, it is deviational transitive.*

PROOF. We only need to show the case where u_i is nondecreasing in x_i. The proof of the case where u_i is non-increasing in x_i is similar.

We need to show that $u_i(z_i, y_{-i}) > u_i(y)$ and $u_i(y_i, x_{-i}) > u_i(x)$ imply $u_i(z_i, x_{-i}) > u_i(x)$. Indeed, by monotonicity of u_i, we have $z_i > y_i$ when $u_i(z_i, y_{-i}) > u_i(y)$ and $y_i > x_i$ when

[15] A game $G = (X_i, u_i)_{i \in I}$ is said to be *strongly diagonal transfer quasiconcave* if for any finite subset $\{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $\{x^1, ..., x^m\} \subset X$ such that for any subset $\{x^{k^1}, x^{k^2}, ..., x^{k^s}\} \subset X^m$, $1 \leq s \leq m$, and any $x \in co\{x^{k^1}, x^{k^2}, ..., x^{k^s}\}$, there exists $y^h \in \{y^{k^1}, ..., y^{k^s}\}$ satisfying $u_i(y_i^{k^h}, x_{-i}) \leq u_i(x)$ for all $i \in I$.

$u_i(y_i, x_{-i}) > u_i(x)$, and thus we have $z_i > y_i > x_i$. Then, by monotonicity of u_i, we have $u_i(z_i, x_{-i}) \geq u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$, which means ψ is deviational transitive. ■

Since better-reply security and its strength[16] imply weak transfer quasi-continuity which in turn implies 1-recursive weak transfer quasi-continuity, by Proposition 3.2 and Lemma 3.2, we can have the following corollary.

COROLLARY 3.2 *Suppose $G = (X_i, \succcurlyeq_i)_{i \in I}$ is compact, and either deviational transfer transitive or diagonally monotonic. Then, $G = (X_i, \succcurlyeq_i)_{i \in I}$ possesses a pure strategy Nash equilibrium if any of the following conditions is satisfied:*

(a) It is weakly transfer quasi-continuous.

(b) It is better-reply secure.

(c) It is payoff secure and reciprocally upper semi-continuous.

(d) It is payoff secure and weakly reciprocal upper semi-continuous.

Again, these are new results that use weak notions of continuity conditions and assumes neither the convexity and convexity of strategy space nor any form of quasiconcavity.

EXAMPLE 3.6 (BAYE AND KOVENOCK; BAYE, TIAN, AND ZHOU) Consider the two-player quasi-symmetric game studied by Baye and Kovenock (1993), and Baye, Tian, and Zhou (1993). Two duopolists have zero costs and set prices (p_1, p_2) on $Z = [0, T] \times [0, T]$. The payoff functions are (for $0 < c < T$):

$$u_i(p_1, p_2) = \begin{cases} p_i & \text{if } p_i \leq p_{-i} \\ p_i - c & \text{otherwise} \end{cases} .$$

One can interpret the game as a duopoly in which each firm has committed to pay brand loyal consumers a penalty of c if the other firm beats its price.[17] These payoffs are neither quasiconcave nor continuous. However, the game is diagonally monotonic and weakly transfer quasi-continuous, and thus it is 1-recursively diagonal transfer continuous. Thus, by Corollary 4.1, this game possesses a symmetric pure strategy equilibrium.

4 Full Characterization of Symmetric Pure Strategy Nash Equilibria

The techniques developed in the previous section can be used to fully characterize the existence of symmetric pure strategy Nash equilibrium. Throughout this section, we assume that the strat-

[16] Reny (1999) shows that a game $G = (X_i, u_i)_{i \in I}$ is better-reply secure if it is payoff secure and reciprocally upper semi-continuous. Bagh and Jofre (2006) further show that $G = (X_i, u_i)_{i \in I}$ is better-reply secure if it is payoff secure and weakly reciprocal upper semi-continuous.

[17] See Baye and Kovenock (1993) for an alternative formulation with both brand loyal and price conscious consumers, whereby a firm commits to pay a penalty if it does not provide the best price in the market.

egy spaces for all players are the same. As such, let $X_0 = X_1 = \ldots = X_n$. If in addition, $u_1(y, x, \ldots, x) = u_2(x, y, x, \ldots, x) = \ldots, u_n(x, \ldots, x, y)$ for all $x, y \in X$, we say that $G = (X_i, u_i)_{i \in I}$ is a quasi-symmetric game.

DEFINITION 4.1 A Nash equilibrium $(x_1^*, \ldots, x_n^*)$ of a game G is said to be *symmetric* if $x_1^* = \ldots = x_n^*$.

For convenience, we denote, for each player i, and for all $x, y \in X_0$, $u_i(x, \ldots, y, \ldots, x)$ the function u_i evaluated at the strategy in which player i chooses y and all others choose x.

Define a quasi-symmetric function $\psi : X_0 \times X_0 \to \mathbb{R}$ by

$$\psi(y, x) = u_i(x, \ldots, y, \ldots, x). \tag{5}$$

Since G is quasi-symmetric, x^* is a symmetric pure strategy Nash equilibrium if and only if $\psi(y, x^*) \leq \psi(x^*, x^*)$ for all $y \in X_i$.

DEFINITION 4.2 $\psi : X_0 \times X_0 \to \mathbb{R}$ is said to be *recursively diagonal transfer continuous* if, whenever $x \in X$ is not equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that $\psi(z, \mathcal{V}_x) > \psi(\mathcal{V}_x, \mathcal{V}_x)$ for any z that recursively upsets y^0.

We then have the following theorem.

THEOREM 4.1 *Suppose a game $G = (X_i,\ u_i)_{i \in I}$ is quasi-symmetric and compact. Then it possesses a symmetric pure strategy Nash equilibrium if and only if $\psi : X_0 \times X_0 \to \mathbb{R}$ is recursively diagonal transfer continuous on X.*

PROOF. The proof is the same as that of Theorems 3.1 and 3.2 provided U is replaced by ψ, thus it is omitted here. ■

Theorem 4.1 strictly generalizes all the existing results on the existence of symmetric pure strategy Nash equilibrium such as those in Reny (1999).

Similar to Theorem 3.3, we can get a full characterization result on the existence of symmetric pure strategy Nash equilibrium for arbitrary (possibly) noncompact strategy space.

EXAMPLE 4.1 (HENDRICKS AND WILSON) Consider the concession quasi-symmetric game between two players studied by Hendricks and Wilson (1983), Simon (1987), and Reny (1999). The players must choose a time $x_1, x_2 \in [0, 1]$ to quit the game. The player who quits last wins, although conditional on winning, quitting earlier is preferred. If both players quit at the same time, the unit prize is divided evenly between them. Then payoffs are:

$$u_i(x_1, x_2) = \begin{cases} -x_i, & \text{if } x_i < x_{-i} \\ 1/2 - x_i, & \text{if } x_i = x_{-i} \\ 1 - x_i, & \text{if } x_i > x_{-i} \end{cases}.$$

Note that the payoffs are not quasiconcave (nor are they quasiconcave along the diagonal of the unit square) although U is diagonally transfer continuous by Proposition 2.(e) in Baye, Tian, and Zhou (1993). We now show that ψ is not recursively diagonal transfer continuous, and thus the game does not possess a symmetric pure strategy equilibrium. To see this, consider $x = 0$. It is clear that $\psi(y, x) = u_i(y, 0) > u_i(0, 0)$ implies that $0 < y < 1/2$. We then cannot find any $y^0 \in X_0$ and neighborhood $\mathcal{V}_x$ of x such that $\psi(z, x') > \psi(x', x')$ for every deviation profile z that is upset by y^0 for all $x' \in \mathcal{V}_x$. We show this by considering two cases.

Case 1. $y^0 \neq 0$. Then, for any neighborhood $\mathcal{V}_0$ of 0, choose a strategy profile $z \in [0, 1]$ and a strategy profile $x' \in \mathcal{V}_0$ such that $\max\{1/2 + \epsilon, y^0\} < z < 1/2 + y^0$ and $x' < \epsilon$, where $0 < \epsilon < \min\{1/2, y^0\}$. Then, by $z > y^0$ and $1 - z > 1/2 - y^0$, we have $\psi(z, y^0) > \psi(y^0, y^0)$. However, since $z > x'$ and $1/2 + x' < 1/2 + \epsilon < z$, we have $1 - z < 1/2 - x'$, and consequently $\psi(z, x') < \psi(x', x')$.

Case 2. $y^0 = 0$. Note that $\psi(z, y^0) > \psi(y^0, y^0)$ if and only if $0 < z < 1/2$. Then, for any neighborhood $\mathcal{V}_0$ of 0, choosing a positive number ϵ such that $(\epsilon/2, \epsilon) \subset \mathcal{V}_0$, $z = \epsilon/2$ and a strategy profile $x' \in \mathcal{V}_0$ such that $x' \in (\epsilon/2, \epsilon)$, we have $\psi(z, y^0) > \psi(y^0, y^0)$ but $\psi(z, x') = -z < 1/2 - x' = \psi(x', x')$.

Thus, we cannot find any $y^0 \in [0, 1]$ and any neighborhood $\mathcal{V}_0$ of 0 such that $\psi(z, x') > \psi(x', x')$ for every deviation profile z that is upset by y^0 for all $x' \in \mathcal{V}_x$. Therefore, ψ is not recursively diagonal transfer continuous on X_0, and thus, by Theorem 4.1, there is no symmetric pure strategy Nash equilibrium on X.

EXAMPLE 4.2 (BAGH AND JOFRE) The following two-person concession quasi-symmetric game on the unit square considered by Bagh and Jofre (2006) is a special case of a class of timing games on the unit square considered by Reny (1999). The payoffs are:

$$u_i(x_1, x_2) = \begin{cases} 10, & \text{if } x_i < x_{-i} \\ 1, & \text{if } x_i = x_{-i} < 0.5 \\ 0, & \text{if } x_i = x_{-i} \geq 0.5 \\ -10, & \text{if } x_i > x_{-i} \end{cases}.$$

Note that the payoffs are not quasiconcave (nor are they quasiconcave along the diagonal of the unit square). We now show that ψ is recursively diagonal transfer continuous, and thus the game possesses a symmetric pure strategy equilibrium. Indeed, let $\psi(y, x) = u_i(y, x)$. Suppose $\psi(y, x) > \psi(x, x)$ for $x \in X_0$ and $y \in X_0$. Let $y^0 = 0$ and $\mathcal{V}_x$ be a neighborhood of x. It is clear that $\psi(y, y^0) \leq \psi(y^0, y^0)$ for all $y \in X$, and thus it is impossible to find any securing strategy profile y^1 such that $\psi(y^1, y^0) > \psi(y^0, y^0)$. Hence, the recursive diagonal transfer continuity holds, and thus by Theorem 4.1, this game has a pure strategy Nash equilibrium.

Besides, since all the games in Examples 3.1-3.5 are quasisymmetric, it is even easier to show the existence/nonexistence of pure strategy (symmetric) Nash equilibrium by working on a single

payoff function ψ, instead of the aggregate payoff function U that is the sum of individual payoff functions.

Similar to Proposition 3.1, we have the following proposition.

PROPOSITION 4.1 *Suppose a game $G = (X_i,\ u_i)_{i\in I}$ is quasi-symmetric, compact, and deviational transitive. Then it possesses a pure strategy symmetric Nash equilibrium if and only if $\psi(x, y)$ defined by (4) is 1-recursively diagonal transfer continuous on X.*

We now provide some sufficient conditions for a game $G = (X_i, u_i)_{i\in I}$ to be deviational transitive and 1-recursively diagonal transfer continuous.

Similarly, we can show that diagonal better-reply security and its strength imply diagonal transfer continuity which in turn implies 1-recursively diagonal transfer continuity.[18] The following shows that under diagonal monotonicity, upper semi-continuity also implies 1-recursively diagonal transfer continuity.

DEFINITION 4.3 A quasi-symmetric game $G = (X_i,\ u_i)_{i\in I}$ with $X_i \subset \mathbb{R}$ is diagonally monotonic if either (i) for each $\bar{x} \in X_0$, $u_i(\bar{x}, \ldots, x, \ldots, \bar{x})$ is decreasing in x for all $x < \bar{x}$ and $x \geqq \bar{x}$ or (ii) for each $\bar{x} \in X_0$, $u_i(\bar{x}, \ldots, x, \ldots, \bar{x})$ is increasing in x for all $x \leqq \bar{x}$ and $x > \bar{x}$.

LEMMA 4.1 *Suppose X_i is a subset of $\mathbb{R}$. If a quasi-symmetric game $G = (X_i,\ u_i)_{i\in I}$ is diagonally monotonic, ψ is deviational transitive.*

The proof is the same as that of Lemma 3.2, and omitted here.

LEMMA 4.2 *Suppose X_i is a subset of $\mathbb{R}$ and a game $G = (X_i,\ u_i)_{i\in I}$ is diagonally monotonic and upper semi-continuous on X. Then it is 1-recursively diagonal transfer continuous in x.*

PROOF. We only need to prove the case where ψ is increasing in x. The proof of the case where ψ is decreasing in x is similar.

Suppose $\psi(y, x) > \psi(x, x)$ for $x, y \in X_0$. We need to show that there exists a point $y^0 \in X_0$ and a neighborhood $\mathcal{V}_x$ of x such that $\psi(z, \mathcal{V}_x) > \psi(\mathcal{V}_x, \mathcal{V}_x)$ whenever $\psi(z, y^0) > \psi(y^0, y^0)$. Indeed, since $\psi(y, x) > \psi(x, x)$, we have $y > x$ by diagonal monotonicity of ψ. Let $y^0 = x + \delta < y$ for some positive $\delta > 0$. We have $\psi(y^0, x) > \psi(x, x)$ by diagonal monotonicity of ψ. Then, by upper semi-continuity, there is a neighborhood $\mathcal{V}_x = \{x' \in X_0 : |x' - x| < \epsilon\}$ such that $\psi(y^0, x) > \psi(x', x)$ for all $x' \in \mathcal{V}_x$. Since $u_i(\bar{x}, \ldots, x, \ldots, \bar{x})$ is increasing in x on $X_0 \setminus \bar{x}$ for all $\bar{x} \in X_0$, we particularly have $\psi(y^0, x') > \psi(x', x')$ for all $\bar{x} = x' \in \mathcal{V}_x$. Thus, whenever $\psi(z, y^0) > \psi(y^0, y^0)$, we have $z > y^0$ by diagonal monotonicity of ψ, and therefore, we have

[18] $G = (X_i,\ u_i)_{i\in I}$ diagonally better-reply secure and diagonally payoff secure if ψ is better-reply secure and diagonally payoff secure, respectively (cf. Reny (1999)). Reny (1999) also shows that diagonally payoff secure and upper semi-continuous imply diagonal better-reply security.

$\psi(z, x') > \psi(y^0, x') > \psi(x', x')$ for all $x' \in \mathcal{V}_x$, which means ψ is 1-recursively diagonal transfer continuous in x. ■

Then, by Proposition 4.1 and Lemmas 4.1 - 4.2, we can have the following corollary.

COROLLARY 4.1 *Suppose that a game $G = (X_i,\ u_i)_{i \in I}$ is quasi-symmetric, compact, diagonally monotonic, and X_i is a subset of $\mathbb{R}$. Then it possesses a pure strategy symmetric Nash equilibrium if any of the following conditions is satisfied:*

(a) It is diagonally better-reply secure.

(b) It is diagonally transfer continuous.

(c) It is upper semi-continuous.

5 Economic Applications

The approach and main results developed in the paper can also allow us to ascertain the existence of equilibria in important classes of economic games or optimization problems. As illustrations, in this section, we show how they can be employed to fully characterize the existence of competitive (or Walrasian) equilibrium for economies with excess demand functions, stable matchings, and greatest and maximal elements of weak and strict preferences.

5.1 Existence of Competitive Equilibrium

One of the great achievements of economic theory in the last sixty years is the general equilibrium theory. The proof of the existence of a competitive equilibrium is generally considered one of the most important and robust results of economic theory. There are many different ways of establishing the existence of competitive equilibria, including the "excess demand approach" by showing that there is a price at which excess demand can be non-positive.

The significance of such an approach lies partly in the fact that demand and/or supply may not be continuous or even not be necessarily derived from profit maximizing behavior of price taking firms, but is determined by prices in completely different ways. It is well known that Walrasian equilibrium precludes the existence of an equilibrium in the presence of increasing returns to scale and assumes price-taking and profit-maximizing behavior. Some other alternative pricing rules then have been proposed such as loss-free, average cost, marginal cost, voluntary trading, and quantity-taking pricing rules in the presence of increasing returns to scale or more general types of non-convexities— cf. Beato (1982), Bonnisseau and Cornet (1990), Quinzii(1992), Tian (2009) and the references therein. There is a large literature on the existence results using the excess demand approach, such as those in Gale (1955), Nikaido (1956, 1968, 1970), Debreu (1970, 1974, 1982), Sonnenschein (1972, 1973), Hildenbrand (1974), Hildenbrand and Kirman (1975),

Grandmont (1977), Neuefeind (1980), Aliprantis and Brown (1983), Hüsseinov (1999), Momi (2003), Quah (2008), etc.

In the following, we provide a complete solution to the existence of competitive equilibrium in economies with general excess demand functions,[19] in which commodities may be indivisible and excess demand functions may be discontinuous or do not have any structure except Walras' law. We introduce a condition, called *recursive transfer lower semi-continuity*, which is necessary and sufficient for the existence of general equilibrium in such economies. Thus, our results strictly generalize all the existing results on the existence of equilibrium in economies with excess demand functions.

Let Δ be the closed $L-1$ dimensional unit simplex defined by

$$\Delta = \{p \in \Re^L_+ : \sum_{l=1}^{L} p^l = 1\}, \tag{6}$$

and let $\hat{z}(\cdot) : \Delta \to \mathbb{R}^L \cup \{\pm\infty\}$ denote the excess demand function of some economy. A very important property of excess demand function is Walras' law, which can take one of the following three forms:

(1) the strong form of Walras' law given by

$$p \cdot \hat{z}(p) = 0 \quad \text{for all } p \in \Delta;$$

(2) the weak form of Walras' law given by

$$p \cdot \hat{z}(p) \leq 0 \quad \text{for all } p \in \Delta;$$

(3) the interior form of Walras' law given by

$$p \cdot \hat{z}(p) = 0 \quad \text{for all } p \in \text{int } \Delta,$$

where int Δ denotes the set of interior points of Δ.

A price vector p^* is a *competitive or Walrasian* equilibrium if $\hat{z}(p^*) \leq 0$.

The equilibrium price problem is to find a price vector p which clears the markets for all commodities (i.e., the excess demand functions $\hat{z}(p) \leq 0$ for the free disposal equilibrium price or $\hat{z}(p) = 0$) under the assumption of Walras' law.

We say that price p *upsets* price q if p gives a higher value to q's excess demand, i.e., $p \cdot \hat{z}(q) > q \cdot \hat{z}(q)$.

[19] In the case of strictly convex preferences and production sets, we obtain excess demand functions rather than correspondences.

DEFINITION 5.1 (Recursive Upset Pricing) Let $\hat{z}(\cdot) : \Delta \to \mathbb{R}^L$ be an excess demand function. We say that a non-equilibrium price vector $p^0 \in \Delta$ is *recursively upset by* $p \in \Delta$ if there exists a finite set of price vectors $\{p^1, p^2, \ldots, p\}$ such that $p^1 \cdot \hat{z}(p^0) > 0$, $p^2 \cdot \hat{z}(p^1) > 0$, ..., $p \cdot \hat{z}(p^{m-1}) > 0$.

In words, a non-equilibrium price vector p^0 is recursively upset by p means that there exist finite upsetting price vectors $p^1, p^2, \ldots, p^m$ with $p^m = p$ such that p^0's excess demand is not affordable at p^1, p^1's excess demand is not affordable at p^2, and p^{m-1}'s excess demand is not affordable at p^m. This implies that p^0 is upset by p^1, p^1 is upset by p^2, ..., p^{m-1} is upset by p.

DEFINITION 5.2 (Recursive Transfer Lower Semi-Continuity) An excess demand function $\hat{z}(\cdot) : \Delta \to \mathbb{R}^L$ is said to be *recursively transfer lower semi-continuous* on Δ if, whenever $q \in \Delta$ is not a competitive equilibrium price vector, there exists some price $p^0 \in \Delta$ (possibly $p^0 = q$) and a neighborhood $\mathcal{V}_q$ such that $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for any p that recursively upsets p^0.

Roughly speaking, recursive transfer lower semi-continuity of $\hat{z}(\cdot)$ means that, whenever q is not a competitive equilibrium price vector, there exists another non-competitive equilibrium price vector p^0 such that all excess demands in some neighborhood of q are not affordable at any price vector p that recursively upsets p^0. This implies that, if a competitive equilibrium fails to exist, then there is some non-equilibrium price vector q such that for every other price vector p^0 and every neighborhood of q, excess demand of some price vector q' in the neighborhood becomes affordable at price vector p that recursively upsets p^0.

REMARK 5.1 While continuity does not imply nor is implied by recursive diagonal transfer continuity, recursive transfer lower semi-continuity is weaker than lower semi-continuity. Indeed, when $\hat{z}(\cdot)$ is lower semi-continuous, $p \cdot \hat{z}(\cdot)$ is also lower semi-continuous for any nonnegative vector p, and thus we have $p^m \cdot \hat{z}(q') > 0$ for all $q' \in N(q)$ and $p \in \Delta$.

We then have the following theorem on the existence of competitive equilibrium in economies that have single-valued excess demand functions.

THEOREM 5.1 *Suppose an excess demand function $\hat{z}(\cdot) : \Delta \to \mathbb{R}^L$ satisfies either the strong or the weak form of Walras' law. Then there exists a competitive price equilibrium $p^* \in \Delta$ if and only if $\hat{z}(\cdot)$ is recursively transfer lower semi-continuous on Δ.*

PROOF. *Sufficiency* ($\Leftarrow$). Define a function $\phi : \Delta \times \Delta \to \Re$ by $\phi(p, q) = p \cdot \hat{z}(q)$ for $p, q \in \Delta$. If q is not a competitive equilibrium, there is a $p \in \Delta$ such that $p \cdot \hat{z}(q) > 0$. Since $q \cdot \hat{z}(q) \leq 0$ for all $q \in \Delta$ by Walras' law, we have $\phi(p, q) > \phi(q, q)$ for any chosen pair $p, q \in \Delta$. Then, by recursive transfer lower semi-continuity of $\hat{z}(\cdot)$, ϕ is recursively diagonal transfer continuous on

Δ.[20] Thus, by Theorem 3.2, there exists $p^* \in \Delta$ such that $p \cdot \hat{z}(p^*) = \phi(p, p^*) \leq \phi(p^*, p^*) \leq 0$ for all $p \in \Delta$. Letting $p^1 = (1, 0, \ldots, 0)$, $p^2 = (0, 1, 0, \ldots, 0)$, and $p^L = (0, 0, \ldots, 0, 1)$, we have $\hat{z}^l(p^*) \leq 0$ for $l = 1, \ldots, L$ and thus p^* is a competitive price equilibrium.

Necessity ($\Rightarrow$). Suppose p^* is a competitive price equilibrium and $p \cdot \hat{z}(q) > 0$ for $q, p \in \Delta$. Let $p^0 = p^*$ and $N(q)$ be a neighborhood of q. Since $p \cdot \hat{z}(p^*) \leq 0$ for all $p \in \Delta$, it is impossible to find any sequence of finite price vectors $\{p^1, p^2, \ldots, p^m\}$ such that $p^1 \cdot \hat{z}(p^0) > 0, p^2 \cdot \hat{z}(p^1) > 0, \ldots, p^m \cdot \hat{z}(p^{m-1}) > 0$. Hence, the recursive transfer lower semi-continuity holds trivially. ■

Theorem 5.1 assumes that the excess demand function is well defined for all prices in the closed unit simplex Δ, including zero prices. However, when preferences are strictly monotone, excess demand functions are not well defined on the boundary of Δ. Then, some boundary condition has been used to show the existence of competitive equilibrium in the case of an open set of price systems for which excess demand is defined —cf. Neuefeind (1980). In this case, we naturally cannot use Theorem 5.1 to fully characterize the existence of competitive equilibrium.

Nevertheless, like Theorems 3.1 and 3.2, Theorem 5.1 can be extended to the case of any set, especially the positive price open set, of price systems for which excess demand is defined. To do so, we introduce the following version of recursive transfer lower semi-continuity.

DEFINITION 5.3 Let D be a subset of $\operatorname{int}\Delta$. An excess demand function $\hat{z}(\cdot) : \operatorname{int}\Delta \to \mathbb{R}^L$ is said to be *recursively transfer lower semi-continuous* on int Δ with respect to D if, whenever $q \in \operatorname{int}\Delta$ is not a competitive equilibrium price vector, there exists some price $p^0 \in \operatorname{int}\Delta$ (possibly $p^0 = q$) and a neighborhood $\mathcal{V}_q$ such that (1) whenever p^0 is upset by a price vector in $\operatorname{int}\Delta \setminus D$, it is upset by a price vector in D, and (2) $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for any $p \in D$ that recursively upsets p^0.

Now we have the following theorem that fully characterizes the existence of competitive equilibrium in economies with possibly indivisible commodity spaces and discontinuous excess demand functions.

THEOREM 5.2 *Suppose an excess demand function $\hat{z}(\cdot) : \operatorname{int}\Delta \to \mathbb{R}^L$ satisfies Walras' law: $p \cdot \hat{z}(p) = 0$ for all $p \in \operatorname{int}\Delta$. Then there is a competitive price equilibrium $p^* \in \operatorname{int}\Delta$ if and only if there exists a compact subset $D \subseteq \operatorname{int}\Delta$ such that $\hat{z}(\cdot)$ is recursively transfer lower semi-continuous on $\operatorname{int}\Delta$ with respect to D.*

PROOF. The same as the proof of Theorem 3.3, we can show that there exists a price system $p^* \in \Delta$ such that $p \cdot \hat{z}(p^*) = \phi(p, p^*) \leq \phi(p^*, p^*) \leq 0$ for all $p \in \operatorname{int}\Delta$. We want to show that p^* is a competitive price equilibrium. Note that $\operatorname{int}\Delta$ is open and D is a compact subset of $\operatorname{int}\Delta$.

[20] The reverse may not be true under the weak form of Walras' law. However, when the strong form of Walras' law holds, $\hat{z}$ is recursively transfer lower semi-continuous if and only if ϕ is recursively diagonal transfer continuous.

One can always find a sequence of price vector $\{q_n^l\} \subseteq \text{int}\,\Delta \setminus D$ such that $q_n^l \to p^l$ as $n \to \infty$, where $p^l = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the unit vector that has only one argument - the lth argument - with value 1 and others with value 0. Since $p \cdot \hat{z}(q)$ is continuous in p, we have $\hat{z}^l(p^*) \leq 0$ for $l = 1, \ldots, L$ and thus p^* is a competitive price equilibrium. ■

Theorem 5.2 then strictly generalizes all the existing results on the existence of competitive equilibrium in economies with single-valued excess demand functions, such as those in Gale (1955), Nikaido (1956), Debreu (1970, 1982), Hildenbrand (1974), Hildenbrand and Kirman (1975), Grandmont (1977), Neuefeind (1980), Aliprantis and Brown (1983), Hüsseinov (1999), Momi(2003), and Quah (2008).

5.2 Full Characterization on Stable Matchings

Our main results can be also used to fully characterize the existence of stable matchings by giving a necessary and sufficient condition for general economic environments.[21] We generalize the existing results on the existence of stable matchings to allow any number of agents that could be finite, countable or even uncountable. This gives a way to characterize the essence of equilibrium with a finite choice set. The method adopted here can also be used to study other matching problems.

The stable matchings (roommates) problem involves pairing-up a set of agents. Gale and Shapley (1962) showed that stable matchings may not exist in roommates problem while they always exist in the marriage problem.[22] When agents' preferences are strict, Tan (1991) provides a necessary and sufficient condition for the existence of stable roommate matchings for an economy with finite agents. In the general case where agents' preferences are allowed to be weak, Chung (2000) identifies a condition called "no odd rings" that is sufficient for the existence of stable roommate matchings in the weak preferences case. However, Chung's result is only sufficient but not necessary. In this section, we fully characterize the existence of stable matchings by providing a necessary and sufficient condition in an economy with an arbitrary number of agents and general preferences.

The notion used below is largely taken from Chung (2000). Let I denote the set of agents. For simplicity, we assume the set I is finite. The case of infinite or even uncountable agents can be similarly analyzed as in the precious sections. Each agent i has a weak preference $\succcurlyeq_i$ which may be nontotal or nontransitive over I. If agent x weakly prefers y to z, we write $y \succcurlyeq_x z$. If agent x strictly prefers y to z, we write $y \succ_x z$. If agent x weakly prefers y to himself (i.e., $y \succcurlyeq_x x$), x weakly prefers having y as roommate to living alone. If x strictly prefers living alone to y (i.e., $x \succ_x y$), y is "unacceptable" to x as a roommate. A preference profile is denoted by

[21] I greatly thank Kim Sau Chung for raising this issue to me and making constructive suggestions.

[22] In "the marriage problem," agents can be divided into two types. An agent of one type can only be matched to an agent of another type, or can remain single.

$\succcurlyeq = (\succcurlyeq_1, \succcurlyeq_2, \ldots, \succcurlyeq_i)$.

A "matching" μ is a one-to-one mapping from I onto itself such that for all $\{x, y\} \subset I$, $\mu(x) = y$ if and only if $\mu(y) = x$. If $\mu(x) = x$, x is "single" under matching μ. Denote by M the set of all matchings, with μ and ν as typical elements, which is endowed with the discrete topology.

DEFINITION 5.4 We say that a matching $\nu \in M$ is "stable" if (1) $\nu(x) \succcurlyeq_x x$ for every $x \in I$ and (2) there does not exist any pair of agents $\{x, y\} \subset I$ such that $x \succ_y \nu(y)$ and $y \succ_x \nu(x)$.

DEFINITION 5.5 We say that a matching μ *upsets* another matching ν, denoted by $\mu P \nu$, if there exist a pair of agents x and y (possibly $x = y$) such that (1) $x = \mu(y)$ and $y = \mu(x)$, and (2) $x \succ_y \nu(y)$ and $y \succ_x \nu(x)$.

In words, the matching μ upsets the matching ν if there exists a pair of agents x and y who form a couple under matching μ, and both strictly prefer this (new) μ-partner to his (old) ν-partner. Observe that strict preference implies that x and y cannot possibly be partners under matching ν.

Thus, P defines a "upsetting relation" on M. Denote by R the completion of P (see Footnote 5).

We immediately have the following lemma

LEMMA 5.1 *A matching $\nu \in M$ is a stable matching if and only if there is no other matching $\mu \in M$ such that $\mu P \nu$.*

DEFINITION 5.6 We say that μ recursively upsets μ^0 if for any subset $\{\mu^0, \mu^1, \ldots, \mu^{m-1}, \mu\} \subset I$, $\mu P \mu^{m-1} \ldots \mu^1 P \mu^0$ imply that $\mu P \mu^0$.

DEFINITION 5.7 We say that the upsetting relation P is recursively transfer transitive on M if, whenever $\mu' P \nu$ for $\mu', \nu \in M$, there exists a matching μ^0 (possibly $\mu^0 = \nu$) such that $\mu \, P \, \nu$ for any μ that recursively upsets μ^0.

We then have the following theorem that fully characterizes the existence of stale matchings in general economies.

THEOREM 5.3 *A stable matching exists if and only if P is recursively transfer transitive on I.*

PROOF. Since I is a finite set, so is M. Then, in this finite choice set, the notion of recursive transfer transitivity is the same as that of recursive diagonal transfer continuity of R. Thus, by Theorems 3.1 and 3.2, we know the existence of greatest element of R. ■

The above results generalizes all the existing results on the existence of stable matchings to allow an arbitrary number of agents and general preference profile $\succcurlyeq$ that may be nontotal or nontransitive

5.3 Full Characterization of Preference Maximization

The notion of preference relations or utility functions is a foundational concept in economics, particularly, in consumer theory.

Tian (1992, 1993), Tian and Zhou (1992, 1995), and Zhou and Tian (1992) have developed the basic transfer method to study the maximization of binary relations that may be nontotal or non-transitive. Especially Zhou and Tian (1992) develop three types of transfers: transfer continuities, transfer convexities, and transfer transitivities to systematically study the maximization of various binary preferences relations. Various notions of transfer continuities, transfer convexities and transfer transitivities provide complete solutions to the question of the existence of maximal elements for complete preorders and interval orders (cf. Tian (1993) and Tian and Zhou (1995)). In this subsection we fully characterizes the existence of greatest and maximal elements of weak and strict preferences that may not be total, transitive, or convex. By the duality of weak preferences R and strict preferences P, again, we only need to consider the existence of greatest elements of weak preferences.

Let Z be a nonempty subset of a topological space E, and R the weak binary relation defined on Z. Let P denote the asymmetric part of R, that is, for any $x, y \in Z$, $y\,P\,x$ if and only if $\neg\, x\,R\,y$.

DEFINITION 5.8 (Recursive Transfer Upper Continuity) A weak preference relation R defined on Z is said to be *recursively transfer upper continuous* on Z if, whenever x is not a greatest element, there exists an element z (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ such that $z\,P\,\mathcal{V}_x$ for any z that recursively upsets y^0.

By Theorem 3.3 and Lemma 3.1, we then have the following result that fully characterizes the existence of greatest elements of the weak binary relation defined on arbitrary set Z.

THEOREM 5.4 *Let Z be a nonempty subset of a topological space E and let R defined on Z be a weak binary relation. Then R has a greatest element on Z if and only if there exists a compact set $B \subseteq X$ such that R is recursively transfer upper continuous on X with respect to B.*

As a consequence of Theorem 5.4, we have

COROLLARY 5.1 *Let Z be a nonempty subset of a topological space E and let R defined on Z be a weak binary relation. Then recursive transfer upper continuity is necessary, and further under compactness of Z, sufficient for the weak binary relation R to have a greatest element on Z.*

Theorem 5.4 and Corollary 5.1 then strictly generalizes all the existing results on the existence of greatest element of weak preferences and maximal element of strict preferences, such as those

in Schmeidler (1969), Sonnenschein (1971), Shafer (1974), Yu (1974), Shafer and Sonnenschein (1975), Walker (1977), Bergstrom (1975), Bergstrom and Rader (1976), Borwein (1983), Yannelis and Prabhakar (1983), Luc (1989), Campbell and Walker (1990), Tian (1992, 1993), Tian and Zhou (1992, 1995), and Zhou and Tian (1992).

6 Conclusion

The existing results only give sufficient conditions for the existence of equilibrium, and no complete solution to the question of the existence of equilibrium in general games has been given in the literature. This paper fills this gap by providing a full characterization of equilibrium in general games. We fully characterize the existence of pure strategy Nash equilibrium in games with arbitrary number of players that may be finite, infinite, or even uncountable; arbitrary general topological strategy spaces that may be discrete, continuum, non-compact or non-convex; payoffs (resp. preferences) that may be discontinuous or do not have any form of quasi-concavity (resp. nontotal, nontransitive, discontinuous, nonconvex, or nonmonotonic).

We establish a condition, called *recursive diagonal transfer continuity* for aggregate payoffs or *recursive weak transfer quasi-continuity* for individuals' preferences, which is necessary, and further, under compactness, sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs. We also provide a stronger version of recursive diagonal transfer continuity (or recursive weak transfer quasi-continuity) that is necessary and sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary (possibly noncompact) strategy spaces and payoff functions. As such, it strictly generalizes all the existing theorems on the existence of pure strategy Nash equilibrium. Our recursive transfer continuities also permit full characterization results on the existence of symmetric pure strategy, mixed strategy Nash, and Bayesian Nash equilibria in games with general strategy spaces and payoffs.

The approach and main results developed in the paper can also allow us to get new sufficient conditions for the existence of equilibrium and to ascertain the existence of equilibria in important classes of economic games. As illustrations, we provide several new sufficient conditions for the existence of equilibrium without imposing any form of quasiconcavity and also show how they can be employed to fully characterize the existence of competitive equilibrium for economies with excess demand functions, stable matchings, and greatest and maximal elements of weak and strict preferences. Our results can help us to understand the essence of Nash equilibrium. We use many known economic examples to illustrate that it is useful to employ recursive diagonal transfer continuity to check the existence of equilibrium, especially the nonexistence of pure strategy Nash equilibrium. Moreover, we introduce new techniques and methods for possibly studying other optimization problems and extending some basic mathematics results such as fixed point theorems, variational inequalities, etc. The method of proof employed to obtain our main results is also new.

Moreover, a remarkable advantage of the proof is that it is simple and elementary without using advanced math.

We end the paper by remarking that characterization results are mainly for the purpose of identifying whether or not a game has an equilibrium. Nessah and Tian (2008) develop some very weak sufficient conditions that are relatively easy to check and generalize most of the existing results for the existence of equilibrium in discontinuous games. A potential future work may be attempted to find more sufficient conditions for recursive diagonal transfer continuity.

Appendix

Appendix A: Full Characterization of Mixed Strategy Nash Equilibria

Fully characterization of the existence of mixed strategy Nash equilibrium can be obtained as corollaries of the pure strategy existence results derived in the previous sections. We assume throughout this section that each u_i is both bounded and measurable, and X_i is a compact Hausdorff space so we call $G = (X_i, u_i)_{i \in I}$ a compact, Hausdorff game. Consequently, if M_i denotes the set of (regular, countably additive) probability measures on the Borel subsets of X_i , M_i is compact in the weak* topology. Extend each u_i to $\prod_{i \in I} M_i$ by defining $u_i(\mu) = \int_X u_i(x) d\mu$ for all $\mu \in M$, and let $\bar{G} = (M_i, u_i)_{i \in I}$ denote the mixed extension of G, where $M = \prod_{i \in I} M_i$.

The definitions of recursive diagonal transfer continuity, etc. given in the previous sections, apply in the obvious ways to the mixed extension $\bar{G}$.

We now present the mixed strategy implications of Theorems 3.1 and 3.2.

THEOREM 6.1 *Suppose that $G = (M_i, u_i)_{i \in I}$ is a compact, Hausdorff game. Then G possesses a mixed strategy Nash equilibrium if and only if its mixed extension, $\bar{G}$, is recursively weakly transfer quasi-continuous.*

This theorem strictly generalizes all the existence results on the mixed strategy equilibrium in the literature such as those in Nash (1950), Glicksberg (1952), Mas-Colell (1984), Dasgupta and Maskin (1986), Robson (1994), Simon (1987), Reny (1999), Monteiro and Page (2007), and Nessah and Tian (2008). Any sufficient conditions imposed in the existing theorems on the existence of mixed strategy Nash equilibrium imply the recursive diagonal transfer continuity of $\bar{U}(\cdot)$. To illustrate this, we present here the results of Monteiro and Page (2007), and Nessah and Tian (2008) as corollaries of Theorem 6.1. We first state some definitions introduced by them.

We now provide a full characterization on the existence of symmetric mixed strategy Nash equilibrium for quasi-symmetric games. For the following result only, let M_0 denote the common set of mixed strategies for each player i.

Define an extended quasi-symmetric function $\bar{\psi} : M_0 \times M_0 \to \mathbb{R}$ by

$$\psi(\nu, \mu) = u_i(\mu, \ldots, \nu, \ldots, \mu). \tag{7}$$

Since $\bar{G}$ is quasi-symmetric, μ^* is a symmetric mixed strategy Nash equilibrium if and only if $\bar{\psi}(\nu, \mu^*) \leq \bar{\psi}(\mu^*, \mu^*)$ for all $\nu \in M_0$.

THEOREM 6.2 *Suppose that $G = (M_i, u_i)_{i \in I}$ is a compact, quasi-symmetric, and Hausdorff game. Then G possesses a symmetric mixed strategy Nash equilibrium if and only if its mixed extension payoff $\bar{\psi}(\nu, \mu)$ defined by (5) is recursively diagonal transfer continuous (or recursively diagonal transfer continuous) on M_0.*

This result covers Corollary 5.3 of Reny (1999) as a special case.

Appendix B: Full Characterization of Bayesian Nash Equilibrium

We can also provide full characterization of Bayesian Nash equilibrium in an ex ante formulation of a Bayesian game, in which each player's beliefs are common prior. The existence of Bayesian Nash equilibrium in this formulation has been studied by Radner and Rosenthal (1982), Milgrom and Weber (1985), Vives (1990), and Van Zandt and Vives (2007). The full characterization of Bayesian Nash equilibrium in an interim or incomplete-information formulation of a Bayesian game studied by Van Zandt (2007) can be similarly investigated.

Let X_i be a compact metric space of feasible actions and T_i the set of types of player i, a non-empty complete separable metric space. After observing his type, each player i selects an action $x_i \in X_i$. Denote by T the Cartesian product of the sets of types of the players, $T = \prod_{i \in I} T_i$. The common beliefs of the players are represented by μ, a probability measure on the Borel subsets of T. The measure μ_i will represent the marginal on T_i. The payoff to player i is given by $u_i : X \times T \to \mathbb{R}$, which is Borel measurable and bounded. A (pure) strategy for player i is a Borel measurable mapping $\sigma_i : T_i \to X_i$ that assigns an action to every possible type of the player. This has the interpretation that when player i learns that his type is t_i, he selects his action $\mu_i(t_i)$. Strategies are taken to be equivalence classes, modulo being equally μ -a.e, i.e., strategies σ_i and τ_i are treated the same if they are equally μ_i-almost surely. Let Σ_i denote the strategy space of player i.

Let

$$\Pi_i(\sigma) = \int_T u_i(\sigma_1(t_1), \ldots, \sigma_n(t_n), t_i) d\mu(dt) \tag{8}$$

be the expected payoff to player i when agent j uses strategy σ_j, $j \in I$.

A strategy σ^* is a *Bayesian Nash equilibrium* of a game $\Gamma = (\Sigma_i,\ \Pi_i)_{i \in I}$ if

$$\Pi_i(\sigma^*) \geqq \Pi_i(\sigma_i, \sigma^*_{-i}) \quad \forall \sigma_i \in \Sigma_i.$$

There are several results available in the literature on the existence of pure strategy equilibria in Bayesian games [e.g. Radner and Rosenthal (1982), Milgrom and Weber (1985), and Vives (1990)].

As a direct corollary of Theorems 3.1 and 3.2, the following result strictly generalizes all the existing results on the existence of Bayesian Nash equilibrium.

THEOREM 6.3 *Suppose a Bayesian-Nash game $\Gamma = (\Sigma_i,\ \Pi_i)_{i \in I}$ is compact. Then the game possesses a Bayesian Nash equilibrium if and only if it is recursively weakly transfer quasi-continuous (or recursively diagonal transfer continuous) on Σ.*

This theorem strictly generalizes the existing results such as those in Ray and Rosenthal (1982), Milgrom and Weber (1985), Vives (1990), Athey (2001), Reny (2006), Van Zandt (2007), and Van Zandt and Vives (2008) as special cases.

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