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A characterization of the existence of optimal dominant strategy mechanisms

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Abstract. This paper provides two theorems which characterize the domains of valuation functions for which there exist Pareto efficient and truth dominant strategy mechanisms (balanced Groves mechanisms). Theorem 1 characterizes the existence of balanced Groves mechanisms for a general class of valuation functions. Theorem 2 provides new balance-permitting domains of valuation functions by reducing the problem of solving partial differential equations to the problem of solving a polynomial function. It shows that a balanced Groves mechanism exists if and only if each valuation function in the family under consideration can be obtained by solving a polynomial function with order less than $n-1$, where n is the number of individuals.

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1 Introduction

Pareto efficiency and dominant strategy equilibrium are both highly desirable properties in designing an incentive compatible mechanism. The importance of these properties is attributed respectively to what may be regarded as a minimal welfare criterion, and an allowance for truth-telling behavior.[1] It is well known

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[1] The attractiveness of dominant strategy behavior is that it is the most informationally efficient in the sense that each participant can ignore others' actions. A fundamental axiom of noncooperative behavior is that if an individual has a dominant strategy available, he will use it.

that, by the Revelation Principle (Myerson, 1979), if there exists a mechanism that implements Pareto efficient allocations in dominant strategy, there is a direct revelation mechanism that implements Pareto efficient allocations with truth-telling strategy as a Nash (dominant strategy) equilibrium.[2] Groves and Loeb (1975) and Green and Laffont (1977) further established the equivalence between truth-telling dominant strategy mechanisms and Groves mechanisms on simple public goods economies with additively separable utility functions. Thus, the problem of finding a mechanism which simultaneously yields Pareto efficient allocations and provides individual agents with incentives to report their true preferences honestly reduces to the problem of finding a balanced Groves mechanism.[3] Because of this equivalence result, economists have paid much attention to Groves mechanisms.

However, most results for optimal dominant strategy mechanisms in the literature are negative. Hurwicz (1975), Green and Laffont (1979) and Walker (1980) proved that there is no balanced Groves mechanism when the set of admissible valuation functions of public goods is uncountable and, in general, sufficiently rich. Hurwicz and Walker (1990), and Zhou (1991) further extended this impossibility result to pure exchange economies without public goods. Because of these impossibility results for dominant incentive compatibility, economists turn to other equilibrium solution concepts such as Nash equilibrium and its refinements, and get positive results such as those in Groves and Ledyard (1977), Maskin (1977), Hurwicz (1979), Walker (1981), Moore and Repullo (1988), Tian (1989), Abreu and Sen (1990), and Palfrey and Srivastava (1991) among many others.

Although the results for dominant strategy incentive compatibility were negative, one might hope to obtain balanced Groves mechanisms by restricting the domain of admissible valuation functions. Indeed, when the set of economic environments is "thin", an optimal dominant strategy mechanism may exist. Such a positive result is of greatly compelling interest since in many cases, especially in applications, one only needs to use a particular form of a valuation function. Groves and Loeb (1975) were the first to give an optimal truth dominant mechanism for the class of quadratic valuation functions when they studied optimal taxation issues.

Green and Laffont (1979) and Laffont and Maskin (1980) provided a necessary and sufficient condition for checking the existence of balanced Groves mechanisms by using a differential approach. Although this characterization result allows for a direct check for the possibility of balanced Groves mechanisms for a given family of admissible valuation functions without attempting to construct the transfers, it is difficult to use this result to find a new class of valuation

[2] By the result of Gibbard (1973), any truth-telling Nash equilibrium of a direct revelation mechanism is also a dominant-strategy equilibrium of the mechanism. The direct revelation mechanism means that the message space of the mechanism consists of profiles of preferences so that all participants announce their individual preferences or characteristics.

[3] This equivalence on any "reasonably rich" set of environments is guaranteed through the generalization of Green and Laffont (1977) by Walker (1978), Holmstrom (1979), and Roberts (1979).

functions that permits balanced Groves mechanisms. While optimal incentive compatible mechanisms in incomplete information has been shown to exist in a relatively large set of economic environments by d'Aspremont and Gérard-Varet (1979), Laffont and Maskin (1979) and others, the positive result for the existence of optimal dominant strategy mechanism has been given only for the class of quadratic valuation functions. In fact, there is a conjecture made by Laffont and Maskin (1980), and believed by many others, that the quadratic case is the only nontrivial case that permits optimal truth dominant mechanisms for public goods economic environments. Recently, Tian (1996) showed the failure of this conjecture by finding a wider class of admissible valuation functions that allows for balanced Groves mechanisms. A more interesting result for the existence of optimal dominant strategy mechanisms was obtained by Banerjee (1995). He showed that, for every finite-commodity and finite-individual pure exchange economies with any domain of countably many admissible valuation functions which are continuous and strictly monotonic, it is always possible to design a direct revelation mechanism that simultaneously yields Pareto efficient allocations and truth-telling as a dominant strategy equilibrium, although the mechanisms may not be continuous.

This paper further extends the scope of balanced Groves mechanisms to an even larger class of admissible valuation functions. We provide two theorems which characterize the domains of valuation functions for which there exist Pareto efficient and truth dominant strategy mechanisms. Theorem 1 provides an alternative general characterization result formulated in terms of partial differentials, which is different from the one given by Laffont and Maskin (1980). Each partial differential equation used to characterize balanced Groves mechanisms only involves one individual's valuation function, but not the sum of valuation functions across individuals. Thus the characterization may be more manageable. The characterization result in Theorem 2 allows us to generate new balance-permitting domains of valuations functions. It shows that continuously differentiable Pareto efficient and truth dominant strategy mechanisms (balanced Groves mechanisms) for public good economies can be obtained for a family of admissible valuation functions which are at least as many as points in R_+^{n-2}, where $n > 2$ is the number of individuals. Moreover, Theorem 2 presents an algebraic characterization that is constructive and identifies the scope of balanced Groves mechanisms. It will be shown that there exists a balanced Groves mechanism if and only if each valuation function in the family under consideration can be obtained by solving a polynomial function with order less than $n-1$. Specifically, we show that any division of the integral of a nonnegative and continuous inverse function of any polynomial function with order m less than $n-1$ permits a continuously differentiable balanced Groves mechanism. In fact, this condition is also necessary. Thus the set of admissible valuation functions is much larger.

The remainder of the paper is organized as follows. Section 2 sets forth a public goods economy model and establishes notation and definitions. Section 3 provides a general characterization on the class of admissible valuation functions for the existence of balanced Groves mechanisms. In Sect. 4 we consider the existence of balanced Groves mechanisms by identifying a specific class of admissible valuation functions. Finally, concluding remarks are offered in Sect. 5.

2 The framework

Following the framework in Laffont and Maskin (1980), we consider a public good economy with n agents, one private good, and one public good. Denote by $N = \{1, 2, \ldots, n\}$ the set of agents. The utility function of agent i, $u_i(t_i, y)$, is additively separable between the public good y and the private good t_i, and thus without loss of generality, we assume it is transferable or quasi-linear:

$$u_i(t_i, y) = t_i + V_i(y, \theta_i),$$

where V_i is a real-valued function and is called agent i's valuation function, θ_i lying in a space Θ_i is a parameter of the valuation function, and where for simplicity we assume that $y \in R_+$. Many interesting economic models considered in the literature result in quasi-linear utility functions. For example, for a standard adverse selection model involving a principal and n-agents, in which the principal seeks to implement some public project(s) y, agent i obtains a benefit $V_i(y, \theta_i)$ (or alternatively, bears a cost $-V_i(y, \theta_i)$) of the project and pays (receives) a monetary transfer tax t_i (payment $-t_i$) for consuming the project when it is implemented, agents' payoff functions are quasi-linear.

Since the endowments of the private good play no role in the paper, we will always interpret t_i as the increment (or transfer) of the private good accruing to agent i; thus, we define the feasible states to be whatever satisfies the condition[4]

$$\sum_{i=1}^{n} t_i \leq 0. \tag{1}$$

Then Pareto optimal states can be characterized as those which maximize the aggregate valuation $\sum_{i=1}^{n} V_i : R_+ \times \Theta \to R$, and also, at the same time, satisfy the feasibility constraint (1) with equality.

Throughout the paper, we make the following assumption that was first adopted by Laffont and Maskin (1980):

Assumption 1. *For any $i \in N$, let Θ_i be an open interval in R and $V_i : R_+ \times \Theta_i \to R$ be a continuously differentiable function such that for any $\theta \in \Theta = \prod_{i=1}^{n} \Theta_i$, there exists $y^*(\theta) \in R_+$ for which: (i) $\sum_{i=1}^{n} V_i(y^*(\theta), \theta_i) = \max_{y>0} \sum_{i=1}^{n} V_i(y, \theta_i)$, and (ii) $y^*(\theta)$ is continuously differentiable.*

We assume that the mechanism designer knows the form of $V_i(\cdot, \cdot)$, but is ignorant of the true parameter value $\hat{\theta}_i, i = 1, \cdots, n$. The purpose of a mechanism is to choose an optimal level of the public good and give individuals incentives to truthfully reveal their true parameters in this framework of imperfect information.

[4] Here, like Green and Laffont (1979), Groves (1979), Laffont and Maskin (1980), we have ignored the lower bound restrictions of individuals' transfers.

The decision rule, the so-called direct revelation mechanism, is a mapping, $(n+1)$-tuple $(d, t_1, \ldots, t_n)$ of functions defined on $\Theta = \prod_{i=1}^n \Theta_i$, with the outcome determined by $y = d(\theta)$ and $t_i = t_i(\theta)$.

A mechanism $(d, t_1, \ldots, t_n) : \Theta \to R_+ \times R^n$ is truth-dominant (or strongly individually incentive compatible) if the truth $\hat{\theta} \in \Theta$ is a dominant strategy for each individual; that is, if for any $i \in N$, any $\theta \in \Theta$

$$V_i(d(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i) + t_i(\hat{\theta}_i, \theta_{-i}) \geq V_i(d(\theta_i, \theta_{-i}), \hat{\theta}_i) + t_i(\theta_i, \theta_{-i})$$

where $\theta_{-i} \in \Theta_{-i} = \prod_{j \neq i} \Theta_j$.

Groves (1973) and Groves and Loeb (1975) have discovered how to construct mechanisms that are truth dominant when all admissible valuations attain a maximum. Such mechanisms are called Groves mechanisms. Formally, a mechanism $(d, t_1, \ldots, t_n) : \Theta \to R_+ \times R^n$ is said to be *a Groves mechanism* if

1) $d(\theta)$ maximizes the aggregate valuation $\sum_{i=1}^n V_i$, i.e.,

$$\sum_{i=1}^{n} V_i(d(\theta), \theta_i) \geq \sum_{i=1}^{n} V_i(y, \theta_i) \quad \text{for all } y \in R_+ \text{ and } \theta \in \Theta. \tag{2}$$

2) the transfer outcomes are given by

$$t_i(\theta) = \sum_{j \neq i} V_j(d(\theta), \theta_j) + h_i(\theta_{-i}), \tag{3}$$

where $h_i(\cdot)$ is an arbitrary function from Θ_{-i} to R.

Note that, by Assumption 1 and the definition of a Groves mechanism, we have $d(\theta) \equiv y^*(\theta)$.

A mechanism $(d, t_1, \ldots, t_n) : \Theta \to R_+ \times R^n$ is said to be *balanced* if for all $\theta \in \Theta$,

$$\sum_{i=1}^{n} t_i(\theta) = 0.$$

Thus, from (3) and the definition of a balanced mechanism, one can see that a Groves mechanism is Pareto efficient if and only if it is balanced, or equivalently if and only if

$$(n-1) \sum_{i=1}^{n} V_i(d(\theta), \theta_i) + \sum_{i=1}^{n} h_i(\theta_{-i}) \equiv 0. \tag{4}$$

3 Characterization of balanced groves mechanisms

Green and Laffont (1979) and Laffont and Maskin(1980) proved that under Assumption 1 there exists a balanced Groves mechanisms for the class of admissible valuation functions $V = \{V_1(\cdot, \theta_1), \ldots, V_n(\cdot, \theta_n) | \theta \in \Theta\}$ if and only if

$$\sum_{i=1}^{n} \frac{\partial^{n-1}}{\partial \theta_{-i}} \left[\frac{\partial V_i}{\partial y} \frac{\partial y^*}{\partial \theta_i} \right] \equiv 0. \tag{5}$$

Here the notation $\frac{\partial^{n-1}}{\partial \theta_{-i}}$ stands for $\frac{\partial^{n-1}}{\partial \theta_1 \dots \partial \theta_{i-1} \partial \theta_{i+1} \dots \partial \theta_n}$. Note that these partial derivatives are taken not just through the second argument θ_i but also through the first argument $y^*(\theta)$ of $V_i(y, \theta_i)$. However, as mentioned in the introduction, although this characterization permits a direct check of balance for a given family of valuation functions without actually attempting to specify the transfers, it is not convenient enough to derive specific balance-permitting families in a constructive way. Until Tian (1996), the class of admissible valuation functions permitting the existence of balanced Groves mechanisms were obtained only for the quadratic functions of form

$$V_i(y, \theta_i) = \theta_i y - y^2/2$$

for $i \in N$ with $n \geq 3$.[5]

Tian (1996) recently extended the quadratic case to the family of valuation functions with form of

$$\{V_i(y, \theta_i) = \psi_i(\theta_i)\phi(y) - (b\phi(y) + c)^\alpha\}_{i=1}^n \tag{6}$$

for $n > 2$, and showed that, under some technical conditions on ϕ and ψ_i, there exists a balanced Groves mechanism if and only if $\alpha \in \{2, \frac{3}{2}, \dots, \frac{n-1}{n-2}\}$.

In the following we provide a different version of characterization on balanced Groves mechanisms. Each partial differential equation involves directly only one individual's valuation function but not the sum of partial differentials. This characterization result is not only simpler but also pursues a more constructive way to identify the functional forms of valuation functions that permit balanced Groves mechanisms.

Theorem 1. *For economic environments with parameterized valuation functions $\{V_i(y, \theta_i)\}_{i=1}^n$, which satisfy Assumption 1, there exists a balanced Groves mechanism if and only if*

$$\frac{\partial^{n-1}}{\partial \theta_{-i}} \frac{\partial V_i(y^*(\theta), \theta_i)}{\partial \theta_i} \equiv 0, \quad \forall i, \tag{7}$$

where the partial derivatives with respect to θ_{-i} are taken through $y^*(\theta)$.

Proof. (Necessity) If $\{V_i(y, \theta_i)\}_{i=1}^n$ is a balance-permitting family, then there exists $h_i(\theta_{-i}), i = 1, \cdots, n$, such that (4) is satisfied. Replacing $\sum_{j=1}^n \frac{\partial V_j(d(\theta), \theta_j)}{\partial y}$ with $\sum_{j=1}^n \frac{\partial V_j(y^*(\theta), \theta_j)}{\partial y}$ in (4) and then taking total differentiation in (4) with respect to θ_i and noting that $\sum_{j=1}^n \frac{\partial V_j(y, \theta_j)}{\partial y} = 0$ at the interior maximum y^*, we have

$$(n-1) \frac{\partial V_i(d(\theta), \theta_i)}{\partial \theta_i} + \sum_{j \neq i} \frac{\partial h_j(\theta_{-j})}{\partial \theta_i} \equiv 0.$$

[5] See, Groves and Loeb (1975), Green and Laffont (1979), and Laffont and Maskin (1980).

Taking total differentiation again in the above equation with respect to, one after another, θ_j for $j \neq i$, we arrive at (7) by noting that the second term disappears after differentiation.

(Sufficiency) Suppose (7) is satisfied for a family $\{V_i(y, \theta_i)\}_{i=1}^n$. Let $M(\theta) = \sum_{j=1}^n V_j(d(\theta), \theta_j)$. Differentiating $M(\theta)$ with respect to θ_i and using the envelope theorem (noting $\sum_{j=1}^n \frac{\partial V_j(y,\theta_j)}{\partial y} = 0$), we have

$$\frac{\partial M(\theta)}{\partial \theta_i} = \frac{\partial V_i(d(\theta), \theta_i)}{\partial \theta_i}.$$

Differentiating the above equation with respect to θ_{-i} and using (7), we then have

$$(n-1)\frac{\partial^{n-1}}{\partial \theta_{-i}}\frac{\partial M(\theta)}{\partial \theta_i} \equiv 0,$$

or

$$-(n-1)\frac{\partial^n}{\partial \theta} M(\theta) \equiv 0. \tag{8}$$

Integrating the partial differential equation (8) for θ, we have $-(n-1)M(\theta)$ on the left side and n terms on the right side of the resulting equation from integration, each of which depends at most on $n-1$ variables, i.e.,

$$-(n-1)M(\theta) \equiv \sum_{i=1}^n h_i(\theta_{-i}), \tag{9}$$

where $h_i(\theta_{-i}), i = 1, \cdots, n$ is an arbitrary function from Θ_{-i} to R. One can verify this by differentiating both sides of the above equation with respect to θ and will go back to the partial differential equation (8). Thus, we have

$$(n-1)\sum_{i=1}^n V_i(d(\theta), \theta_i) + \sum_{i=1}^n h_i(\theta_{-i}) \equiv 0,$$

which is exactly (4). Note that $d(\theta) = y^*(\theta)$ and define

$$t_i(\theta) = \sum_{j \neq i} V_j(y^*(\theta), \theta_j) + h_i(\theta_{-i}).$$

We then have

$$\sum_{i=1}^n t_i(\theta) = 0.$$

Hence $\{V_i(y, \theta_i)\}_{i=1}^n$ is a balance-permitting family. Q.E.D.

Note that, integrating (7) for θ_{-i}, we have

$$\frac{\partial V_i(y^*, \theta_i)}{\partial \theta_i} = \sum_{j \neq i} \bar{h}_i^j(\theta_{-j}), \quad \forall i \in N \tag{10}$$

which is a necessary condition for an indirect valuation function to permit a balanced Groves mechanism. Here $\bar{h}_i^j(\theta_{-j})$ is an arbitrary function that is independent of θ_j.

4 The existence of balanced groves mechanisms

In this section we investigate the existence of Pareto efficient and truth dominant mechanisms for a family of admissible valuation functions that has the form of

$$V_i(y,\theta_i) = \psi_i(\theta_i)\phi(y) - G_i(y) + \varphi_i(\theta_i)$$

for any $i \in N$.

The following lemma shows that, under the monotonicity of ψ_i and ϕ, this form of valuation functions is equivalent to the family of admissible valuation functions

$$V_i(y,\theta_i) = \theta_i y - g_i(y) \quad \forall i \in N \tag{11}$$

with $g_i(y) = G_i(\phi^{-1}(y))$ in the sense of guaranteeing the existence of balanced Groves mechanisms, i.e., they both satisfy (7).

Lemma 1. *Under Assumption 1, the family of valuation functions* $\{\bar{V}_i(\bar{y},\bar{\theta}_i) = \psi_i(\bar{\theta}_i)\phi(\bar{y}) - G_i(\bar{y}) + \varphi_i(\bar{\theta}_i)|\bar{\theta}_i \in \bar{\Theta}_i\}$ *satisfies (7) if and only if the family of valuation functions* $\{V_i(y,\theta_i) = \theta_i y - g_i(y)|\theta_i \in \Theta_i\}_{i=1}^n$ *satisfies (7), i.e.,* $\frac{\partial^{n-1}}{\partial\bar{\theta}_{-i}}\frac{\partial \bar{V}_i(\bar{y}^*(\bar{\theta}),\bar{\theta}_i)}{\partial\bar{\theta}_i} \equiv 0$ *if and only if* $\frac{\partial^{n-1}}{\partial\theta_{-i}}\frac{\partial V_i(y^*(\theta),\theta_i)}{\partial\theta_i} \equiv 0$*, where* $\phi(y)$ *and* $\psi_i(\theta_i), i = 1,\cdots,n$ *both satisfy strict differentiable monotonicity (i.e.,* $\phi'(y) > 0$, $\psi_i'(\theta) > 0$*),* φ_i *is differentiable,* $\phi(0) = 0$*, and* $g_i(y) = G_i(\phi^{-1}(y))$.

Proof. Let $\theta_i = \psi_i(\bar{\theta}_i)$, $\theta = \psi(\bar{\theta}) = (\psi_1(\bar{\theta}_1),\ldots,\psi_n(\bar{\theta}_n))$, and $y = \phi(\bar{y})$. Since $g_i(y) \equiv G_i(\phi^{-1}(y))$, we have

$$\begin{aligned}\bar{V}_i(\bar{y},\bar{\theta}_i) &= \psi_i(\bar{\theta}_i)\phi(\bar{y}) - G_i(\bar{y}) + \varphi_i(\bar{\theta}_i)\\ &\equiv \theta_i y - g_i(y) + \varphi_i(\psi_i^{-1}(\theta_i))\\ &= V_i(y,\theta_i) + \varphi_i(\psi_i^{-1}(\theta_i)),\end{aligned}$$

and thus

$$\begin{aligned}\frac{\partial \bar{V}_i(\bar{y}^*,\bar{\theta}_i)}{\partial\bar{\theta}_i} &= \psi_i'(\bar{\theta}_i)\phi(\bar{y}^*) + \varphi_i'(\bar{\theta}_i)\\ &= y^*\psi_i'(\bar{\theta}_i) + \varphi_i'(\bar{\theta}_i)\\ &= \frac{\partial V_i(y^*,\theta_i)}{\partial\theta_i}\psi_i'(\bar{\theta}_i) + \varphi_i'(\bar{\theta}_i).\end{aligned} \tag{12}$$

Then, by using the chain rule and noting the fact $\frac{\partial^{n-1}}{\partial\theta_{-i}}\varphi_i'(\bar{\theta}_i) = 0$, we have

$$\begin{aligned}&\frac{\partial^{n-1}}{\partial\bar{\theta}_{-i}}\frac{\partial \bar{V}_i(\bar{y}^*(\bar{\theta}),\bar{\theta}_i)}{\partial\bar{\theta}_i}\\ &= \frac{\partial^{n-1}}{\partial\bar{\theta}_{-i}}\frac{\partial V_i(y^*(\psi(\bar{\theta})),\theta_i)}{\partial\theta_i}\psi_i'(\bar{\theta}_i) \quad \text{(by Eq. (12))}\\ &= \frac{\partial^{n-1}}{\partial\theta_{-i}}\frac{\partial V_i(y^*(\theta),\theta_i)}{\partial\theta_i}\psi_i'(\bar{\theta}_i)\prod_{j\neq i} d\theta_j/d\bar{\theta}_j \quad \text{(by the chain rule)}\\ &= \frac{\partial^{n-1}}{\partial\theta_{-i}}\frac{\partial V_i(y^*(\theta),\theta_i)}{\partial\theta_i}\prod_{j=1}^{n}\psi_j'(\bar{\theta}_j).\end{aligned}$$

Therefore, by $\prod_{j=1}^{n} \psi_j'(\bar{\theta}_j) \neq 0$, we have

$$\frac{\partial^{n-1}}{\partial \bar{\theta}_{-i}} \frac{\partial \bar{V}_i(\bar{y}^*(\bar{\theta}), \bar{\theta}_i)}{\partial \bar{\theta}_i} \equiv 0 \Longleftrightarrow \frac{\partial^{n-1}}{\partial \theta_{-i}} \frac{\partial V_i(y^*(\theta), \theta_i)}{\partial \theta_i} \equiv 0.$$

Hence, $\bar{V}_i(\bar{y}^*(\bar{\theta}), \bar{\theta}_i)$ satisfies (7) if and only if $V_i(y^*(\theta), \theta_i)$ satisfies (7). Q.E.D.

Thus, without loss of generality, we only concentrate on the families of the form $\{V_i(y, \theta_i) = \theta_i y - g_i(y)\}$ with $g_i'(0) = 0$, and the parametric space $\Theta_i = (0, \infty)$ in the remainder of the paper.[6] The possibility results for the more general form $\{V_i(y, \theta_i) = \psi_i(\theta_i)\phi(y) - G_i(y) + \varphi_i(\theta_i)\}$ and any open interval parametric space can be obtained by applying Lemma 1.

The following proposition establishes that each valuation function in the family can be obtained by solving a polynomial function with order less than $n - 1$.

Proposition 1. *For any $a = (a_1, \cdots, a_m) \in R^m$ with $a_m > 0$ and $1 \le m \le n-2$, if there exists a non-negative and continuous inverse function, $x = Q(y)$, for the polynomial function*

$$y = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x \tag{13}$$

with $Q(0) = 0$, then for any division of $G(y) = \int_0^y Q(z)dz$ with $\sum_{i=1}^{n} g_i(y) = G(y)$, the family of admissible valuation functions of the form $\{V_i(y, \theta_i) = \theta_i y - g_i(y)\}_{i=1}^{n}$ allows for balanced Groves mechanisms.

Proof. For the family of valuation functions with the form $\{V_i(y, \theta_i) = \theta_i y - g_i(y)\}_{i=1}^{n}$, since $Q(y) \to 0$ as $y \to 0$ (by noting that $Q(y)$ is continuous and $Q(0) = 0$) and $Q(y) \to \infty$ as $y \to \infty$ (by noting that $a_m > 0$), then, for each $\theta_i \in (0, \infty)$, $\frac{\partial}{\partial y}[y \sum_{i=1}^{n} \theta_i - G(y)] = \sum_{i=1}^{n} \theta_i - Q(y) > 0$ for sufficiently small y, and $\sum_{i=1}^{n} \theta_i - Q(y) < 0$ for sufficiently large y. Then $y^*(\theta) = \arg\max_y \{y \sum_{i=1}^{n} \theta_i - G(y)\}$ must be an interior point of $[0, \infty)$, and thus $y^*(\theta)$ must satisfy

$$\sum_{i=1}^{n} \theta_i - Q(y^*(\theta)) = 0. \tag{14}$$

Since $Q(y)$ satisfies

$$y \equiv a_m [Q(y)]^m + a_{m-1}[Q(y)]^{m-1} + \cdots + a_1 Q(y),$$

we have by (14)

$$y^*(\theta) \equiv a_m (\sum \theta_i)^m + a_{m-1}(\sum \theta_i)^{m-1} + \cdots + a_1 \sum \theta_i.$$

Thus, we have

$$\frac{\partial^{n-1} y^*(\theta)}{\partial \theta_{-i}} \equiv 0 \quad \forall i \in N$$

[6] We can always transform any open interval $\bar{\Theta}_i = (\alpha, \beta)$ to $\Theta = (0, \infty)$ by making the monotonic transformation, $\theta_i = \frac{\bar{\theta}_i - \alpha}{\beta - \bar{\theta}_i}$ for $\bar{\theta}_i \in (\alpha, \beta)$.

by noting $m \leq n-2$, which in turn implies

$$\frac{\partial^{n-1}}{\partial \theta_{-i}} \frac{\partial V_i(y^*(\theta), \theta_i)}{\partial \theta_i} \equiv 0 \quad \forall i \in N$$

by noting $\frac{\partial V_i(y^*(\theta),\theta_i)}{\partial \theta_i} \equiv y^*(\theta)$. Therefore, by Theorem 1, balanced Groves mechanisms exist. Q.E.D.

Due to the constructive nature of Proposition 1, one can identify balance-permitting families by solving algebraic equation (13) with various $(a_1, \cdots, a_m)$. One may want to ask how big the set of admissible valuation functions that permit balanced Groves mechanisms is. The following Claim shows that the number of such admissible valuation functions are at least as many as the points in R_+^{n-2}.

Claim 1. *For any $a = (a_1, \cdots, a_m) \in R_+^m$ with $a_m > 0$ and $1 \leq m \leq n-2$, there exists a unique inverse function $Q(y)$ for the polynomial function given in (13), which is non-negative continuously differentiable, and satisfies $Q(0) = 0$.*

Proof. We first note that, when $a \in R_+^m$ and $x \in R_+$, the polynomial function is monotonically increasing. Thus, the monotonicity (one to one mapping) of the polynomial function guarantees the existence of a unique inverse function and the monotonicity of the inverse function. Since the polynomial function is continuously differentiable and equal to zero when $x = 0$, the inverse function $x = Q(y)$ is also continuously differentiable and satisfies $Q(0) = 0$. Finally, $Q(y)$ is nonnegative by the monotonicity of $Q(y)$ and $Q(0) = 0$. Thus we have shown $x = Q(y)$ is a non-negative single-valued continuously differentiable function with $Q(0) = 0$. Q.E.D.

Proposition 2 below constitutes the reverse of Proposition 1.

Proposition 2. *Under Assumption 1, if a balanced Groves mechanism exists for a family of admissible valuation functions of the form $\{V_i(y, \theta_i) = \theta_i y - g_i(y)\}_{i=1}^n$, then there exists $(a_1, a_2, \cdots, a_m) \in R^m$ with $1 \leq m \leq n-2$ such that*

$$a_m[G'(y)]^m + a_{m-1}[G'(y)]^{m-1} + \cdots + a_1 G'(y) \equiv y,$$

where $G(y) = \sum_{i=1}^n g_i(y)$.

Proof. Since the balanced Groves mechanism exists, we have from (7)

$$\frac{\partial^{n-1}}{\partial \theta_{-i}} \frac{\partial V_i(y^*(\theta), \theta_i)}{\partial \theta_i} \equiv \frac{\partial^{n-1}}{\partial \theta_{-i}} y^*(\theta) \equiv 0 \quad \forall i \in N. \tag{15}$$

Since $y^*(\theta)$ is the interior solution by Assumption 1, it satisfies

$$\sum_{i=1}^n \theta_i \equiv G'(y^*(\theta)).$$

Also, by Assumption 1, the optimal solution from the above first order condition, $y^*(\theta)$, is single-valued and continuously differentiable for all $\theta \in \Theta$, and thus

$G'(y^*)$ must be monotonic along the optimal path $y^*(\theta)$. Therefore, G' has a unique inverse function, denoted by $G'^{-1}(\cdot)$, and thus we have

$$y^*(\theta) \equiv G'^{-1}(\sum_{i=1}^{n} \theta_i). \tag{16}$$

Hence, from (15) and (16), we have

$$0 \equiv \frac{\partial^{n-1}}{\partial \theta_{-i}} y^*(\theta) \equiv \frac{d^{n-1} G'^{-1}(x)}{dx^{n-1}} \equiv 0,$$

where $x = \sum_{i \in N} \theta_i$. (Note that, by definition, $G'^{-1}(G'(x)) = x$.) This implies that $G'^{-1}(\cdot)$ is a polynomial function of order m less than $n-1$. That is, there exists $(a_1, \cdots, a_m) \in R^m$ such that

$$G'^{-1}(x) \equiv a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x.$$

Since $x = \sum_{i \in N} \theta_i = G'(y)$, we have

$$y \equiv a_m [G'(y)]^m + a_{m-1}[G'(y)]^{m-1} + \cdots + a_1 G'(y). \quad Q.E.D.$$

Thus, from Propositions 1 and 2, we have the following theorem that completely characterizes the existence of balanced Groves mechanisms for the family of valuation functions given by (11).

Theorem 2. *Under Assumption 1, there exists a balanced Groves mechanism for the family of valuation functions* $\{V_i(y, \theta_i) = \theta_i y - g_i(y)\}_{i=1}^n$ *if and only if there exists a point* $a = (a_1, \cdots, a_m) \in R^m$ *with* $1 \le m \le n-2$ *such that*

$$a_m [G'(y)]^m + a_{m-1}[G'(y)]^{m-1} + \cdots + a_1 G'(y) \equiv y \tag{17}$$

where $G(y) = \sum_{i=1}^n g_i(y)$.

Compared to (5) or (7), Theorem 2 gives us an algebraic characterization of the existence of balanced Groves mechanisms that is in principle constructive rather than descriptive. Each admissible valuation function in the family considered in this section can be obtained by solving a polynomial function with order less than $n-1$. Thus Theorem 2 gives us a way to find new balance-permitting valuation functions and identifies the scope of balanced Groves mechanisms. It may be remarked that many of the existing results can be easily obtained from Theorem 2, and reduce to special cases of ours. For instance, for the case of $n = 2$, since there is no m such that $1 \le m \le n-2 = 0$ and (17) holds only for $m \ge 1$, there is no balanced Groves mechanism for the family of valuation functions specified in this section. For $n = 3$, (17) becomes $a[G'(y)] \equiv y$ to which the solution is $G(y) = \frac{1}{2a} y^2$ that is the quadratic case.

For $n \ge 3$, consider the following polynomial function

$$a[G'(y)]^m \equiv y$$

for $1 \leq m \leq n-2$. Then we have

$$G'(y) = (y/a)^{\frac{1}{m}}$$

and thus

$$G(y) = \frac{my^{\frac{m+1}{m}}}{(m+1)a^{\frac{1}{m}}} \tag{18}$$

for $m = 1, 2, \ldots, n-2$ and for $G(0) = 0$. By Lemma 1, we know that this is equivalent to $G(\bar{y}) = \frac{am(b\bar{y}+c)^{\frac{m+1}{m}}}{m+1}$ by letting $y = ab\bar{y} + ac$, which is the balance-permitting families found by Tian (1996).

It may be remarked that, although $G(y) = \frac{my^{\frac{m+1}{m}}}{(m+1)a^{\frac{1}{m}}}$ with $1 \leq m \leq n-2$, is sufficient to allow for balanced Groves mechanisms, it is not necessary. Other forms of $G(y)$ can be derived from (17) by allowing more than one of $\{a_1, \cdots, a_m\}$ to be non-zero.

Example 1. The case of $n \geq 4$. From

$$aG'(y)^2 + 2bG'(y) \equiv y,$$

we can get the non-negative solution

$$G'(y) = \frac{1}{a}\left[-b + \sqrt{b^2 + ay}\right].$$

Then

$$G(y) = \frac{1}{a}\left[-by + \frac{2}{3a}\left(b^2 + ay\right)^{\frac{3}{2}} - \frac{2b^3}{3a}\right].$$

Note this $G(y)$ is not represented by (6).

Example 2. The case of $n \geq 5$. From

$$aG'(y)^3 + bG'(y)^2 + cG'(y) \equiv y,$$

we can get

$$G'(y) = \left(\frac{q(y)}{2} + \sqrt{\frac{q(y)^2}{4} + \frac{p}{27}}\right)^{\frac{1}{3}} + \left(\frac{q(y)}{2} - \sqrt{\frac{q(y)^2}{4} + \frac{p}{27}}\right)^{\frac{1}{3}} - \frac{b}{3a},$$

where $q(y) = \frac{y}{a} - \frac{2b^3}{27a^3} + \frac{bc}{3a^2}$, and $p = (\frac{c}{a} - \frac{b^2}{3a^2})$. Thus we have

$$G(y) = \int_0^y G'(t)dt.$$

This $G(y)$ is not represented by (6) either.

Note that, for any $G(y)$ obtained in the above examples, we can define $g_i(y)$ arbitrarily as long as they satisfy $\sum_{i \in N} g_i(y) = G(y)$. An obvious choice is to define $g_i(y) = G(y)/n$. In general, for any collection of functions $\{\gamma_i : R \to R :$

$i \in N\}$ such that $\sum_{i \in N} \gamma_i(y) \equiv 1$, we can define $g_i(y) = \gamma_i(y)G(y)$, splitting up $G(y)$ arbitrarily.[7]

Thus, new findings coming from our characterizations are: First, the inverse function of any m order polynomial function (13) with nonnegative coefficients can be used to construct a valuation function. This indicates that the set of balance-permitting $G(y)$ corresponds to the whole space of R_+^{n-2}. Hence, it is of a dimension of $n-2$. Second, what matters for the existence of balanced Groves mechanisms is $G(y) = \sum g_i(y)$ rather than individual $g_i(y)$. Especially, $g_i(y)$ $(i = 1, \cdots, n)$ are not necessarily the same. Hence, this characterization explicitly allows for differences in valuation functions across individuals, compared to the existing symmetric cases.[8]

5 Conclusion

In this paper, we have given an alternative characterization for the existence of balanced Groves mechanisms within the families of $\{V_i(y, \theta_i) = \psi_i(\theta_i)\phi(y) - G_i(y) + \varphi_i(\theta_i)\}$. The major merit of this characterization is that we can use it to construct families of valuation functions that permit balanced Groves mechanisms that are continuously differentiable, while the characterization given by (5) or (7) is merely descriptive. We have shown that any division of the integral, $G(y)$, of a nonnegative and continuous inverse function of any polynomial function with nonnegative coefficients and with order m less than $n-1$ permits a continuously differentiable balanced Groves mechanism. In fact, this condition is also necessary. Thus, a balanced Groves mechanism exists if and only if the derivative of the sum of $G_i(\phi^{-1}(y))$ is a solution of some polynomial function with the order less than $n-1$. In addition, our newly found balance-permitting family does not require the same G_i across individuals, and what matters is the aggregation of $G_i(\cdot)$. With this constructive characterization, we have not only significantly extended the scope of balanced Groves mechanisms, but also given a simple way of finding a new class of balance-permitting valuation functions. As for the existence of families other than $\{V_i(y, \theta_i) = \psi_i(\theta_i)\phi(y) - G_i(y) + \varphi_i(\theta_i)\}$ that permit balanced Groves mechanisms, we could neither prove that such families do not exist nor give an example that permits balanced Groves mechanisms. Hence, it is still an open question.

[7] We thank the anonymous referee for providing such a division of $G(y)$.

[8] One of the remarkable features of both the traditional quadratic family and Tian's more general families that permit balance is the symmetry in valuation functions across individuals.

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