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Implementation of the Lindahl Correspondence by a Single-Valued, Feasible, and Continuous Mechanism

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This paper considers the problem of designing mechanisms whose Nash allocations coincide with the Lindahl allocations for public goods economies with more than one private good. Unlike previous mechanisms, the mechanism presented here has a single-valued, feasible, and continuous outcome function. Furthermore, when there are no public goods in economies, feasible and continuous implementation of the (constrained) Walrasian correspondence can be obtained as a corollary of our Theorem 1.

1. INTRODUCTION

For public goods economies with more than one private good, the mechanisms in the literature have some undesirable properties. The so-called mixed competitive mechanisms proposed by Groves and Ledyard (1977), Walker (1981), and Tian (1987) have a common defect: the outcome functions for allocating private goods are not single-valued even though they are single-valued in the case of one private good. This kind of mechanism can only determine the level of public goods and taxing rules, but not that of private goods. Private goods are assumed to be allocated through competitive markets. As Groves and Ledyard (1987) pointed out, many problems arise if the outcome function is not single-valued. For example, agents may be unable to coordinate actions or a single agent may be unable to evaluate the consequences of his actions even if he knows the actions of others. Also, from the view point of game theory, mechanisms that are not single-valued are not well-defined. Because of this, it is highly desirable to have a single-valued outcome function. There also exist other problems for this kind of mechanism. For example, for a small number of agents, there is an incentive-compatibility problem since private goods are assumed to be allocated through competitive markets. We know that the first welfare theorem states that competitive allocations are Pareto efficient. However, besides several explicit assumptions in the hypothesis of this theorem, there is often an implicit assumption that agents are "price takers". Consequently, this involves the idea that there are many agents, each of which is relatively small. Thus, for a small number of agents, the efficiency of outcomes may not follow from the mixed competitive mechanisms if we do not believe that the agents are "price takers" for private goods. In fact, the above is one of the reasons why, even for the classical private goods economies, people want to seek some alternative mechanisms which yield efficient allocations. In addition to the above problems, the Groves–Ledyard mechanism is neither individually rational nor individually feasible.[1] Walker's mechanism Nash-implements the Lindahl correspondence and thus is individually rational, but not individually feasible. On the other hand, Hurwicz, Maskin, and Postlewaite (1984) did indeed give a mechanism with a single-valued and feasible outcome function which Nash-implements Pareto-efficient and individually-rational social choice correspondences. Their mechanism, however, is not continuous and thus small variations in an agent's strategy choice may lead to large jumps in the resulting allocations. For the motivation behind designing the feasible and continuous mechanisms, see Groves and Ledyard (1987) and Postlewaite and Wettstein (1989).

A key question then is whether or not a mechanism exists which has a feasible, continuous, and single-valued outcome function and which Nash-implements a Pareto-efficient and individually-rational social choice correspondence. This paper will answer this question in the affirmative.

This paper presents a feasible and continuous mechanism with a single-valued outcome function. It Nash-implements the Lindahl correspondence which yields Pareto-efficient and individually-rational allocations. As noted, this mechanism is, in fact, a combination (with suitable modification) of two mechanisms proposed by Postlewaite and Wettstein (1989) and Tian (1987): the level of private goods and taxing rules are determined by Tian's mechanism while private goods are allocated by Postlewaite and Wettstein's mechanism. Moreover, our mechanism, like the one proposed by Hurwicz, Maskin, and Postlewaite (1984), allows the true endowments to be private information. This is a more interesting approach for considering the information aspects of decentralization. This situation would certainly increase the size of the message space but would reduce the information requirements on the designer. It allows each agent to reveal information about his own endowment in that he can understate but not overstate his own endowment and so guarantee feasibility out of equilibrium. For simplicity, we only consider the case where the underreported endowments are cancelled (destroyed) rather than consumed.[2] In this case, the mechanism presented in this paper has the advantage that, unlike Hurwicz, Maskin and Postlewaite's and Postlewaite and Wettstein's mechanisms, each agent is required to announce only his own endowment but not others' endowments and thus it uses a message space of much lower dimension than theirs. It will be shown that no destruction would occur in any equilibrium even though agents may underreport their endowments at non-equilibrium.

The plan of the paper is as follows. Section 2 sets out a public goods model and presents a feasible and continuous mechanism which has a single-valued outcome function when the true endowments are private information. Before the formal presentation of the mechanism, we describe and explain the mechanism in order to give readers an intuitive idea about our mechanism. In Section 3, we give the main results on Nash-implementation of the Lindahl correspondence and their proofs.

2. PUBLIC GOODS MODEL AND MECHANISM

2.1. *Economic environments*

The economy under consideration has L private goods ($L>1$) and K public goods, x

1. If a mechanism is not individually feasible, it may yield an allocation which is not in the consumption set out of equilibrium.

2. Following the technique used by Hurwicz, Maskin, and Postlewaite (1984), we may design a mechanism which allows the underreported endowments to be consumed by agents, but a higher cost must be paid in terms of informational requirements since this situation would greatly increase the size of the message space.

being private and y public. There are n agents. Denote by $N=\{1,2,\ldots,n\}$ the set of agents. Throughout this paper, the subscript is used to index agents and the superscript is used to index goods. Each agent's characteristic is denoted by $e_i=(C_i, \hat{w}_i, R_i)$, where C_i is the consumption set with element (x_i, y), $\hat{w}_i$ the true initial endowment of private goods, and R_i the preference ordering (P_i denotes the asymmetric part of R_i). We assume that there is no initial endowment of public goods, but the public goods can be produced from the private goods. Let $\mathcal{Y}$ be the production possibility set. Its element is (r, y), where $r \in -\mathbb{R}^L_+$ is the vector of private goods inputs and $y \in \mathbb{R}^K_+$ is the vector of public goods outputs. An economy is the full vector $e=(e_1,\ldots,e_n,\mathcal{Y})$ and the set of all such economies is denoted by E.

2.2. *Assumptions on economies*

Throughout this paper we make the following assumptions on E:

Assumption 1. $n \geqq 3$.

Assumption 2. $\hat{w}_i > 0$[3] *for all* $i \in N$.[4]

Assumption 3. $C_i = \mathbb{R}^{L+K}_+$.

Assumption 4. The preference R_i is reflexive, transitive, complete, continuous, and convex[5] as well as strictly monotone increasing on $\mathbb{R}^{L+K}_+$.

Assumption 5. For all $i \in N$, $(x_i, y) P_i (x'_i, y')$ for all $x_i \in \mathbb{R}^L_{++}$, $y, y' \in \mathbb{R}^K_+$, and $x'_i \in \partial\mathbb{R}^L_+$, where $\partial\mathbb{R}^L_+$ is the boundary of $\mathbb{R}^L_+$.[6]

Assumption 6. The production possibility set $\mathcal{Y}$ is a closed, convex cone; $0 \in \mathcal{Y}$; $(-\mathbb{R}^L_+, 0) \subseteq \mathcal{Y}$ (free disposal); and for any $y \in \mathbb{R}^K_+$, there is an $r \in -\mathbb{R}^L_+$ such that $(r, y) \in \mathcal{Y}$.

Remark 1. From Theorem 3($i'a$) of Tian (1988), we know that even with Assumptions 1–4 and 6, the (constrained) Lindahl correspondence may violate the Maskin monotonicity condition which is a necessary condition for Nash implementation. Thus we know to impose further restrictions on E and/or the social choice correspondence F (like Assumption 5) to guarantee Nash-implementability of the (constrained) Lindahl correspondence[7] by feasible mechanisms. Hurwicz, Maskin, and Postlewaite (1984) pointed out that Assumption 5 and the rationality condition imply the condition that every allocation in F has a positive private goods bundle. They use the condition to obtain implementability of the (constrained) Lindahl correspondence.

3. As usual, vector inequalities are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s=1,\ldots,m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s=1,\ldots,m$.

4. In the case of the true endowment known to the designer, Assumption 2 can be replaced by $\hat{w}_i \geq 0$ and $\hat{w}=\sum_{i=1}^n \hat{w}_i > 0$.

5. Preference R_i is convex if, for a, b, c in C_i and $0<\lambda \leqq 1$, $c=\lambda a+(1-\lambda)b$, the relation aP_ib implies cP_ib.

6. If Assumption 5 is replaced by the assumption that $(x_i, y)P_i(x'_i, y')$ for all $(x_i, y) \in \mathbb{R}^{L+K}_{++}$ and $(x'_i, y') \in \partial\mathbb{R}^{L+K}_+$, the dimensions of the message space of the mechanism presented in this paper can be reduced by nK dimensions. For details, see Tian (1987).

7. A constrained Lindahl (Walrasian) allocation differs from an ordinary Lindahl (Walrasian) allocation only in the way that each agent maximizes his preferences not only subject to his budget constraint but also subject to total endowments available to the economy (see, Hurwicz (1986) or Tian (1988)).

Remark 2. Observe that under Assumption 6, if $(r, y) \in \mathcal{Y}$, then $(r', y) \in \mathcal{Y}$ for any $r' \leqq r$ (cf. Debreu (1959)).

2.3. *Lindahl allocations*

An allocation $(x, y) = (x_1, \ldots, x_n, y)$ is feasible if $(x, y) \in \mathbb{R}_+^{nL+K}$ and $(\sum_{i=1}^n x_i - \hat{w}, y) \in \mathcal{Y}$, where $\hat{w} = \sum_{i=1}^n \hat{w}_i$.

An allocation (x^*, y^*) is a *Lindahl allocation* for an economy e, if it is feasible and there is a price vector $p^* \in \mathbb{R}_+^L$ and price vectors $q_i^* \in \mathbb{R}_+^K$, one for each i, such that

(1) $p^* \cdot x_i^* + q_i^* \cdot y^* = p^* \cdot \hat{w}_i$ for all $i \in N$;
(2) $(x_i, y) P_i (x_i^*, y^*)$ implies $p^* \cdot x_i + q_i^* \cdot y > p^* \cdot \hat{w}_i$ for all $i \in N$;
(3) $q^* \cdot y^* + p^* \cdot (\sum_{i=1}^n x_i^* - \sum_{i=1}^n \hat{w}_i) \geqq p^* \cdot r + q^* \cdot y$ for all $(r, y) \in \mathcal{Y}$, where $q^* = \sum_{i=1}^n q_i^*$. Denote by $L(e)$ the set of all such allocations.

Remark 3. We may note that conditions 1 and 3 in the preceding definition imply that $q^* \cdot y^* + p^* \cdot (\sum_{i=1}^n x_i^* - \sum_{i=1}^n \hat{w}_i) = 0$. This is the familiar zero-profit condition under constant returns.

Remark 4. There is the following relationship between the Lindahl (Walrasian) allocation and the constrained Lindahl (Walrasian) allocation: every Lindahl (Walrasian) allocation is a constrained Lindahl (Walrasian) allocation; every constrained Lindahl (Walrasian) allocation with a positive private goods bundle for all agents is a Lindahl (Walrasian) allocation if preferences satisfy convexity (cf. Theorems 2 and 3 in Tian (1988)). Thus the Lindahl (Walrasian) allocations coincide with the constrained Lindahl (Walrasian) allocations under the assumptions given in this paper.

2.4. *Message space and mechanism*

Let M_i denote the i-th message (strategy) domain. Its element is written as m_i and is called the message. Let $M = \prod_{i=1}^n M_i$ denote the message (strategy) space. Let $h: M \to \mathbb{R}^{nL+K}$ denote the outcome function, or more explicitly, $h_i(m) = (X_i(m), Y(m))$.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is a *Nash equilibrium* (NE) of the mechanism $(M; h)$ for an economy e if for any $i \in N$,

$$h_i(m^*) R_i h_i(m^*/m_i, i) \quad \text{for all} \quad m_i \in M_i, \tag{1}$$

where $(m^*/m_i, i) = (m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. The $h(m^*)$ is then called a *Nash (equilibrium) allocation.* Denote by $N_{M,h}(e)$ the set of all Nash (equilibrium) allocations.

In the following we will construct a single-valued, feasible, and continuous mechanism[8] which (fully) Nash-implements[9] the Lindahl correspondence. Before the formal presentation of this mechanism, we give a description as follows. The designer first determines the "trade" (market) prices of private and public goods and the personalized prices for public goods according to agents' announcements about these prices (which are given in (3) and (4) below). Then define a feasible choice correspondence B (given in (5) below) for public goods that can be produced with the claimed total endowments and can be purchased with the endowment claimed by all agents. The outcome $Y(m)$

8. That is, the outcome function of the mechanism is single-valued and continuous, and yields feasible allocations for all $m \in M$.

9. A mechanism is said to fully Nash-implement a social choice correspondence F on E if, for all $e \in E$, $N_{M,h}(e) \neq \varnothing$ and $N_{M,h}(e) = F(e)$.

for public goods will be chosen from $B(m)$ so that it is the closest to the sum of the contributions that each agent is willing to pay (see (6) below). That is, the designer tries to satisfy the agents' desires as best as he can. After that, for each agent i, define a feasible choice correspondence $B_i(m)$ (given in (9)) for private goods under his budget constraint with the claimed endowment and the constraint that private goods should not exceed the remaining part of the claimed total endowments after the public goods are produced. Now choose a "tentative" bundle of private goods $x_i'(m)$ (given in (10)) from $B_i(m)$ for each agent i so that it is the closest to the proposed trade of private goods. If the "tentative' bundle $x'(m)$ with the outcome $Y(m)$ is not feasible under the claimed total endowments, shrink the level of the "tentative" bundle until the final holdings of all agents are feasible under the claimed total endowments. Such a shrunk outcome for private goods $X_i(m)$ is given in (12). It will be seen that each agent's final holdings $(X_i(m), Y(m))$ satisfy his own budget set and the final allocation $(X(m), Y(m))$ resulting from the message m is feasible, continuous, and single-valued for all $m \in M$.

We now turn to the formal construction of the mechanism. For each $i \in N$, it is assumed that his message domain is of the form

$$M_i = (0, \hat{w}_i] \times \mathbb{R}_{++}^{L+K} \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}_+^L \times \mathbb{R}_{++}. \tag{2}$$

A generic element of M_i is $(w_i, p_i, \phi_i, y_i, z_i, \gamma_i)$, where w_i denotes a claim about agent i's endowment. The inequality $0 < w_i \leqq \hat{w}_i$ means that the agent can understate but not overstate his own endowment, the claimed endowment w_i (like the true endowment $\hat{w}_i$) must be positive; p_i denotes the price vector of private and public goods proposed by agent i; ϕ_i denotes the price vector of public goods proposed by agent i for use in other agents' budget constraints; y_i denotes the tax or the contribution that agent i is willing to pay; z_i denotes the proposed trade of private goods for agent i; and γ_i is a real positive number.

Define the price vector $(p, q): M \rightarrow \mathbb{R}_{++}^{L+K}$ by

$$(p(m), q(m)) = \begin{cases} \sum_{i=1}^n \dfrac{a_i}{a} p_i & \text{if } a > 0 \\ \sum_{i=1}^n \dfrac{1}{n} p_i & \text{if } a = 0 \end{cases}, \tag{3}$$

which is continuous. Here $a_i = \sum_{j,s \neq i}^n \|p_j - p_s\|$, $a = \sum_{i=1}^n a_i$, and $\|\cdot\|$ is a Euclidian norm. Notice that even though $(p(m), q(m))$ is a function of the component $(p_1, \ldots, p_n)$ of the message m only, we can write it as the function of m without loss of generality.

Define the personalized price of each public good k for the i-th consumer by

$$q_i^k(m) = b_i^k(m) + \sum_{j=1}^n a_{ij}^k \phi_j^k, \tag{4}$$

where $\sum_{i=1}^n b_i^k(m) = q^k(m)$ (which is the k-component of $q(m)$), $\sum_{i=1}^n a_{ij}^k = 0$, $a_{ii}^k = 0$, and $\sum_{j=1}^n |a_{ij}^k| > 0$ for $i \in N$ and $k = 1$ to K. In addition, a_{ij}^k are chosen so that any personalized price vector q_i can be attained from the equation (4) by suitably choosing $(\phi_1, \ldots, \phi_n)$. Observe that, by construction, $\sum_{i=1}^n q_i(m) = q(m)$ for all $m \in M$ and each agent's personalized prices are independent of his own message (i.e. $q_i(m^*) = q_i(m^*/m_i, i)$ for any $m_i \in M_i$). Here $q_i(m) = (q_i^1(m), \ldots, q_i^K(m))$. This form is very general and covers the personalized prices specified by Hurwicz (1979) and Walker (1981) as special cases.

Define the correspondence $B: M \rightarrow\rightarrow \mathbb{R}_+^K$ by

$$B(m) = \{y \in \mathbb{R}_+^K : p(m) \cdot w_i - q_i(m) \cdot y \geqq 0 \quad \text{for all} \quad i \in N \ \& \ (-w, y) \in \mathscr{Y}\}, \tag{5}$$

which is nonempty, closed, convex, and continuous on $m \in M$. Here $w = \sum_{i=1}^n w_i$.

Define the outcome function for public goods $Y: M \to B$ by

$$Y(m) = \{y : \min_{y \in B(m)} \|y - \tilde{y}\|\}, \tag{6}$$

which is the closest to $\tilde{y}$. Here $\tilde{y} = \sum_{i=1}^{n} y_i$. Then $Y(m)$ is single-valued and continuous on M.[10]

For each individual i, define the tax function $T_i: M \to \mathbb{R}$ by

$$T_i(m) = q_i(m) \cdot Y(m). \tag{7}$$

Then

$$\textstyle\sum_{i=1}^{n} T_i(m) = q(m) \cdot Y(m). \tag{8}$$

To determine the level of private goods for each individual i, define the correspondence $B_i: M \to\to \mathbb{R}_+^L$ by

$$B_i(m) = \{x_i \in \mathbb{R}_+^L : p(m) \cdot x_i + T_i(m) \leqq p(m) \cdot w_i \;\&\; (x_i - w, Y(m)) \in \mathcal{Y}\}, \tag{9}$$

which is nonempty, closed, convex, and continuous.[11]

Define $x_i': M \to B_i$ by

$$x_i'(m) = \{x_i : \min_{x_i \in B_i(m)} \|x_i - z_i\|\}, \tag{10}$$

which is the closest to z_i.

Define the shrinking correspondence $A: M \to\to \mathbb{R}_+$ by

$$A(m) = \{\gamma \in \mathbb{R}_+ : \gamma\gamma_i \leqq 1 \quad \text{for all} \quad i \in N \;\&\; (\gamma \textstyle\sum_{i=1}^{n} \gamma_i x_i(m) - w, Y(m)) \in \mathcal{Y}\}. \tag{11}$$

Let $\bar{\gamma}$ be the greatest element of A, i.e. $\bar{\gamma} \geqq \gamma$ for all $\gamma \in A(m)$.

Now define agent i's outcome function for private goods $X_i: M \to \mathbb{R}_+^L$ by

$$X_i(m) = \bar{\gamma}\gamma_i x_i'(m), \tag{12}$$

which is the agent's final (total) holdings for private goods resulting from the strategic configuration m.

We can easily see that the outcome function specified above is single-valued, feasible, and continuous on M from (9)–(11).

Remark 5. Observe that in the case of $L = 1$, the part for the private goods game is not needed. Also, if there are no public goods (i.e. $K = 0$), public goods economies reduce to private goods economies and thus feasible and continuous Nash-implementation of the (constrained) Walrasian correspondence can be obtained as a corollary of Theorem 1 below.

3. MAIN RESULTS

Our main results are given in the following theorem.

Theorem 1. *For public goods economies with more than one private good, there exists a single-valued, feasible, and continuous mechanism which fully Nash-implements the Lindahl correspondence.*

10. This is because $Y(m)$ is an upper hemi-continuous correspondence by Berge's Maximum Theorem (see Debreu (1959, p. 19)) and single-valued (see Mas-Colell (1985, p. 28)).

11. $B_i(m)$ is clearly continuous at m if $T_i(m) < p(m) \cdot w_i$. $B_i(m)$ is also continuous at m even if $T_i(m) = p(m) \cdot w_i$. In fact, $B_i(m) = \{0\}$ at m (since $P(m) > 0$) and for any sequence $\{m_t\}$ with $m_t \to m$, $0 \in B_i(m_t)$. So $B_i(m)$ is lower hemi-continuous at m. On the other hand, $B_i(m)$ is clearly upper hemi-continuous at m.

Proof. The proof of Theorem 1 consists of the following lemmas. All we need to do is to show that the set of Nash allocations coincides precisely with the set of the Lindahl allocations. That is, $N_{M,h}(e)=L(e)$ for all $e\in E$ satisfying Assumptions 1–6.

Lemma 1. *Suppose $(x_i'(m), Y(m))P_i(x_i, y)$. Then agent i can choose a very large γ_i such that $(X_i(m), Y(m))P_i(x_i, y)$.*

Proof. If agent i declares a large enough γ_i, then $\bar{\gamma}$ becomes very small (since $\bar{\gamma}\gamma_i\leqq 1$) and thus almost nullifies the effect of other agents in $(\bar{\gamma}\sum_{j=1}^n \gamma_j x'(m)-w, Y(m))\in \mathcal{Y}$. Thus $X_i(m)$ can arbitrarily approach $x_i'(m)$ as agent i wishes. From $(x_i'(m), Y(m))P_i(x_i, y)$ and continuity of preferences, we have $(X_i(m), Y(m))P_i(x_i, y)$ if agent i chooses a very large γ_i. ||

Lemma 2. *If $(X(m^*), Y(m^*))\in N_{M,h}(e)$, then $X_i(m^*)\in\mathbb{R}^L_{++}$ for all $i\in N$.*

Proof. Suppose, by way of contradiction, that $X_i(m^*)\in\partial\mathbb{R}^L_+$ for some $i\in N$. Since $p(m^*)\cdot w_i^*>0$, there is some $x_i\in\mathbb{R}^L_{++}$ such that $p(m^*)\cdot x_i\leqq p(m^*)\cdot w_i^*$, $x_i<w^*$, and $(x_i, 0)P_i(X_i(m^*), Y(m^*))$ by Assumption 5. Thus if agent i chooses $y_i=-\sum_{j\neq i}^n y_j^*$ and $z_i=x_i$ and keeps other components of the message unchanged, then $(x_i'(m^*/m_i, i), Y(m^*/m_i, i))=(x_i, 0)$. Therefore, by Lemma 1, $(X_i(m^*/m_i, i), Y(m^*/m_i, i))P_i(X_i(m^*), Y(m^*))$ if agent i chooses a very large γ_i. This contradicts $(X(m^*), Y(m^*))\in N_{M,h}(e)$. ||

Lemma 3. *Suppose $X(m^*)\in\mathbb{R}^{nL}_{++}$ for some $m^*\in M$ and there is $(x_i, y)\in\mathbb{R}^{L+K}_+$ for some $i\in N$ such that $p(m^*)\cdot x_i+q_i(m^*)\cdot y\leqq p(m^*)\cdot w_i^*$ and $(x_i, y)P_i(X_i(m^*), Y(m^*))$. Then there is some $m_i\in M_i$ such that $(X_i(m^*/m_i, i), Y(m^*/m_i, i))P_i(X_i(m^*), Y(m^*))$.*

Proof. By Assumption 6, we know that for y there is an input vector r such that $(r, y)\in\mathcal{Y}$. Let $(x_{i\lambda}, y_\lambda, r_\lambda)=(\lambda x_i+(1-\lambda)X_i(m^*), \lambda y+(1-\lambda)Y(m^*), \lambda r+(1-\lambda)r(m^*))$. Here $r(m^*)=\sum_{j=1}^n X_j(m^*)-w^*$. Then by convexity of preferences and production possibility set we have $(x_{i\lambda}, y_\lambda)P_i(X_i(m^*), Y(m^*))$ and $(r_\lambda, y_\lambda)\in\mathcal{Y}$ for any $0<\lambda<1$. Also $(x_{i\lambda}, y_\lambda)\in\mathbb{R}^{L+K}_+$ and $p(m^*)\cdot x_{i\lambda}+q_i(m^*)\cdot y_\lambda\leqq p(m^*)\cdot w_i^*$.

Since $X(m^*)\in\mathbb{R}^{nL}_{++}$, we must have $p(m^*)\cdot w_j-q_j(m^*)\cdot Y(m^*)>0$ for all $j\in N$ and $r(m^*)=\sum_{j=1}^n X_j(m^*)-w^*>X_i(m^*)-w^*$. Then, we have $p(m^*)\cdot w_j-q_j(m^*)\cdot y_\lambda>0$ for all $j\in N$ and $x_{i\lambda}-w^*<r_\lambda$ as λ is a sufficiently small positive number.[12]

Now suppose that player i chooses $y_i=y_\lambda-\sum_{j\neq i}^n y_j^*$ and $z_i=x_\lambda$ and keeps other components of the message unchanged. Then $Y(m^*/m_i, i)=y_\lambda$ and $x_i'(m^*/m_i, i)=x_{i\lambda}$. By Lemma 1, we have $(X_i(m^*/m_i, i), Y(m^*/m_i, i))P_i(X_i(m^*), Y(m^*))$ if agent i chooses a very large γ_i. ||

Remark 6. From Lemma 3 and monotonicity of preferences, we can know that $q_i(m^*)\in\mathbb{R}^K_{++}$ for all $i\in N$ at any Nash equilibrium $m^*\in M$.

Lemma 4. *If $(X(m^*), Y(m^*))\in N_{M,h}(e)$, then $w_i^*=\hat{w}_i$ for all $i\in N$.*

Proof. Suppose, by way of contradiction, that $w_i^*\leq\hat{w}_i$ for some $i\in N$. Then $p(m^*)\cdot X_i(m^*)+q_i(m^*)\cdot Y(m^*)<p(m^*)\cdot\hat{w}_i$ (since $p(m^*)>0$ and $w_i^*\leq\hat{w}_i$), and thus

12. From Remark 2, we know that $(x_{i\lambda}-w^*, y_\lambda)\in y$ and $(-w^*, y_\lambda)\in\mathcal{Y}$ since $-w^*<x_{i\lambda}-w^*<r_\lambda$.

there is $(x_i, y)\in\mathbb{R}_+^{L+K}$ for some $i\in N$ such that $p(m^*)\cdot x_i+q_i(m^*)\cdot y\leqq p(m^*)\cdot\hat{w}_i$ and $(x_i, y)P_i(X_i(m^*), Y(m^*))$ by monotonicity of preferences. Since $X_i(m^*)\in\mathbb{R}_{++}^L$ for all $i\in N$ (by Lemma 2), there is some $m_i\in M_i$ such that $(X_i(m^*/m_i, i), Y(m^*/m_i, i))P_i(X_i(m^*), Y(m^*))$ by Lemma 3. This contradicts $(X(m^*), Y(m^*))\in N_{M,h}(e)$. ‖

This lemma shows that the true initial endowment must be revealed at Nash equilibrium even if agents may destroy part of their initial endowments at non-equilibrium points.

Lemma 5. *If* $(X(m^*), Y(m^*))\in N_{M,h}(e)$, *then* $p(m^*)\cdot X_i(m^*)+q_i(m^*)\cdot Y(m^*)=p(m^*)\cdot\hat{w}_i$.

Proof. Suppose, by way of contradiction, that $p(m^*)\cdot X_i(m^*)+q_i(m^*)\cdot Y(m^*)<p(m^*)\cdot\hat{w}_i$. Then there is (x_i, y) for some $i\in N$ such that $p(m^*)\cdot x_i+q_i(m^*)\cdot y\leqq p(m^*)\cdot\hat{w}_i$ and $(x_i, y)P_i(X_i(m^*), Y(m^*))$ by monotonicity of preferences. From Lemma 2 we know that $X_i(m^*)\in\mathbb{R}_{++}^L$ for all $i\in N$. Thus, by Lemma 3, there is some $m_i\in M_i$ such that $(X_i(m^*/m_i, i), Y(m^*/m_i, i))P_i(X_i(m^*), Y(m^*))$. This contradicts $(X(m^*), Y(m^*))\in N_{M,h}(e)$. ‖

Lemma 6. *If* $(X(m^*), Y(m^*))\in N_{M,h}(e)$, *then* $\bar{\gamma}^*\gamma_i^*=1$ *for all* $i\in N$ *and thus* $X(m^*)=x'(m^*)$.

Proof. This is a direct corollary of Lemma 5. Suppose $\bar{\gamma}^*\gamma_i^*<1$ for some $i\in N$. Then $X(m^*)=\bar{\gamma}^*\gamma_i^*x'(m^*)<x'(m^*)$, and therefore $p(m^*)\cdot X_i(m^*)+q_i(m^*)\cdot Y(m^*)<p(m^*)\cdot\hat{w}_i$. But this is impossible by Lemma 5. ‖

We now turn to show that every Nash allocation is a Lindahl allocation.

Lemma 7. *If the mechanism has a Nash equilibrium* m^*, *then* $(X(m^*), Y(m^*))$ *is a Lindahl allocation with the price system* $(p(m^*), q_1(m^*), \ldots, q_n(m^*))$, *i.e.* $N_{M,h}(e)\subseteq L(e)$.

Proof. Let m^* be a Nash equilibrium. Now we prove that $(X(m^*), Y(m^*))$ is a Lindahl allocation with the vector system $(p(m^*), q_1(m^*), \ldots, q_n(m^*))\in\mathbb{R}_{++}^{L+nK}$. By Lemma 5, we know $p(m^*)\cdot X_i(m^*)+q_i(m^*)\cdot Y(m^*)=p(m^*)\cdot\hat{w}_i$, and thus the zero-profit condition holds. Also by the definition of the mechanism, $(X(m^*), Y(m^*))$ is feasible. So we only need to show that each individual is maximizing his/her preferences. Suppose, by way of contradiction, that there is some $(x_i, y)\in\mathbb{R}_+^{L+K}$ such that $(x_i, y)P_i(X_i(m^*), Y(m^*))$ and $p(m^*)\cdot x_i+q_i(m^*)\cdot y\leqq p(m^*)\cdot\hat{w}_i$. Since $X_i(m^*)>0$ (by Lemma 2), there is some $m_i\in M_i$ such that $(X_i(m^*/m_i, i), Y(m^*/m_i, i))P_i(X_i(m^*), Y(m^*))$ by Lemma 3. This contradicts $(X(m^*), Y(m^*))\in N_{M,h}(e)$. ‖

Finally, we show every Lindahl allocation is a Nash allocation.

Lemma 8. *If* (x^*, y^*) *is a Lindahl allocation with the price system* $(p^*, q_1^*, \ldots, q_n^*)$, *then there is a Nash equilibrium* m^* *of the mechanism such that* $Y(m^*)=y^*$, $X_i(m^*)=x_i^*$, $q_i(m^*)=q_i^*$, *for all* $i\in N$, $p(m^*)=p^*$, *i.e.* $L(e)\subseteq N_{M,h}(e)$.

Proof. We need to show that there is a message m^* such that (x^*, y^*) is a Nash allocation. Let $w_i^* = \hat{w}_i$, $p_i^* = (p^*, q^*)$ with $q^* = \sum_{j=1}^n q_j^*$, $z_i^* = x_i^*$, $\gamma_i^* = 1$ and let $(y_1^*, \ldots, y_n^*, \phi_1^*, \ldots, \phi_n^*)$ be the solution of the following linear equation system:

$$\sum_{i=1}^n y_i = y^*,$$
$$q_i^k = b_i^k(m^*) + \sum_{j=1}^n a_{ij}^k \phi_{kj} \tag{13}$$

for $k = 1, \ldots, K$. Then, $p(m^*) = p^*$, $q(m^*) = q^*$, $w_i^* = \hat{w}_i$, $\bar{\gamma}^* = 1$, $X_i(m^*) = x_i^*$, $Y(m^*) = y^*$, and $q_i(m^*) = q_i^*$ for all $i \in N$. Notice that $(p(m^*/m_i, i),\ q_i(m^*/m_i, i)) = (p(m^*), q_i(m^*))$ for all $m_i \in M_i$, $(X(m^*/m_i, i),\ Y(m^*/m_i, i)) \in \mathbb{R}_+^{L+K}$, and $p(m^*) \cdot X_i(m^*/m_i, i) + q_i(m^*) \cdot Y(m^*/m_i, i) \leq p(m^*) \cdot \hat{w}_i$ for all $i \in N$ and $m_i \in M_i$. From $(x^*, y^*) \in L(e)$, we have $(X_i(m^*),\ Y(m^*)) R_i (X_i(m^*/m_i, i),\ Y(m^*/m_i, i))$. ||

From Lemmas 7-8, we know that $N_{M,h}(e) = L(e)$ and thus the proof of Theorem 1 is completed. ||

When $K = 0$, we have the following corollary of Theorem 1:

Corollary 1. *For private goods economies satisfying Assumptions* 1-5 (*with* $K = 0$), *there exists a single-valued, feasible, and continuous mechanism which fully Nash-implements the* (*constrained*) *Walrasian correspondence.*

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