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Necessary and Sufficient Conditions for Maximization of a Class of Preference Relations

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This paper provides necessary and sufficient conditions for the existence of greatest and maximal elements of weak and strict preferences, and unifies two very different approaches used in the related literature (the convexity and acyclicity approaches). Conditions called transfer FS-convexity and transfer SS-convexity are shown to be necessary and, in conjunction with transfer closedness and transfer openness, sufficient for the existence of greatest and maximal elements of weak and strict preferences, respectively. The results require neither the continuity nor convexity of preferences, and are valid for both ordered and unordered binary relations. Thus, the results generalize almost all of the theorems on existence of maximal elements of preferences that appear in the literature.

1. INTRODUCTION

The notion of preference relations or utility functions is a central concept in economics, particularly, in consumer theory. Historically, the concept has been included in writings on economics and psychology for a few hundred years, at least as far back as Jeremy Bentham (1789). For the representability of a preference relation, Wold (1943–44) provided the first rigorous study, and the complete solution to this problem was given by Eilenberg (1941), Debreu (1954, 1964), Rader (1963) and Bowen (1968) among others. The complete solution of the existence of maximal (or greatest) elements of a preference relation, however, has not been achieved in the sense that all the conditions for existence are sufficient, and no unified conditions have been obtained for ordered preferences (i.e., for total, reflexive and transitive preferences) and non-ordered preferences. Indeed, one can easily find many simple examples of binary relations which have or do not have a maximal element, but none of the existing theorems can reveal these. One might therefore conjecture that the existing theorems can be generalized. The results obtained in the paper reveal that this conjecture is indeed correct.

For non-total/non-transitive preference relations, some economists (Sonnenschein (1971), Shafer and Sonnenschein (1975), Yannelis and Prabhakar (1983)) have worked only with preferences which are SS-convex or FS-convex or have convex upper contour sets;[1] some economists (Brown (1973), Bergstrom (1975), Walker (1977)) have worked only with acyclic preferences; and others (such as Schmeidler (1969), Rader (1972) among others) have considered existence only for incomplete preferences. All of these studies assume that weakly upper contour sets are closed or strictly lower contour sets are open.

1. The SS is for Shafer and Sonnenschein and the FS is for Fan and Sonnenschein. The definitions of SS and FS convexities will be given below.

Also, the conditions such as SS-convexity, FS-convexity, convexity, and acyclicity, as one will see, are stronger than is necessary to prove existence. Moreover, when a preference relation becomes an ordering, these conditions are redundant and thus only the closedness of weakly upper contour sets are sufficient for existence since, as shown in Tian (1992b), SS-convexity, FS-convexity, and convexity are equivalent to one another. In addition, the convexity conditions and acyclicity conditions seem quite different and thus are considered as two basic approaches to proving the existence of maximal elements in the literature (cf. Border (1985, p. 32)). The unified conditions, to our best knowledge, have not yet been given. A question is then whether or not there exist unified conditions which can be used to prove existence under vaious cases and which automatically hold when a preference relation becomes an ordering, and if so, what the minimal possible conditions for existence are.

This paper answers the above questions in the affirmative and gives necessary and sufficient conditions for the existence of greatest and maximal elements of both weak and strict preference relations, which may be ordered or unordered. Conditions which are called transfer FS-convexity and transfer closedness (resp. transfer SS-convexity and transfer openness) characterize the existence of greatest elements of weak preferences (resp. maximal elements of strict preferences) on compact and convex choice sets. We prove that transfer FS-convexity and transfer SS-convexity are necessary and, in conjunction with transfer closedness and transfer openness, also sufficient for the existence of greatest and maximal elements of weak and strict preferences, respectively. The transfer FS-convexity and transfer SS-convexity conditions not only give a new and interesting class of preference relations but are also extremely weak. Any one of the following conditions which are widely used in the literature implies transfer FS-convexity or transfer SS-convexity: (1) a preference relation is an ordering—in particular, it can be represented by a utility function; (2) a preference relation is antisymmetric, negatively transitive, and irreflexive; (3) a reference relation is acyclic—in particular, it has slightly stronger order properties than acyclicity such as those introduced in Campbell and Walker (1990); (4) a preference relation is SS-convex—in particular, it is irreflexive and its upper contour sets are convex; (5) a preference relation is FS-convex; and (6) a preference relation is everywhere locally KF-majorized. Moreover, even if the compactness and convexity of choice sets are weakened in some way, these conditions continue to be necessary and sufficient for existence so that we can deal with the existence of maximal elements with more general problems, such as those with indivisibility of commodities and non-compact feasible sets.

The plan of this paper is as follows. Section 2 states some notation and definitions about weak and strict preference relations. In Section 3 we provide several existence theorems which give necessary and sufficient conditions for the existence of greatest and maximal elements of weak and strict preferences. Lemma 1, which we use to prove our main results, actually generalizes the KKM lemma by relaxing both closedness and FS-convexity of a correspondence. It may be remarked that the range and domain of the correspondence are not required to be the same and can be two independent sets. The concluding remarks are given in Section 4.

2. NOTATION AND DEFINITIONS

There are two approaches to dealing with the non-transitive/non-total case: One is through "weak" (i.e., reflexive) preferences (see, e.g., Rader (1972), Sonnenschein (1971), Tian

(1992b)); the other is through "strict" (i.e., irreflexive) preferences (see, e.g. Schmeidler (1969), Brown (1973), Mas-Colell (1974), Bergstrom (1975), Walker (1977), Yannelis and Prabhakar (1983), Campbell and Walker (1990), Tian (1992b) among others). The distinction becomes important when preferences are not total.[2]

Suppose that a preference relation which is either weak or strict is defined on a topological space Z and is a subset $Z \times Z$. Here Z may be considered as a consumption space. Let $\succcurlyeq$ denote a weak preference relation and $\succ$ a strict preference relation. Denote weakly upper, weakly lower, strictly upper, and strictly lower contour sets of $\succcurlyeq$ and $\succ$ by, for each x, $U_w(x) = \{y \in Z: y \succcurlyeq x\}$, $L_w(x) = \{y \in Z: x \succcurlyeq y\}$, $U_s(x) = \{y \in Z: y \succ x\}$, and $L_s(x) = \{y \in Z: x \succ y\}$, respectively.

In many cases, not all points in Z can be chosen. So let $B \subset Z$ be a feasible choice set, which may be, say, the budget set or the set of feasible allocations. A weak binary relation $\succcurlyeq$ is said to have a *greatest element* on the subset B of Z if there exists a point $x^* \in B$ such that $x^* \succcurlyeq x$ for all $x \in B$, or equivalently, $B \cap [\bigcap_{x \in B} U_w(x)] \neq \emptyset$. A strict binary relation $\succ$ is said to have a *maximal element* on the subset B of Z if there exists a point $x^* \in B$ such that $\neg x \succ x^*$ for all $x \in B$, or equivalently, $B \cap U_s(x^*) = \emptyset$, where "$\neg$" stands for "it is not the case that."

Let S be a subset of a topological space T and let $D \subset S$. Denote the collections of all subsets, closure, interior, and convex hull of the set D by 2^D, cl D, int D, and co D, respectively. Denote by $\mathrm{cl}_S\, D$ and $\mathrm{int}_S\, D$ the closure and interior of the set D relative to S. Let E be a Hausdorff topological vector space.

A weak binary relation $\succcurlyeq$ is said to be *FS-convex* on B if $U_w: B \to 2^Z$ is FS-convex, i.e. for every finite subset $\{x_1, x_2, \ldots, x_m\}$ of B, we have $\mathrm{co}\, \{x_1, x_2, \ldots, x_m\} \subset \bigcup_{j=1}^m U_w(x_j)$. The FS-convexity of a correspondence was introduced and used by Fan (1961) and Sonnenschein (1971) to generalize the classical KKM lemma by relaxing the finiteness of the index set. A strict binary relation $\succ$ is said to be *SS-convex* on Z if $x \notin \mathrm{co}\, U_s(x)$ for all $x \in Z$. A strict binary relation $\succ$ is said to be *acyclic* on the subset B of Z if for any finite subset $\{x_1, \ldots, x_m\} \subset B$, $x_m \succ x_{m-1}$, $x_2 \succ x_1$ implies that $\neg x_1 \succ x_m$. Let Z be a convex subset of a topological vector space.

There are two basic ways to prove the existence of maximal elements of a strict preference relation. One is based on convexity assumptions. Sonnenschein (1971) and Shafer and Sonnenschein (1975) proved that a maximal element of $\succ$ exists on Z if (1) Z is a non-empty convex compact set of the Euclidean space, (2) the strictly lower contour sets are open, and (3) $\succ$ is SS-convex on Z. Yannelis and Prabhakar (1983) generalized the results of Sonnenschein (1971) and Shafer and Sonnenschein (1975) to Hausdorff topological vector spaces. The second approach is based on acyclicity assumptions and involves no convexity assumptions. Bergstrom (1975) and Walker (1977) proved that a maximal element of $\succ$ exists on Z if (1) Z is a non-empty compact set (2) the weakly lower contour sets are open, and (3) $\succ$ is acyclic on Z. For the existence of greatest elements of weak preferences, Kim and Richter (1986) showed that a greatest element of $\succcurlyeq$ exists on Z if (1) Z is a non-empty convex compact set of the Euclidean space, (2) the weakly upper contour sets are closed, and (3) $\succcurlyeq$ is FS-convex on Z. Tian (1992b) generalized the result of Kim and Richter (1986) to Hausdorff topological vector spaces and non-closed weakly upper contour sets. Unfortunately, all the results mentioned above invariably set forth only sufficient conditions for existence.

2. A preference $\succcurlyeq$ is said to be complete if, for any $x, y \in X$, either $x \succcurlyeq y$ or $y \succcurlyeq x$. A preference $\succcurlyeq$ is said to be total if, for any $x, y \in X$, $x \neq y$ implies $x \succcurlyeq y$ or $y \succcurlyeq x$.

3. THE EXISTENCE OF MAXIMAL ELEMENTS OF PREFERENCES

In this section we present several existence theorems which give necessary and sufficient conditions for the greatest and maximal elements of weak and strict preferences under various assumptions on Z. We first define the following transfer continuity and convexity concepts.

Definition 1 (Transfer Closedness). Let X and Y be two topological spaces. A correspondence $G: X \to 2^Y$ is said to be ***transfer closed-valued*** on X if for every $x \in X$, $y \notin G(x)$ implies that there exists $x' \in X$ such that $y \notin \mathrm{cl}\, G(x')$.

Definition 2 (Transfer Openness). Let X and Y be two topological spaces. A correspondence $P: X \to 2^Y$ is said to be ***transfer open-valued*** on X if for every $x \in X$, $y \in P(x)$ implies that there exists a point $x' \in X$ such that $y \in \mathrm{int}\, P(x')$.

Remark 1. Observe that a correspondence is transfer closed-valued if it is closed-valued; a correspondence is transfer open-valued if it is open-valued by letting $x' = x$. Also a correspondence $P: X \to 2^Y$ is transfer open-valued if and only if the correspondence $G: X \to 2^Y$, defined by, for every $x \in X$, $G(x) = Y \backslash P(x)$, is transfer closed-valued on X.

Definition 3 (Transfer FS-Convexity). Let X be a topological space and let Z be a non-empty convex subset in E. A correspondence $G: X \to 2^Z$ is said to be ***transfer FS-convex*** on X, if for any finite subset $\{x_1, x_2, \ldots, x_m\} \subset X$, there exists a corresponding finite subset $\{y_1, y_2, \ldots, y_m\} \subset Z$ such that for any subset $\{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\} \subset \{y_1, y_2, \ldots, y_m\}$, $1 \leq k \leq m$, we have $\mathrm{co}\{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\} \subset \bigcup_{j=1}^{k} G(x_{i_j})$. A weak binary relation $\succcurlyeq$ defined on Z is said to be ***transfer FS-convex*** on B if $U_w: B \to 2^Z$ is transfer FS-convex on B.

Remark 2. It is clear that FS-convexity implies transfer FS-convexity by letting $X \subset Z$ and $y_i = x_i$. However, transfer FS-convexity is a much weaker condition thatn FS-convexity. For example, for any ordering preference relation $\succcurlyeq$, the weakly upper contour correspondence $U_w B \to 2^Z$ is transfer-FS-convex (by letting $y_1 =, \ldots, = y_m = y^*$ where y^* is the greatet element of the finite subset $\{x_1, x_2, \ldots, x_m\}$). But, U_w is not FS-convex unless it is convex valued. Thus, as long as $\succcurlyeq$ is an ordering, it is FS-convex.

Definition 4 (Transfer SS-Convexity). Let Z be a convex subset of a Hausdorff topological vector space E and let $\emptyset \neq B \subset Z$. A strict binary relation $\succ$ is said to be ***transfer SS-convex*** on B, if for any finite subset $\{x_1, x_2, \ldots x_m\} \subset B$, there exists a corresponding finite subset $\{y_1, y_2, \ldots, y_m\} \subset Z$ such that for any subset $\{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\}$ of $\{y_1, y_2, \ldots, y_m\}$, $1 \leq k \leq m$ and for any $y_{i_0} \in \mathrm{co}\{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\}$, we have $x_{i_j} \notin U_s(y_{i_0})$.

Remark 3. Observe that when $B = Z$, the SS-convexity of $\succ$ implies the transfer SS-convexity of $\succ$ by letting $y_i = x_i$ for $i = 1, \ldots, m$. Transfer SS-convexity is also a very weak condition. For instance, one can easily verify that $\succ$ is transfer SS-convex if it is acyclic or if it is antisymmetric, irreflexive and negatively transitive on B.[3]

3. A preference relation $\succ$ is antisymmetric if, for any, $x, y \in Z, x \neq y$ implies $\neg x \succ y$ or $\neg y \succ x$. A preference relation $\succ$ is negatively transitive if, for any $x, y, z \in Z, \neg x \succ y$ and $\neg y \succ z$ imply $\neg x \succ z$. Kim and Richter (1986, p. 349) showed that the canonical conjugate $\succcurlyeq$ of $\succ$ is an ordering on B.

We first have the following result which generalizes the result of Kim and Richter (1986) by relaxing the closedness and FS-convexity of weakly upper contour sets, and the results of Tian (1992b) by relaxing the FS-convexity of weakly contour sets.

Theorem 1. *Let Z be a non-empty compact convex subset of a Hausdorff topological vector space E and let $\succcurlyeq$ defined on Z be a weak binary relation such that $U_w: Z \to 2^Z$ is transfer closed-valued on Z. Then the set of greatest elements of $\succcurlyeq$ on Z is non-empty and compact if and only if $\succcurlyeq$ is transfer FS-convex on Z.*

Proof. *Necessity* ($\Rightarrow$). Suppose the set of greatest elements of $\succcurlyeq$ on Z, which is given by $\bigcap_{x\in X} U_w(x)$, is non-empty and compact. Then, for any finite subset $\{x_1, x_2, \dots, x_m\} \subset X, \bigcap_{i=1}^m U_w(x_i) \neq \emptyset$. Taking $y^* \in \bigcap_{i=1}^m U_w(x_i)$ and letting $y_i = y^*$ for $i = 1, \dots, m$, we have $\operatorname{co}\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\} = \{y^*\} \subset \bigcap_{i=1}^m U_w(x_{i_j}) \subset \bigcup_{j=1}^k U_w(x_{i_j})$ for any finite subset $\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\} \subset \{y_1, y_2, \dots y_m\}$. So $\succcurlyeq$ is transfer FS-convex on Z.

Sufficiency ($\Leftarrow$). To prove the sufficiency, we first prove the following lemma. ‖

Lemma 1. *Let X be a topological space and Z be a non-empty compact convex subset in a Hausdorff topological vector space E. Suppose a correspondence $G: X \to 2^Z$ is transfer closed-valued and transfer FS-convex on X. Then, $\bigcap_{x\in X} G(x)$ is non-empty and compact.*

Proof. We first prove $\bigcap_{x\in X} \operatorname{cl}_Z G(x) = \bigcap_{x\in X} G(x)$. It is clear that $\bigcap_{x\in X} G(x) \subset \bigcap_{x\in X} \operatorname{cl}_Z G(x)$. So we only need to show $\bigcap_{x\in X} \operatorname{cl}_Z G(x) \subset \bigcap_{x\in X} G(x)$. Suppose, by way of contradiction, that there is some y in $\bigcap_{x\in X} \operatorname{cl}_Z G(x)$ but not in $\bigcap_{x\in X} G(x)$. Then $y \notin G(x)$ for some $x \in X$. By the transfer closedness condition, there is some $x' \in X$ such that $y \notin \operatorname{cl}_Z G(x')$, a contradiction.

For $x \in X$, let $\bar{G}(x) = \operatorname{cl}_Z G(x)$. Thus to prove $\bigcap_{x\in X} G(x)$ is non-empty and compact is equivalent to proving that $\bigcap_{x\in X} \bar{G}(x)$ is non-empty and compact. Since Z is compact and $\bar{G}$ is closed-valued in Z, $\bigcap_{x\in X} \bar{G}(x)$ is clearly compact. So we only need to show it is non-empty. We first show the family of sets $\{\bar{G}(x): x \in X\}$ has the finite intersection property.

Suppose, by way of contradiction, that $\{\bar{G}(x): x \in X\}$ does not have the finite intersection property, i.e., there exists some finite subset $\{x_1, x_2, \dots, x_m\} \subset X$ such that $\bigcap_{i=1}^m \bar{G}(x_i) = \emptyset$. By the transfer FS-convexity of $\bar{G}$, for this given finite set $\{x_1, x_2, \dots, x_m\}$, there exists a corresponding subset $\{y_1, y_2, \dots, y_m\} \subset Z$ such that for any $\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\} \subset \{y_1, y_2, \dots, y_m\}$, $\operatorname{co}\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\} \subset \bigcup_{j=1}^k \bar{G}(x_{i_j})$. In particular, we have $\operatorname{co}\{y_1, y_2, \dots, y_m\} \subset \bigcup_{i=1}^m \bar{G}(x_i)$. Let $S = \operatorname{co}\{y_1, y_2, \dots, y_m\}$ and $L = \operatorname{span}\{y_1, y_2, \dots, y_m\}$. Then $S \subset L$. Since $\bar{G}(x)$ is closed, $\bar{G}(x_i) \cap L$ is also closed. Let d be the Euclidean metric on L. It is easy to see that $d(y, L \cap \bar{G}(x_i)) > 0$ if and only if $y \notin L \cap \bar{G}(x_i)$. Now define a continuous function $f: S \to [0, \infty)$ as follows:

$$f(y) = \sum_{i=1}^m d(y, L \cap \bar{G}(x_i)) \tag{1}$$

for all $y \in S$. It follows from $\bigcap_{i=1}^m \bar{G}(x_i) = \emptyset$ that for each $y \in S$, $f(y) > 0$. Define a continuous function $g: S \to S$ by, for each $y \in S$,

$$g(y) = \sum_{i=1}^m \frac{1}{f(y)} d(y, L \cap \bar{G}(x_i)) y_i. \tag{2}$$

Then, by the Brouwer fixed point theorem, there exists a $y^* \in S$ such that

$$y^* = g(y^*) = \sum_{i=1}^{m} \frac{1}{f(y^*)} d(y^*, L \cap \bar{G}(x_i)) y_i^*. \tag{3}$$

Denote $I = \{i \in \{1, \ldots, m\}: d(y^*, L \cap \bar{G}(x_i)) > 0\}$. Then for each $i \in I$, $y^* \notin L \cap \bar{G}(x_i)$. Since $y^* \in L$, so $y^* \notin \bar{G}(x_i)$ for any $i \in I$ and thus

$$y^* \notin \bigcup_{i \in I} \bar{G}(x_i). \tag{4}$$

From (3), we have $y^* = \sum_{i \in I} [1/f(y^*)] d(y^*, L \cap \bar{G}(x_i)) y_i^* \in \text{co}\{y_i^*: i \in I\}$. However, since $\bar{G}: X \to 2^Z$ is transfer FS-convex, we have $y^* \in \text{co}\{y_i^*: i \in I\} \subset \bigcup_{i \in I} \bar{G}(x_i)$, which contradicts (4). Hence $\{\bar{G}(x): x \in X\}$ has the finite intersection property. Since Z is compact, $\bigcap_{x \in X} G(x) \neq \emptyset$. ||

We now prove the sufficiency of Theorem 1. Since the weakly upper contour correspondence $U_w: Z \to 2^Z$ is transfer closed-valued and transfer FS-convex on X, by Lemma 1, we know that $\bigcap_{x \in Z} U_w(x)$ is non-empty and compact. Hence the set of greatest elements of $\succcurlyeq$ on Z is non-empty and compact. The proof of Theorem 1 is completed. ||

Remark 4. From the proof of necessity of Theorem 1, we can see that U_w must satisfy transfer FS-convexity if it has the finite intersection property. Thus transfer FS-convexity is a necessary (weakest) condition for guaranteeing that $\bigcap_{x \in B} U_w(x)$ is non-empty. Also we can see from the proof that under conditions of Theorem 1 the intersection is non-empty if and only if U_w is transfer FS-convex. This is a strengthening conclusion in the sense that non-emptiness alone, without compactness, implies transfer FS-convexity.[4] Further, one can see that the transfer closedness of G is also a necessary condition for $\bigcap_{x \in X} G(x) = \bigcap_{x \in X} \text{cl}_Z G(x)$ (the sufficiency is given in the proof of Lemma 1). Thus it is the weakest condition which enables us to use the finite intersection property to show the non-emptiness of $\bigcap_{x \in X} G(x)$ on a compact set. Note that Lemma 1 generalizes the results of Fan (1961, 1984), Sonnenschein (1971), and Tian (1992b) by relaxing the FS-convexity and/or closedness of $G(x)$. They also generalize the results of Chang and Zhang (1989) by relaxing the convexity of X and the closedness of $G(x)$. Further, unlike the results mentioned above, we do not require that the domain set X be a subset of the range set Z.

Theorem 1 can be used to characterize the existence of maximal elements for strict preferences $\succ$ defined on Z. From $\succ$, we can define a weak preference $\succcurlyeq$ on Z as follows: $y \succcurlyeq x$ if and only if $\neg x \succ y$. The preference $\succcurlyeq$ defined in such a way is called the 'canonical conjugate' of $\succ$ by Kim and Richter (1986). Then we can easily verify that $U_w(x) = Z \backslash L_s(x)$ for all $x \in B \subset Z$ and thus x^* is a maximal element of $\succ$ if and only if it is a greatest element of $\succcurlyeq$. Also, U_w is transfer closed-valued and transfer FS-convex on B if and only if L_s is transfer open-valued and transfer SS-convex on B. Then we have the following theorem which generalizes the results of Sonnenschein (1971), Rader (1972), Yannelis and Prabhakar (1983), and Tian (1992b) by relaxing the SS-convexity of $\succ$ and/or the openness of strictly lower contour sets.

Theorem 2. *Let Z be a non-empty compact convex subset of a Hausdorff topological vector space E and let $\succ$ defined on Z be a strict binary relation such that L_s is transfer*

4. I thank an anonymous referee for pointing out this fact to me.

open-valued on Z. Then the set of maximal elements of $>$ *on Z is non-empty and compact if and only if* $>$ *is transfer SS-convex on Z.*

Theorems 1 and 2 can be generalized to a non-compact and non-convex choice set B so that the existence of greatest (maximal) elements can be proven even when commodities are indivisible and/or feasible sets are non-compact.

Theorem 3. *Let Z be a convex subset of a Hausdorff topological vector space E, let B be a non-empty subset of Z, and let* $\succcurlyeq$ *(resp.* $>$*) defined on Z be a weak (resp. strict) binary relation such that* $U_w: B \to 2^Z$ *(resp.* $L_s: B \to 2^Z$*) satisfies the following conditions:*

(*a*) *U_w (resp. L_s) is transfer closed-valued (resp. transfer open-valued) on B;*

(*b*) *there exists a finite subset* $\{x_{01}, \ldots, x_{0n}\} \subset B$ *such that* $\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})$ *is compact* ($U_w(x) = Z \backslash L_s(x)$ *for the strict preference*);

(*c*) *for each* $y \in Z \backslash B$ *there exists* $x \in B$ *with* $y \notin U_w(x)$ *(resp.* $y \in L_s(x)$*).*

Then the set of greatest (resp. maximal) elements of $\succcurlyeq$ *(resp.* $>$*) on B is non-empty and compact if and only if* $\succcurlyeq$ *(resp.* $>$*) is transfer FS-convex (resp. transfer SS-convex) on B.*

Proof. We only need to prove the theorem for the case of weak preferences. The proof of necessity is the same as that of Theorem 1. So we only need to show sufficiency. Since $U_w: B \to 2^Z$ is transfer FS-convex on Z, from the proof of Lemma 1, $\mathrm{cl}_z\, U_w$ has the finite intersection property and thus the family of sets $\{\mathrm{cl}_Z\, U_w(x) \cap [\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})]: x \in B\}$ also has the finite intersection property on the compact set $\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})$. Now since $\{\mathrm{cl}_Z\, U_w(x) \cap [\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})]: x \in B\}$ is a family of compact sets in $\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})$, we have $\emptyset \neq [\bigcap_{x \in B} \mathrm{cl}_Z\, U_w(x)] \cap [\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})] = \bigcap_{x \in B} \mathrm{cl}_Z\, U_w(x) = \bigcap_{x \in B} U_w(x)$. The intersection is clearly compact. Finally, we show $\bigcap_{x \in B} U_w(x) \subset B$. Indeed, for any $y^* \in \bigcap_{x \in B} U_w(x)$, we must have $y^* \in B$, for otherwise $y^* \notin U_w(x)$ for some $x \in B$ by condition (c). Hence $y^* \in B$. ‖

Remark 5. Note that Condition (c) in the above theorem is satisfied if $B = Z$. This assumption is similar to an assumption in Borden (1985, p. 35) and means that any point not in the feasible choice set B must not be a greatest and maximal element of $\succcurlyeq$ and $>$, respectively.

Remark 6. From the above discussions, we know that transfer FS-convexity and transfer FS-convexity are necessary conditions for the existence of greatest and maximal elements of *any* type of weak and strict preferences. Thus many preference relations in the literature satisfy these types of convexities. For instance, the local KF-majorized preference relation introduced by Borglin and Keiding (1976) and the preference relations with slightly stronger order properties than acyclicity introduced in Campbell and Walker (1990) satisfy transfer SS-convexity.

We know that when $\succcurlyeq$ is transfer FS-convex, transfer closedness is a sufficient condition for the set of greatest elements to be non-empty and compact. Even though this condition, in general, is not necessary, it is, however, very weak. As shown in Tian and Zhou (1990), when a weak preference is an ordering, this condition completely characterizes the non-empty and compactness of greatest elements of $\succcurlyeq$. This result in fact can be obtained as a consequence of Theorem 3.

Corollary 1. *Let B be a non-empty compact subset in a Hausdorff topological vector space E and let* $\succcurlyeq$ *be an ordering. Then the set of greatest elements of* $\succcurlyeq$ *is non-empty and compact if and only if* U_w *is transfer closed-valued on B.*

open-valued on Z. Then the set of maximal elements of $>$ on Z is non-empty and compact if and only if $>$ is transfer SS-convex on Z.

Theorems 1 and 2 can be generalized to a non-compact and non-convex choice set B so that the existence of greatest (maximal) elements can be proven even when commodities are indivisible and/or feasible sets are non-compact.

Theorem 3. *Let Z be a convex subset of a Hausdorff topological vector space E, let B be a non-empty subset of Z, and let $\succcurlyeq$ (resp. $>$) defined on Z be a weak (resp. strict) binary relation such that $U_w: B \to 2^Z$ (resp. $L_s: B \to 2^Z$) satisfies the following conditions:*

(a) U_w (resp. L_s) is transfer closed-valued (resp. transfer open-valued) on B;

(b) there exists a finite subset $\{x_{01}, \ldots, x_{0n}\} \subset B$ such that $\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})$ is compact ($U_w(x) = Z \setminus L_s(x)$ for the strict preference);

(c) for each $y \in Z \setminus B$ there exists $x \in B$ with $y \notin U_w(x)$ (resp. $y \in L_s(x)$).

Then the set of greatest (resp. maximal) elements of $\succcurlyeq$ (resp. $>$) on B is non-empty and compact if and only if $\succcurlyeq$ (resp. $>$) is transfer FS-convex (resp. transfer SS-convex) on B.

Proof. We only need to prove the theorem for the case of weak preferences. The proof of necessity is the same as that of Theorem 1. So we only need to show sufficiency. Since $U_w: B \to 2^Z$ is transfer FS-convex on Z, from the proof of Lemma 1, $\mathrm{cl}_z\, U_w$ has the finite intersection property and thus the family of sets $\{\mathrm{cl}_Z\, U_w(x) \cap [\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})]: x \in B\}$ also has the finite intersection property on the compact set $\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})$. Now since $\{\mathrm{cl}_Z\, U_w(x) \cap [\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})]: x \in B\}$ is a family of compact sets in $\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})$, we have $\emptyset \neq [\bigcap_{x \in B} \mathrm{cl}_Z\, U_w(x)] \cap [\bigcap_{i=1}^{n} \mathrm{cl}_Z\, U_w(x_{0i})] = \bigcap_{x \in B} \mathrm{cl}_Z\, U_w(x) = \bigcap_{x \in B} U_w(x)$. The intersection is clearly compact. Finally, we show $\bigcap_{x \in B} U_w(x) \subset B$. Indeed, for any $y^* \in \bigcap_{x \in B} U_w(x)$, we must have $y^* \in B$, for otherwise $y^* \notin U_w(x)$ for some $x \in B$ by condition (c). Hence $y^* \in B$. ||

Remark 5. Note that Condition (c) in the above theorem is satisfied if $B = Z$. This assumption is similar to an assumption in Borden (1985, p. 35) and means that any point not in the feasible choice set B must not be a greatest and maximal element of $\succcurlyeq$ and $>$, respectively.

Remark 6. From the above discussions, we know that transfer FS-convexity and transfer FS-convexity are necessary conditions for the existence of greatest and maximal elements of *any* type of weak and strict preferences. Thus many preference relations in the literature satisfy these types of convexities. For instance, the local KF-majorized preference relation introduced by Borglin and Keiding (1976) and the preference relations with slightly stronger order properties than acyclicity introduced in Campbell and Walker (1990) satisfy transfer SS-convexity.

We know that when $\succcurlyeq$ is transfer FS-convex, transfer closedness is a sufficient condition for the set of greatest elements to be non-empty and compact. Even though this condition, in general, is not necessary, it is, however, very weak. As shown in Tian and Zhou (1990), when a weak preference is an ordering, this condition completely characterizes the non-empty and compactness of greatest elements of $\succcurlyeq$. This result in fact can be obtained as a consequence of Theorem 3.

Corollary 1. *Let B be a non-empty compact subset in a Hausdorff topological vector space E and let $\succcurlyeq$ be an ordering. Then the set of greatest elements of $\succcurlyeq$ is non-empty and compact if and only if U_w is transfer closed-valued on B.*

Proof. *Necessity* ($\Rightarrow$). Suppose that the set of greatest elements of $\succcurlyeq$ is non-empty compact. Since $\succcurlyeq$ is an ordering, then for every $x \in B$, if $y \notin U_w(x)$ for some $y \in B$, $y \notin U_w(x')$, where $x' \in B$ is a greatest element of $\succcurlyeq$ on B. Since the set of greatest elements is compact, there exists a neighbourhood $\mathcal{N}(y)$ of y such that $y' \notin U_w(x')$ for all $y' \in \mathcal{N}(y)$. Thus, $y \notin \text{cl } U_w(x')$. Hence U_w is transfer closed-valued on B.

Sufficiency ($\Leftarrow$). Define an extended preference ordering $\succcurlyeq^*$ on E as follows: All points in E but not in B are indifferent and less preferred to any point in B. If two points are in B, the relation is determined by $\succcurlyeq$. Then, for every $x \in B$, weakly upper contour sets of $\succcurlyeq^*$ are $U_w^*(x) \equiv \{y \in E : y \succcurlyeq^* x\} = \{y \in B : y \succcurlyeq x\} = U_w(x)$. Thus U_w^* is transfer closed-valued and transfer FS-convex on B since U_w is transfer closed-valued on B and $\succcurlyeq^*$ is an ordering. As $U_w^*(x) \subset B$ for all $x \in B$, $\text{cl}_Z\, U_w^*(x_0)$ is compact for all $x_0 \in B$. Also, by the definition of $\succcurlyeq^*$, for each $y \in E \backslash B$ and $x \in B$, we have $y \notin U_w^*(x)$. Then, by Theorem 3, the set of greatest elements of $\succcurlyeq^*$ is non-empty and compact and therefore the set of greatest elements of $\succcurlyeq$ is non-empty and compact on B. ‖

Remark 7. Observe that the transfer closedness of $U_w(x)$ can be equivalently stated as follows: If for each $x \in B$, $x \succ y$ for some $y \in Z$ implies that there exists a point $x' \in B$ and a neighbourhood $\mathcal{N}(y)$ of y such that $x' \succ z$ for all $z \in \mathcal{N}(y)$. Here $\succ$ is the asymmetric part of $\succcurlyeq$. It is interesting to see that if we weaken this condition slightly by replacing $x' \succ z$ for all $z \in \mathcal{N}(y)$ by $x' \succcurlyeq z$ for all $z \in \mathcal{N}(y)$, then the modified condition becomes a necessary condition for existence by letting x' be any greatest element. Further, Tian and Zhou (1990) showed that, by using the finite covering theorem, this condition is in fact a necessary and sufficient condition for a function to attain its maximum on the compact set, even though the set of maximum points is not necessarily compact.

Similarly, by applying Theorem 3 and Corollary 1, we have the following corollary which characterizes the non-emptiness and compactness of the set of maximal elements when $\succ$ is acyclic or antisymmetric, irreflexive and negatively transitive and generalizes the results of Brown (1973), Bergstrom (1975), and Walker (1977) by relaxing the openness of lower contour sets.

Corollary 2. *Let B be a non-empty compact subset of a Hausdorff topological vector space E and let $\succ$ defined on B be a strict binary relation such that $\succ$ is acyclic or antisymmetric, irreflexive and negatively transitive. Then the set of maximal elements of $\succ$ is non-empty and compact if L_s is transfer open-valued on B. Further, the condition is necessary when $\succ$ is antisymmetric, irreflexive and negatively transitive.*

4. CONCLUSION

This paper shows that the closedness of weakly upper contour sets and the FS-convexity of weak preference relations, as well as the openness of strictly lower contours sets and the transfer SS-convexity of strict preference relations, can be significantly weakened. In doing so, we have unified the convexity and acyclicity approaches used in the existing literature on the greatest and maximal elements of binary relations. Specifically, transfer FS-convexity and transfer SS-convexity are proved to be necessary and, in conjunction with transfer closedness and transfer openness, also sufficient for the existence of greatest and maximal elements of weak and strict preferences, respectively. Thus our results, which require neither closedness (resp. openness) nor FS-convexity (resp. SS-convex, or acyclicity) of weakly upper (resp. strictly lower) contour sets, generalize almost all of the existence theorems on maximal elements in the literature. Even though this paper only considers the existence of greatest and maximal elements of preference relations, we think the transfer methods used here can be used to generalize the existence theorems on the competitive mechanisms, Nash equilibrium, and abstract economies, of mappings such as those in Schmeidler (1969), Mas-Colell (1974), Shafer and Sonnenschein (1975), Yannelis and Prabhakar (1983), Tian (1992a, 1993), and others. In this volume, Baye, Tian, and Zhou (1993) use the transfer method to characterize the existence of equilibria in games with discontinuous and nonquasiconcave payoff functions. Zhou and Tian (1992) further developed the transfer method systematically to study the existence of maximal elements of a binary relation on a general topological space which may not be a set of topological *vector* space.

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