Review of Economic Studies (1993) **60**, 935-948
0034-6527/93/00470935$02.00

Characterizations of the Existence of Equilibria in Games with Discontinuous and Non-quasiconcave Payoffs

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First version received April 1991; *final version accepted January* 1993 (*Eds.*)

This paper characterizes pure-strategy and dominant-strategy Nash equilibrium in non-cooperative games which may have discontinuous and/or non-quasiconcave payoffs. Conditions called diagonal transfer quasiconcavity and uniform transfer quasiconcavity are shown to be necessary and, with conditions called diagonal transfer continuity and transfer upper semicontinuity, sufficient for the existence of pure-strategy and dominant-strategy Nash equilibrium, respectively. The results are used to examine the existence or non-existence of equilibrium in some well-known economic games with discontinuous and/or non-quasiconcave payoffs. For example, we show that the failure of the existence of a pure-strategy Nash equilibrium in the Hotelling model is due to the failure of an aggregator function to be diagonal transfer quasiconcave—not the failure of payoffs to be quasiconcave, as has been elsewhere conjectured.

1. INTRODUCTION

This paper presents theorems that completely characterize the existence of pure-strategy and dominant-strategy Nash equilibrium in non-cooperative games where individual payoffs may be non-quasiconcave and/or discontinuous, and where strategy spaces may be non-compact and/or non-convex. We do so by introducing generalized notions of quasiconcavity and continuity. Our diagonal transfer quasiconcavity condition is a natural generalization of quasiconcavity, and turns out to be a necessary condition for the existence of pure-strategy Nash equilibrium without imposing any additional conditions on strategy spaces or payoff functions. Since the conditions imposed in the existing literature on existence thus imply diagonal transfer quasiconcavity, our results extend virtually all of the literature on existence of pure-strategy Nash equilibrium. For this reason, it is useful to provide an overview of the existing literature.

The early theorems of Nash (1950, 1951), Debreu (1952) and Fan (1953) reveal that games possess a pure-strategy Nash equilibrium if (1) the strategy spaces are non-empty, convex and compact, and (2) players have continuous, quasiconcave payoff functions. The theorems say nothing about equilibrium in games with discontinuous and/or non-quasiconcave payoffs. Accordingly, Dasgupta and Maskin (1986) were motivated to establish an existence theorem valid for games with discontinuous payoff functions. Their results reveal that such games possess a pure-strategy equilibrium, provided (1) the strategy spaces are non-empty, convex and compact, and (2) players have payoff functions that are quasiconcave, upper semicontinuous and graph continuous. McManus (1964), Roberts and Sonnenschein (1976), Nishimura and Friedman (1981) and Vives (1990), among others, establish the existence of pure-strategy Nash equilibrium in games with non-quasiconcave payoffs. The results of Nishimura and Friedman (1981) assume, in addition to some conditions on best-reply correspondences, that payoff functions are continuous. Vives (1990) establishes existence in games where payoffs are upper semicontinuous and satisfy certain monotonicity properties by using lattice-theoretic methods and Tarski's fixed point theorem. An advantage of Vives' approach is that it does not require the convexity of strategy sets. To the best of our knowledge, no results are available on the existence of dominant-strategy equilibria. This is surprising, given the growing literature on incentive-compatible dominant-strategy mechanisms.[1]

The paper is organized as follows. In Section 2 we introduce the basic terminology used in our study of non-cooperative games. Section 3 defines the aggregator function that underlies our analysis of non-cooperative games, and introduces the concepts of diagonal transfer quasiconcavity and diagonal transfer continuity used in our characterization of pure-strategy Nash equilibrium. These concepts are applied to the aggregator function to obtain our main theorems on existence. We also provide two propositions that give sufficient conditions for these generalized quasiconcavity and continuity conditions. Section 4 presents theorems on the existence of dominant-strategy equilibrium, which are based on the transfer upper semicontinuity and uniform transfer quasiconcavity of individual payoff functions. Section 5 applies our results to some well-known discontinuous and/or non-quasiconcave economic games. For example, we show that diagonal transfer quasiconcavity is to be credited for the existence of equilibrium in Tullock's (1980) model of rent-seeking, while the lack of diagonal transfer quasiconcavity is to be blamed for the failure of existence in the Hotelling (1929) and Varian (1980) models. The proofs of our theorems are relegated to Section 6, and a few concluding remarks are offered in Section 7.

2. NON-COOPERATIVE GAMES

We assume that the topologial space under consideration, denoted by $\Re^L$, is Euclidian.[2] Let D be a subset of L-dimensional space $\Re^L$. Denote the convex hull and closure by co D and cl D, respectively. Let S be a subset of $\Re^L$ and let $D \subset S$. Denote by $\text{cl}_S D$ the relative closure of D in S.

Let I be a finite set of players,[3] and suppose that each agent i's strategy set is $Z_i \subset \Re^{L_i}$. Denote by Z the (Cartesian) product $\prod_{j\in I} Z_j$ and Z_{-i} the product $\prod_{j\in I\setminus\{i\}} Z_j$. Each player i has a payoff function $u_i : Z \to \Re$. Variables with superscripts or without subscripts, such as x^1, x^k, x, y, will be used to denote elements of Z. Subscripts on variables will associate the variable with a particular player or group of players. For example, x_i and y_i will be used to denote elements of Z_i, while x_{-i} and y_{-i} will be used to denote elements of Z_{-i}. A *game* $\Gamma = (Z_i, u_i)_{i\in I}$ is simply a family of ordered tuples (Z_i, u_i).

A point $y^* \in Z$ is said to be a *pure-strategy Nash equilibrium* for Γ if $u_i(y^*) \geqq u_i(x_i, y^*_{-i})$ for all $x_i \in Z_i$ and for all $i \in I$. A point $y^* \in Z$ is said to be a *dominant-strategy* (*Nash*) *equilibrium* for Γ if, for all $i \in I$, $u_i(y_i^*, x_{-i}) \geqq u_i(x_i, x_{-i})$ for all $(x_i, x_{-i}) \in Z$.

1. Examples of dominant-strategy mechanisms in the incentive-compatibility literature include Vickery (1961), Hurwicz (1972, 1986), Groves (1973), Groves and Loeb (1975), Groves and Ledyard (1987) and Tian (1992b).

2. However, all of our results hold for any Hausdorff topological vector space.

3. Again, all the results in the paper hold for a countable infinity of players.

3. CHARACTERIZATION OF PURE-STRATEGY NASH EQUILIBRIUM

Our strategy is to consider a mapping of individual payoffs into an aggregator function, and then determine the restrictions on the aggregator function that guarantee the existence of Nash equilibrium. This kind of approach was pioneered by Nikaido and Isoda (1955). Dasgupta and Maskin (1986) also use a similar approach to prove the existence of mixed-strategy Nash equilibrium in games with discontinuous payoff functions.

Our theorems on the existence of pure-strategy Nash equilibrium apply generalized notions of quasiconcavity and continuity to the aggregator function, $U: Z \times Z \to \Re \cup \{\pm\infty\}$ given by[4]

$$U(x, y) = \sum_{i \in I} u_i(x_i, y_{-i}). \tag{1}$$

Specifically, we will show that the diagonal transfer quasiconcavity and continuity conditions defined below imply the existence of a point $y^* \in Z$ such that $U(x, y^*) \leqq U(y^*, y^*)$ for all $x \in Z$. This, in turn, implies that y^* is a Nash equilibrium (by letting $x = (x_i, y^*_{-i})$). Note that $U(y, y) = \sum_{i \in I} u_i(y)$ is simply the sum of the payoffs arising from some profile y, while $U(x, y) = \sum_{i \in I} u_i(x_i, y_{-i})$ is the sum of the payoffs that arise when each player unilaterally deviates from y_i to x_i, given y_{-i}. We will refer to y as a candidate profile and x as a deviation profile. If $U(x, y) > U(y, y)$, we say that deviation profile x upsets candidate profile, y.

Definition 1 (*Diagonal Transfer Continuity*). Let Z be a subset of $\Re^L$ and let A and C be two non-empty subsets of Z. A function $U: Z \times C \to \Re$ is said to be *diagonally transfer continuous in* y *on* A if for every $(x, y) \in A \times C$, $U(x, y) > U(y, y)$ implies that there exist some point $x' \in A$ and some neighbourhood $\mathcal{N}(y) \subset C$ of y such that $U(x', z) > U(z, z)$ for all $z \in \mathcal{N}(y)$. We will simply say U is diagonally transfer continuous in y when $A = Z$.

Diagonal transfer continuity says that if candidate profile y is upset by a deviation profile x, then there is an open set of candidate profiles containing y, all of which can be upset by a single deviation profile x'. This implies among other things that, if equilibrium fails to exist, then there is a finite number of deviation profiles which suffice to upset all candidate profiles.[5] In the definition, *diagonal* refers to the fact that U on the right-hand side of the inequality is evaluated at points along the diagonal (i.e., $U(z, z)$); *transfer continuity* refers to the fact that x may be transferred to some x' in order for the inequality to hold for all points in a neighbourhood of y. The usual notion of continuity would require that the first inequality hold at x for all points in a neighbourhood of y. Thus, diagonal transfer continuity is much weaker than the notions of continuity used in the literature and, as we will show in Section 5, it is satisfied in many economic games with discontinuous payoffs.

Definition 2 (*Diagonal Transfer Quasiconcavity*). Let Z be a convex subset of $\Re^L$, let $\varnothing \neq A \subset Z$, and let C be a non-empty convex subset of Z. A function $U(x, y): Z \times C \to \Re$ is said to be *diagonally transfer quasiconcave* in x on A if, for any finite subset $X^m = \{x^1, \ldots, x^m\} \subset A$, there exists a corresponding finite subset $Y^m = \{y^1, \ldots, y^m\} \subset C$ such

4. When I is a countably infinite set, one may define U according to $U(x, y) = \sum_{i \in I} (1/2^i) u_i(x_i, y_{-i})$. This is a more general formulation.

5. We thank an anonymous referee for helping us with this intuition as well as the one for diagonal transfer quasiconcavity given below.

that for any subset $\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\} \subset Y^m$, $1 \leq s \leq m$, and any $y^{k^0} \in \text{co}\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\}$ we have

$$\min_{1\leq l\leq s} U(x^{k^l}, y^{k0}) \leqq Y(y^{k0}, y^{k0}). \tag{2}$$

We will simply say U is diagonally transfer quasiconcave in x when $A = Z$.

Our choice of the phrase *transfer quasiconcavity* reflects the fact that the y^k's are not restricted to equal the x^k's, as in the usual definition of quasiconcavity. *Diagonal* refers to the fact that U on the right-hand side of the inequality (2) is evaluated at convex combinations along the diagonal (y^{k0}, y^{k0}). Diagonal transfer quasiconcavity says that, given any deviation profile x, there exists a candidate profile y that is not upset by x. Given any two deviation profiles x^1 and x^2, there exist corresponding candidate profiles y^1 and y^2 such that y^1 is not upset by x^1 and y^2 is not upset by x^2; and any weighted average (convex combination) $y = \lambda y^1 + (1-\lambda)y^2$ with $0 \leqq \lambda \leqq 1$ is not upset by x^1 *and* x^2. For the general case, it roughly says that given any finite set X^m of deviation profiles, there exists a corresponding finite set Y^m of candidate profiles such that, for any subset $Y^{\tilde{m}} \subset Y^m (1 \leqq \tilde{m} \leqq m)$, its convex combinations are not upset by *all* of the deviations in $X^{\tilde{m}}$. We will see from Theorem 1 below that diagonal transfer quasiconcavity is necessary for the existence of a pure-strategy Nash equilibrium.

Remark 1. The diagonal transfer quasiconcavity of U is a very weak condition.[6] It can be easily seen that U is diagonal transfer quasiconcave in x if its is quasiconcave or diagonally quasiconcave in x (by letting $y^k = x^k$).[7] Further sufficient conditions for diagonal transfer quasiconcavity and diagonal transfer continuity are provided in Propositions 1 and 2 below.

We now state our main result, which characterizes the existence of pure-strategy Nash equilibria when the strategy space is compact and convex. (All proofs are presented in Section 6.)

Theorem 1. *Suppose that the strategy space Z_i is a non-empty convex compact subset in $\Re^{L_i}$ and $U: Z \times Z \to \Re$ is defined by* (1) *such that $U(x, y)$ is diagonally transfer continuous in y. Then Γ has a pure-strategy Nash equilibrium if and only if $U(x, y)$ is diagonally transfer quasiconcave in x.*

The key role of our generalized quasiconcavity and continuity conditions in Theorem 1 is roughly as follows. When U satisfies the diagonal transfer continuity and diagonal transfer quasiconcavity conditions, $G(x) \equiv \{y \in Z: U(x, y) \leqq U(y, y)\}$ satisfies $\bigcap_{x\in Z} G(x) \neq \emptyset$, which implies that there exists a point $y^* \in Z$ such that $U(x, y^*) \leqq U(y^*, y^*)$ for all $x \in Z$. This means that y^* is a pure-strategy Nash equilibrium. In the proof of Theorem 1, we will see that diagonal transfer continuity implies that $\bigcap_{x\in Z} G(x) = \bigcap_{x\in Z} \text{cl}_Z G(x)$. This, in conjunction with the diagonal transfer quasiconcavity of U, implies that the family of sets $\{\text{cl}_Z G(x): x \in Z\}$ has the finite intersection property and thus $\bigcap_{x\in Z} G(x) = \bigcap_{x\in Z} \text{cl}_Z G(x) \neq \emptyset$ by compactness of Z.

6. Of course, weaker conditions make it harder to verify the conditions. For this reason we present below two propositions that set-forth sufficient conditions for diagonal transfer quasiconcavity and continuity.

7. A function $U: Z \times Z \to \Re$ is said to be *diagonally quasiconcave in* x if for every finite subset X^m of Z and $x^0 \in \text{co}\, X^m$, $\min_k U(x^k, x^0) \leqq U(x^0, x^0)$. Diagonal quasiconcavity, due to Chen and Zhou (1988), requires that the function be quasiconcave in x only for those y's such that $y = x^0$. This is a weaker requirement than quasiconcavity in x, for *all* y.

Remark 2. From the proof of the necessity of Theorem 1 given in Section 6 below, the diagonal transfer quasiconcavity condition is necessary for the existence of equilibrium, even in the absence of any additional conditions on Z or U. Note that the contrapositive of this fact states that if the aggregator function, $U(x, y)$, is not diagonally transfer quasiconcave in x, then the game does not possess a pure-strategy Nash equilibrium. This statement is powerful because it is *not* predicted on individual payoffs being (semi)continuous or strategy spaces being compact.

In general, the weaker the conditions in an existence theorem, the harder it is to verify whether the conditions are satisfied in a particular game. For this reason it is useful to provide sufficient conditions for diagonal transfer quasiconcavity and diagonal transfer continuity. These conditions may be used along with Theorem 1 to provide new sufficient conditions for the existence of pure-strategy Nash equilibrium.

Proposition 1. *Suppose Z_i is a convex subset of $\Re^{L_i}$. Then any one of the following conditions is sufficient for $U(x, y)$ defined in (1) to be diagonally transfer quasiconcave in x:*

1(*a*) *Each $u_i(x_i, x_{-i})$ is concave in x_i;*

1(*b*) *$U(x, y)$ is concave in x;*

1(*c*) *$U(x, y)$ is quasiconcave in x;*

1(*d*) *$U(x, y)$ is diagonally quasiconcave in x.*

1(*e*) *For a two-person game, one player's payoff function is arbitrary and the other player's payoff function is uniformly transfer quasiconcave on a convex compact set Z, and transfer upper semicontinuous in his own strategy.*[8]

In addition, by Remark 2 any of the conditions imposed in the existing theorems on the existence of pure-strategy Nash equilibrium, such as those in Nash (1951), Debreu (1952), Nikaido and Isoda (1955), Nishimura and Friedman (1981), Dasgupta and Maskin (1986), Tian and Zhou (1992) and Vives (1990) imply the diagonal transfer quasiconcavity of $U(x, y)$ in x.

Proposition 2. *Any one of the following conditions is sufficient for $U(x, y)$ defined in (1) to be diagonally transfer continuous:*

2(*a*) *Each $u_i(x, x_{-i})$ is continuous;*

2(*b*) *Each $u_i(x_i, x_{-i})$ is upper semicontinuous in x_i and continuous in x_{-i};*[9]

2(*c*) *$U(x, y)$ is continuous;*

2(*d*) *$\phi(x, y) \equiv U(x, y) - U(y, y)$ is lower semicontinuous in y;*

2(*e*) *For two-person games with $Z \subset \Re^2$, $U(y, y)$ is upper semicontinuous and u_i is given by*

$$u_i(x_1, x_2) = \begin{cases} f_i^-(x) & \text{if } x_i < x_{-i} - d \\ f_i^0(x) & \text{if } |x_1 - x_2| \leqq d \\ f_i^+(x) & \text{if } x_i > x_{-i} + d \end{cases} \tag{3}$$

where $d > 0$ and $f_i^-(x)$, $f_i^0(x)$, and $f_i^+(x)$ are all continuous;

2(*f*) *$U(y, y)$ is upper semicontinuous and for $U(x, y) > U(y, y)$, there exists a point x' such that $U(x', y) > U(y, y)$ and $U(x', y)$ is lower semicontinuous in y.*

8. The definitions of uniformly transfer quasiconcavity and transfer upper semicontinuity will be given in Section 4.

9. A function $F: Z \to \Re$ is said to be *upper semicontinuous* if, for each point x', $\limsup_{x \to x'} f(x) \leqq f(x')$. Naturally, a function $f: Z \to \Re$ is said to be *lower semicontinuous* if $-f(x)$ is upper semicontinuous.

As a direct consequence of Theorem 1 and Propositions 1 and 2, any pair of conditions, one each from Propositions 1 and 2, are sufficient for the existence of a pure-strategy Nash equilibrium for games with convex, compact strategy spaces.

In many cases, there may exist profiles of strategies that are not *socially feasible*, so let $A \subset Z$ denote the set of *socially feasible actions.* A *quasi-game* dealing with this situation can be defined as $\Gamma = (Z_i, A, u_i)_{i\in I}$ which is a family of ordered triples (Z_i, A, u_i). A point $y^* \in Z$ is said to be a *pure-strategy Nash equilibrium* for a quasi-game Γ if $y^* \in A$ and $u_i(y^*) \geqq u_i(x_i, y^*_{-i})$ for all $x_i \in Z_i$ with $(x_i, y^*_{-i}) \in A$ and for all $i \in I$.

Theorem 1 can be extended to quasi-games where the strategy space is not compact and the set of socially feasible actions (A) is not a compact, convex, and Cartesian product subset of Z.

Theorem 2. *Let the strategy space Z be a non-empty convex subset in $\Re^L$ and let $\varnothing \neq A \subset Z$. Suppose that $U: Z \times Z \to \Re$ is defined by (1) such that*

(*a*) *$U(x, y)$ is diagonally transfer continuous in y on A;*

(*b*) *there exist $x^1, \ldots, x^n \in A$ such that $\bigcap_{k=1}^n \mathrm{cl}_Z G(x^k)$ is compact where $G(x) = \{y \in Z: U(x, y) \leqq U(y, y)\}$.*

(*c*) *for each $y \in Z \backslash A$ there exists $x \in A$ such that $U(x, y) > U(y, y)$.*

Then Γ has a pure-strategy Nash equilibrium on A if and only if $U(x, y)$ is diagonally transfer quasiconcave in x on A.

Note that condition (*c*) above guarantees that the equilibrium point in the socially feasible set A.

Theorems 1 and 2 raise the following question:[10] Does a game have an equilibrium under other (possibly weaker) conditions than the combination of diagonal transfer continuity and diagonal transfer quasiconcavity? One answer to this question is provided in

Theorem 3. *Let the strategy space Z be a non-empty subset in $\Re^L$. Let $U: Z \times Z \to \Re$ be defined by (1). Then Γ has a pure-strategy Nash equilibrium on Z if and only if there exists a non-empty convex subset $C \subset Z$ such that the restricted mapping $U|_{Z\times C}: Z \times C \to \Re$ satisfies the following conditions*

(*a*) *$U|_{Z\times C}(x, y)$ is diagonally transfer continuous in y;*

(*b*) *$U|_{Z\times C}(x, y)$ is diagonally transfer quasiconcave in x;*

(*c*) *there exist $x^1, \ldots, x^n \in Z$ such that $\bigcap_{k=1}^n \mathrm{cl}_C G(x^k)$ is compact where $G(x) = \{y \in C: U(x, y) \leqq U(y, y)\}$.*

Thus, the above theorem completely characterizes the existence of pure-strategy Nash equilibrium without imposing any preassembly conditions on strategy spaces or payoff functions.

4. CHARACTERIZATION OF DOMINANT-STRATEGY EQUILIBRIUM

We next present theorems that give necessary and sufficient conditions for the existence of dominant-strategy Nash equilibrium. Accordingly, we introduce the notions of uniform transfer quasiconcavity and transfer upper semicontinuity, which we apply to *individual* payoff functions.

10. We thank an anonymous referee for raising this question to us.

Definition 3 (*Uniform Transfer Quasiconcavity*). For each player i, suppose the strategy space Z_i is a convex subset in $\Re^{L_i}$. A payoff function $u_i: Z \to \Re$ is said to be *uniformly transfer quasiconcave* on Z if, for every finite subset $X^m = \{x^1, \ldots, x^m\} \subset Z$, there exists a corresponding finite subset $Y_i^m = \{y_i^1, \ldots, y_i^m\} \subset Z_i$ such that for any subset $\{y_i^{k^1}, y_i^{k^2}, \ldots, y_i^{k^s}\} \subset Y_i^m$, $1 \leqq s \leqq m$ and any $y_i^{k0} \in \text{co}\{y_i^{k^1}, y_i^{k^2}, \ldots, y_i^{k^s}\}$ we have

$$\min_{1 \leqq l \leqq s} [u_i(x_i^{k^l}, x_{-i}^{k^l}) - u_i(y_i^{k0}, x_{-i}^{k^l})] \leqq 0.$$

This concept has an interpretation similar to diagonal transfer quasiconcavity, interpreted in the context of an individual payoff function. It says, roughly, that given any finite set X^m of deviation profiles, there exists a corresponding finite set Y_i^m of candidate strategies for agent i such that, for any subset $Y^{\tilde{m}_i} \subset Y_i^m$ $(1 \leqq \tilde{m} \leqq m)$, its convex combination are not dominated by *all* deviations in $X^{\tilde{m}}$. We will see below that uniform transfer quasiconcavity is necessary for the existence of a dominant strategy.

Definition 4 (*Transfer Upper Semicontinuity*). Let $Z = \prod_{i \in I} Z_i \subset \Re^L$ be the strategy space of the players. The i-th agent's payoff function $u_i: Z \to \Re$ is said to be *transfer upper semicontinuous in* x_i if for every $y_i \in Z_i$ and $x \in Z$, $u_i(x_i, x_{-i}) > u_i(y_i, x_{-i})$ implies that there exists a point $x' \in Z$ and a neighbourhood $\mathcal{N}(y_i)$ of y_i such that $u_i(x') > u(y_i', x_{-i}')$ for all $y_i' \in \mathcal{N}(y_i)$.

Note that transfer upper semicontinuity is defined in the context of an individual payoff function. It says that if strategy y_i is dominated by x_i when others play x_{-i}, then there is an open set of candidate strategies containing y_i, all of which are dominated by a single strategy x_i' when others play x_{-i}'. This implies that, if y_i is not a dominant strategy for agent i, then there is a finite number of strategy profiles which suffice to dominate all others. Note that a sufficient condition for u_i to be transfer upper semicontinuous in x_i is for u_i to be upper semicontinuous in x_i (by taking $x' = x$).

These generalizations of the quasiconcavity and continuity of individual payoffs can be used to characterize the existence of a dominant-strategy equilibrium.

Theorem 4. *Suppose that, for each player i, the strategy space Z_i is a non-empty convex compact subset in $\Re^{L_i}$ and $u_i: Z \to \Re$ is a payoff function such that for $u_i(x)$ is transfer upper semicontinuous in x_i. Then Γ has a dominant-strategy equilibrium $y^* \in Z$ if and only if u_i is uniformly transfer quasiconcave on Z for all $i \in I$.*

Theorem 4 can be generalized by relaxing the compactness of Z.

Theorem 5. *For each player i, let the strategy space Z_i be a non-empty convex subset in $\Re^{L_i}$. Suppose $u_i: Z \to \Re$ satisfies the following conditions:*

(*a*) *u_i is transfer upper semicontinuous in x_i;*

(*b*) *there exist $x^1, \ldots, x^n \in Z$ such that $\bigcap_{k=1}^n G_i(x^k)$ is compact where $G_i(x^k) = \{y_i \in Z_i: u_i(x^k) - u_i(y_i, x_{-i}^k) \leqq 0\}$.*

Then Γ has a dominant-strategy equilibrium if and only if u_i is uniformly transfer quasiconcave on Z.

A complete characterization theorem of the existence of dominant-strategy equilibrium can be obtained similar to Theorem 3.

5. APPLICATIONS TO OLIGOPOLY THEORY

In this section we show how our theorems characterize existence in games that do not satisfy the conditions of existing theorems. Examples 1 and 2 are games that have pure-strategy equilibria that are accounted for by our Theorem 1, but which violate the continuity and/or quasiconcavity conditions required by the theorems of Dasgupta and Maskin (1986) and Nishimura and Friedman (1981). Examples 3 and 4 are games that fail to have an equilibrium because U fails to satisfy our diagonal transfer quasiconcavity condition. Finally, example 5 uses our Theorem 5 to establish the existence of a dominant strategy equilibrium in an oligopoly game.

Example 1. Two duopolists have zero costs and set prices (p_1, p_2) on $Z = [0, T]\times[0, T]$. The payoff functions are (for $0<c<T$);

$$u_i(p_1, p_2) = \begin{cases} p_i & \text{if } p_i \leqq p_{-i}, \\ p_i - c & \text{otherwise.} \end{cases}$$

One can interpret the game as a duopoly in which each firm has committed to pay brand-loyal consumers a penalty of c if the other firm beats its price.[11] These payoffs are not quasiconcave, and thus one cannot use the theorems of Dasgupta and Maskin or of Debreu to examine existence. Moreover, as the payoffs are not continuous, the theorems of Friedman and Nishimura may not be applied. Note, however, that $U(x, y)$ is diagonally transfer continuous and diagonally transfer quasiconcave, and thus by Theorem 1, this game has a pure-strategy equilibrium.[12]

Example 2 (Tullock). Consider a two-person game played on the unit square. Thus $Z_1 = Z_2 = [0, 1]$. The payoffs $u_i(x_1, x_2)$ $(i = 1, 2)$ are given by the functions

$$u_i(x_1, x_2) = \begin{cases} \frac{1}{2} & \text{if } x_1 = x_2 = 0, \\ p_i(x_1, x_2) - x_i & \text{otherwise,} \end{cases}$$

where $p_i(x_1, x_2) = x_i^\alpha/(x_1^\alpha + x_2^\alpha)$ and $1 > \alpha > 0$.

This game has been proposed by Tullock (1980) to model rent-seeking behaviour; p_i is interpreted as the probability player i wins a prize worth \$1 by expanding \$$x_i$ in resources.

It is easy to verify that the payoff functions are quasiconcave but are *not* upper semicontinuous.[13] Thus, the *sufficient* conditions for the existence of a pure-strategy Nash equilibrium set forth in the existing literature *are not satisfied.* However, the conditions of Theorem 1 are satisfied, and thus the game has a pure-strategy Nash equilibrium.[14]

11. See Baye and Kovenock (1993) for an alternative formulation with both brand-loyal and price-conscious consumers, whereby a firm commits to pay a penalty if it does not provide the best price in the market.

12. To see that U is diagonally transfer continuous in y, note that $U(y, y)$ is upper semicontinuous and apply condition 2(f) in Proposition 2. To establish diagonal transfer quasiconcavity, use the definition and, for any finite subset $\{p^1, p^2, \dots, p^m\} \subset Z$, take $y_1^k = y_2^k = b = \max\{p_1^1, p_2^1, \dots, p_1^m, p_2^m\}$ for $k = 1, \dots, m$.

13. The problem occurs when $x_i = 0$ for one of the players.

14. One can verify that $U(x, y)$ is diagonally transfer quasiconcave in x (by taking $y_i^k = \alpha/4$ for $i = 1, 2$). Verification that $U(x, y)$ is diagonally transfer continuous in y is simple; when $y_i > 0$, $U(x, y)$ is continuous and thus diagonally transfer continuous in y. When $y_1 = y_2 = 0$, take $x_1' = x_2' = \delta > 0$ and use y_i' with $0 \leqq y_i' \leqq \delta/n$ $(i = 1, 2)$ in the definition. When $y_1 = 0$ and $y_2 > 0$, take $x_1' = x_2' = y_2/2$.

Example 3 (*Varian*). Consider a two-person game with non-negative price strategies p_1 and p_2. Thus $Z_1 = Z_2 = [0, r]$. The payoffs $u_i(p_1, p_2)$ $(i = 1, 2)$ are given by the functions

$$u_i(p_1, p_2) = \begin{cases} p_i(I+\mu) - k & \text{if } p_i < p_{-i}, \\ p_i(\frac{1}{2}+\mu) - k & \text{if } p_1 = p_2, . \\ p_i\mu - k & \text{if } p_i > p_{-i} \end{cases}$$

This game has a number of interpretations; Varian (1980) interprets I to be the number of informed consumers, who will shop at the firm charging the lowest price, while 2μ is the number of uninformed consumers, who allocate themselves equally across the two firms. Thus each firm sells to μ uninformed consumers automatically, but gets the I informed consumers only if it succeeds in setting the lowest price.[15]

It is well-known that this game has no pure-strategy Nash equilibrium (cf. Varian (1980), Baye, Kovenock and de Vries (1992)). To see that the blame unambiguously lies in the fact that $U(x, y)$ is not diagonally transfer quasiconcave in x, note that $\sum u_i(x)$ is continuous and for $U(x, y) > U(y, y)$, similar to Example 1, we can find a point x' such that $U(x', y) > U(y, y)$ and $U'(x', y)$ is lower semicontinuous in y. Therefore, by Proposition 2(f), $U(x, y)$ is diagonally transfer continuous in y. But by Theorem 1, this means the existence of Nash equilibrium is equivalent to $U(x, y)$ being diagonally transfer quasiconcave in x. Since the game does not have a Nash equilibrium in pure strategies, the blame lies squarely on the fact that $U(x, y)$ is not diagonally transfer quasiconcave in x.

Example 4 (*Hotelling*). Consider a two-person game with non-negative price strategies p_1 and p_2. Thus $Z_1 = Z_2 = [0, +\infty)$. The payoffs are (for $i = 1, 2$)

$$u_i(p_1, p_2) = \begin{cases} lp_i & \text{if } p_1 < p_{-i} - c(l - \alpha_i - \alpha_{-i}) \\ \alpha_i p_i + \frac{1}{2}(l - \alpha_i - \alpha_{-i})p_i + (1/2c)p_1 p_2 - (1/2c)p_i^2 & \text{if } |p_1 - p_2| \leqq c(l - \alpha_i - \alpha_{-i}) \\ 0 & \text{if } p_i > p_{-i} + c(l - \alpha_i - \alpha_{-i}) \end{cases}$$

where $l > 0$ is the length of a line (street), $c > 0$ is the marginal transportation cost, and $\alpha_1 + \alpha_2 \leqq l$.

D'Aspremont, Gabszewicz, and Thisse (1979) have shown by direct argument that this game has a pure-strategy Nash equilibrium if and only if

$$\left(l + \frac{\alpha_i - \alpha_{-i}}{3}\right)^2 \geqq \tfrac{4}{3} l(\alpha_i + 2\alpha_{-i}), \quad i = 1, 2. \tag{4}$$

It is easy to show that $U(p, p)$ is upper semicontinuous. By condition 2(e) in Proposition 2, $U(x, p)$ is diagonally transfer continuous in p. Thus, by Theorem 1, $U(x, p)$ is diagonally transfer quasiconcave in x if and only if equation (4) holds. The existence (or non-existence) of a pure-strategy Nash equilibrium in the Hotelling model must be credited to (or blamed on) $U(x, p)$ being (not being) diagonally transfer quasiconcave in x.

Example 5 (*Case* (*1979*); *Moulin* (*1982*)). Consider a quantity-setting game where the (inverse) demand for a homogeneous good is given by ce^{-S}. S denotes total supply,

15. For an alternative interpretation of the model in the context of a trade model with a "buy British" demand bias, see Baye and de Vries (1992).

production costs are zero, and there are n producers who set quantities $x_1, \dots, x_n$. Here, then $Z_i = [0, +\infty)$ and the payoffs $u_i(x_1, \dots, x_n)$ $(i = 1, \dots, n)$ are given by the functions

$$u_i(x_1, \dots, x_n) = cx_i e^{-(x_1 + \dots + x_n)}$$

It is to verify that the payoff functions are uniformly transfer quasiconcave (by taking $y_i^1 = \dots = y_i^m = y_i^*$ where y_i^* are the maximum elements of the functions $x_i e^{-x_i}$ on the given finite subset $\{x^1, x^2, \dots, x^m\}$). Also u_i are transfer upper continuous in x_i since it is continuous. One can also easily verify that for any $x = (x_1, x_2)$ the set $G_i(x) = \{y_i \in Z: u_i(x) - u_i(y_i, x_{-i}) \leqq 0\}$ is compact. Then, by Theorem 5, the game has a dominant-strategy equilibrium.

6. PROOFS

Now that we have presented and illustrated our main theorems, we present their proofs. We first state

Definition 5 (*Transfer FS-Convexity*). Let X be a subset of $\mathfrak{R}^L$ and let Y be a convex subset of $\mathfrak{R}^M$. A correspondence $G: X \to 2^Y$ is said to be ***transfer FS-convex*** on X, if for any finite subset $X^m = \{x^1, x^2, \dots, x^m\} \subset X$, there exists a corresponding finite subset $Y^m = \{y^1, y^2, \dots, y^m\} \subset Y$ such that for any subset $\{y^{k^1}, y^{k^2}, \dots, y^{k^s}\} \subset Y^m$, $1 \leqq s \leqq m$, we have

$$\text{co}\,\{y^{k^1}, y^{k^2}, \dots, y^{k^s}\} \subset \bigcap_{l=1}^{s} G(x^{k^l}).$$

Transfer FS-convexity is a generalization of the FS-convexity,[16] which has been used elsewhere by Chang and Zhang (1989), Tian (1993) and Zhou and Tian (1992)a).

Remark 3. Diagonal transfer quasiconcavity and uniform transfer quasiconcavity are closely related to transfer *FS*-convexity. For $A \subset Z$, $C \subset Z$, and $U: Z \times C \to \mathfrak{R}$, if we define the mapping $G: A \to 2^C$ by $G(x) = \{y \in C: U(x, y) \leqq U(y, y)\}$ for $x \in A$, it can be easily verified that G is transfer *FS*-convex on A if and only if U is diagonally transfer quasiconcave in x on A. On the other hand, for a payoff $u_i: Z \to \mathfrak{R}$, if we define the mapping $G_i: Z \to 2^{Z_i}$ by $G_i(x) = \{y \in Z_i: u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) \leqq 0\}$, it can be easily verified that u_i is uniformly transfer quasiconcave on Z if and only if G_i is transfer *FS*-convex on Z.

Proof of Theorem 1. Necessity. Suppose the game Γ has a pure-strategy Nash equilibrium $y^* \in Z$. We want to show that U is diagonally transfer quasiconcave in x. Indeed, for any finite subset $X^m = \{x^1, \dots, x^m\} \subset Z$, let the corresponding finite subset $Y^m = \{y^1, \dots, y^m\} = \{y^*\}$. Then for any subset $\{y^{k^1}, y^{k^2}, \dots, y^{k^s}\} \subset Y^m = \{y^*\}$, $1 \leqq s \leqq m$, and $y^{k0} \in \text{co}\,\{y^{k^1}, y^{k^2}, \dots, y^{k^s}\} = \{y^*\}$, we have

$$\min_{1 \leqq l \leqq s} [U(x^{k^l}, y^{k0}) - U(y^{k0}, y^{k0})] \leqq [U(x^{k^l}, y^*) - U(y^*, y^*)]$$

$$= \sum_{i \in I} [u_i(x_i^{k^l}, y_{-i}^*) - u_i(y^*)] \leqq 0.$$

Hence U is diagonally transfer quasiconcave in x.

16. For a convex set Z and $\varnothing \neq X \subset Z$, a correspondence $G: X \to 2^Z$ is said to be *FS-convex* on X if for every finite subset $\{x^1, x^2, \dots, x^m\}$ of X, $\text{co}\,\{x^1, x^2, \dots, x^m\} \subset \bigcup_{k=1}^m G(x^k)$.

Sufficiency. For each $x \in Z$, let $G(x)=\{y \in Z: U(x, y) \leqq U(y, y)\}$. We first prove $\bigcap_{x \in Z} \mathrm{cl}_Z\, G(x)=\bigcap_{x \in Z} G(x)$. It is clear that $\bigcap_{x \in Z} G(x) \subset \bigcap_{x \in Z} \mathrm{cl}_Z\, G(x)$. So we only need to show $\bigcap_{x \in Y} \mathrm{cl}_Y\, G(x) \subset \bigcap_{x \in Z} G(x)$. Suppose, by way of contradiction, that there is some y in $\bigcap_{x \in Z} \mathrm{cl}_Z\, G(x)$ but not in $\bigcap_{x \in Z} G(x)$. Then $y \notin G(x)$ for some $x \in Z$ and thus $U(x, y)>U(y, y)$. By the diagonal transfer continuity of U, there is some $x' \in Z$ and some neighbourhood $\mathcal{N}(y)$ of y such that $U(x', z)>U(z, z)$ for all $z \in \mathcal{N}(y)$. Thus $y \notin \mathrm{cl}_Z\, G(x')$, a contradiction.

For $x \in Z$, let $\bar{G}(x)=\mathrm{cl}_Z\, G(x)$. Then $\bar{G}(x)$ is closed and, by the diagonal transfer quasiconcavity of U, it is transfer *FS*-convex (see Remark 3). From the proof of Lemma 1 in Tian (1993), we know that $\bar{G}$ has the finite intersection property. Since Z is compact, $\bigcap_{x \in Z} G(x)=\bigcap_{x \in Z} \mathrm{cl}_Z\, G(x) \neq \varnothing$. Hence, there exists a $y^* \in Z$ such that $U(x, y^*) \leqq U(y^*, y^*)$ for all $x \in Z$. Now let $x=(x_i, y^*_{-i})$. Then we have

$$U(x, y^*)-U(y^*, y^*)=[u_i(x_i, y^*_{-i})-u_i(y^*)] \leqq 0. \tag{5}$$

for any $x_i \in Z_i$ with (x_i, y^*_{-i}). So y^* is a pure-strategy Nash equilibrium of game Γ. ||

Proof of Theorem 2. The proof of necessity is the same as that of Theorem 1, so we only prove sufficiency. Again, for each $x \in A$, let $G(x)=\{y \in Z: U(x, y) \leqq U(y, y)\}$ and let $\bar{G}(x)=\mathrm{cl}_Z\, G(x)$. Then $\bar{G}(x)$ is closed and, by the diagonal transfer quasiconcavity of U on A, it is transfer FS-convex on A. By Lemma 1 in Tian (1993), $\bar{G}$ has the finite intersection property and therefore $\{\bar{G}(x) \cap [\bigcap_{k=1}^n \bar{G}(x^k)]: x \in A\}$ also has the finite intersection property. Now since $\{\bar{G}(x) \cap [\bigcap_{k=1}^n \bar{G}(x^k)]: x \in A\}$ is a family of compact subsets in the compact set $\bigcap_{k=1}^n \bar{G}(x^k)$, we have $\varnothing \neq \bigcap_{x \in A} \bar{G}(x) \cap [\bigcap_{k=1}^n \bar{G}(x^k)]=\bigcap_{x \in A} \bar{G}(x)=\bigcap_{x \in A} G(x)$. That is, there exists a point $y^* \in Z$ such that $U(x, y^*) \leqq U(y^*, y^*)$ for all $x \in A$. Now we must have $y^* \in A$, for otherwise $U(x, y^*)>U(y^*, y^*)$ for some $x \in A$ by condition (c). Hence $y^* \in A$. Again, by letting $x=(x_i, y^*_{-i}) \in A$, we have $u_i(x_i, y^*_{-i}) \leqq u_i(y^*)$ for any $x_i \in Z_i$ with $(x_i, y^*_{-i}) \in A$. ||

Proof of Theorem 3. Sufficiency. For each $x \in Z$, let $G(x)=\{y \in C: U(x, y) \leqq U(y, y)\}$. The remaning proof of sufficiency is the same as that in the proof of Theorem 2 and thus is omitted.

Necessity. Suppose the game Γ has a pure-strategy Nash equilibrium, say y^*. Let $C=\{y^*\}$. Then the set C is convex, the restricted mapping $U|_{Z \times C}$ clearly is diagonally transfer continuous in y and diagonally transfer quasiconcave in x, and further the set $G(x)=\{y \in C: U(x, y) \leqq U(y, y)\}=\{y^*\}$ is compact for any $x \in Z$. ||

Proof of Theorem 4. Necessity. Suppose the game Γ has a dominant-strategy equlibrium $y^* \in Z$. We need to show that u_i is uniformly transfer quasiconcave on Z. For any finite subset $X^m=\{x^1, \ldots, x^m\} \subset Z$, let the corresponding finite subset $Y_i^m=\{y_i^1, \ldots, y_i^m\}=\{y_i^*\}$. Then for any subset $\{y_i^{k^1}, y_i^{k^2}, \ldots, y_i^{k^s}\} \subset Y_i^m=\{y_i^*\}$, $1 \leqq s \leqq m$, and $y_i^{k0} \in \mathrm{co}\,\{y_i^{k^1}, y_i^{k^2}, \ldots, y_i^{k^s}\}=\{y_i^*\}$, we have

$$\lim_{1 \leqq l \leqq s}\, [u_i(x_i^{k^l}, x_{-i}^{k^l})-u_i(y_i^{k0}, x_{-i}^{k^l})] \leqq [u_i(x_i^{k^l}, x_{-i}^{k^l})-u_i(y_i^*, x_{-i}^{k^l})] \leqq 0.$$

Hence u_i is uniformly transfer quasiconcave on Z.

Sufficiency. Define a correspondence $G_i: Z \rightarrow 2^{Z_i}$ by $G_i(x)=\{y_i \in Z_i: u_i(x)-u_i(y_i, x_{-i}) \leqq 0\}$ for all $x \in Z$. From Remark 3, we know that G_i is transfer FS-convex on Z if and only if u_i is uniformly transfer quasiconcave on Z. Then by the same arguments in Theorem 1, one can show that $\bigcap_{x \in Z} G_i(x) \neq \varnothing$ if u_i is transfer upper semicontinuous in x_i and uniformly transfer quasiconcave on Z. Hence there exists a $y_i^* \in Z_i$ such that $u_i(x) - u_i(y_i^*, x_{-i}) \leqq 0$ for all $x \in Z$. ||

Proof of Theorem 5. The proof of necessity is the same as that of Theorem 4. To show sufficiency, define $G_i(x) = \{y_i \in Z_i : u_i(x) - u_i(y_i, x_{-i}) \leqq 0\}$ for $x \in Z$. Then by the same arguments in Theorem 2, one can show that $\bigcap_{x \in Z} G_i(x) \neq \varnothing$ if u_i is transfer upper semicontinuous in x_i and uniformly transfer quasiconcave on Z. Then there exists a $y_i^* \in Z_i$ such that $u_i(x) - u_i(y_i^*, x_{-i}) \leqq 0$ for all $x \in Z$. ||

Proof of Proposition 1. We prove only 1(e); the other conditions are easily verified. Since one player's (say, player 1) payoff function is uniformly transfer quasiconcave on the compact Z and transfer upper semicontinuous in his own strategies, by the conclusions of Theorem 4, there exists a point $y_1^* \in Z_1$ such that $u_1(x) - u_1(y_1^*, x_2) \leqq 0$ for all $x \in Z$. Now for any finite subset $X^m = \{x^1, x^2, \ldots, x^m\} \subset Z$, if we let $y_1^k = y_1^*$ and $y_2^k = y_2^*$ for $k = 1, \ldots, m$ where y_2^* is the maximum element of $u_2(y_1^*, \cdot)$ on the finite set X^m, then for any subset $\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\} = \{(y_1^*, y_2^*)\} \subset \{y^1, y^2, \ldots, y^m\} = \{(y_1^*, y_2^*)\}$, $1 \leqq s \leqq m$, and $y^{k0} \in \text{co}\,\{y_1^{k^1}, y_1^{k^2}, \ldots, y_1^{k^s}\} = \{(y_1^*, y_2^*)\}$ we have

$$U(x^{k^l}, y^{k0}) - U(y^{k0}, y^{k0}) = [u_1(x_1^{k^l}, y_2^*) - u_1(y_1^*, y_2^*)] + [u_2(y_1^*, x_2^{k^l}) - u_2(y_1^*, y_2^*)] \leqq 0$$

for $1 \leqq l \leqq s$. So $U(x, y)$ is diagonally transfer quasiconcave in x. ||

Proof of Proposition 2. We prove only 2(e); the other conditions are easily verified. We need to show that if $U(x, y) - U(y, y) > 0$ for any $(x, y) \in Z \times Z$, then there exist $x' \in Z$ and $\mathcal{N}(y)$ of y such that $U(x', y') - U'(y', y') > 0$, $\forall y' \in \mathcal{N}(y)$. Note that since $U(y, y)$ is upper semicontinuous, there are only two cases to consider. One is the case where $|x_1 - y_2| = d$ and $|x_2 - y_1| = d$, the other is the case where $|x_1 - y_2| = d$, $|x_2 - y_1| \neq d$ (or $|x_1 - y_2| \neq d$, $|x_2 - y_1| = d$). For all other cases, $U(x, y)$ is continuous and then $U(x, y) - U(y, y)$ is lower semicontinuous in y.

Case (i). $|x_1 - y_2| = d$ and $|x_2 - y_1| = d$: Since $U(x, z)$ is continuous on $\{(x, z) : |x_1 - z_2| \leqq d \;\&\; |x_2 - z_1| \leqq d\}$, for $0 < \varepsilon < U(x, y) - U(y, y)$, there exists a point x' and a neighbourhood $\mathcal{N}_1(y)$ of y satisfying $|x_1' - y_2'| < d$ and $|x_2' - y_1'| < d$, $|U(x', y) - U(x, y)| < \frac{1}{3}\varepsilon$ and $|U(x', y') - U(x', y)| < \frac{1}{3}\varepsilon$ for all $y' = (y_1', y_2') \in \mathcal{N}_1(y)$. Since $U(y, y)$ is upper semicontinuous, there exists a neighbourhood $\mathcal{N}_2(y)$ of y such that $U(y', y') < U(y, y) + \frac{1}{3}\varepsilon$, $\forall y' \in \mathcal{N}_2(y)$. Let $\mathcal{N}(y) \equiv \mathcal{N}_1(y) \cap \mathcal{N}_2(y)$. Then for any $y' \in \mathcal{N}(y)$, $U(x', y') - U(y', y') > U(x, y) - U(y, y) - \varepsilon > 0$.

Case (ii). $|x_1 - y_2\| = d$ and $|x_2 - y_1| \neq d$: Since $U(x, z)$ is continuous on $\{(x, z) : |x_1 - z_2| \leqq d \;\&\; |x_2 - z_1| \neq d\}$, there exists a point x' and a neighborhood $\mathcal{N}_1(y)$ of y satisfying $|x_1' - y_2'| < d$ and $|x_2' - y_1'| \neq d$, $|U(x', y) - U(x, y)| < \frac{1}{3}\varepsilon$ and $|U(x', y') - U(x', y)| < \frac{1}{3}\varepsilon$ for all $y' \in \mathcal{N}_1(y)$. The result then follows from the same arguments as in Case (i). ||

7. CONCLUDING REMARKS

The examples and theorems presented above reveal that the continuity and quasiconcavity conditions used in the literature on the existence of pure-strategy Nash equilibrium can be considerably weakened. Specifically, we proved that the diagonal transfer quasiconcavity of the aggregator function in equation (1) is necessary and, in conjunction with diagonal transfer continuity, also sufficient for the existence of a pure strategy Nash equilibrium. Similarly, the uniform transfer quasiconcavity of individual payoff functions is necessary and, along with transfer upper semicontinuity, sufficient for the existence of a dominant-strategy equilibrium. We presented examples where our theorems point out existence in games that do not satisfy the conditions of previous theorems. We also demonstrated that the blame for the failure of existence in games such as the Hotelling and Varian models lies not with the failure of individual payoffs to be quasiconcave, but rather the failure of $U(x, y)$ defined in equation (1) to be diagonally transfer quasiconcave in x.

We provided a few sufficient conditions for transfer quasiconcavity and transfer continuity here, and noted that the conditions imposed in all of the existing literature implies one or more of the conditions. Since these conditions imply, by our Theorem 1, the existence of pure-strategy Nash equilibrium, a potentially useful avenue of research is to attempt to build upon our Propositions 1 and 2 to find additional sufficient conditions for the existence of pure-strategy Nash equilibrium.

Acknowledgements. We are grateful to Dave Furth, Dan Kovenock, Eric Maskin, Bill Nielson, Jim Friedman, and Eric van Damme for helpful comments and suggestions. We especially wish to thank Ian Jewitt and two anonymous referees for insights and suggestions that led to substantial improvements in the paper. An earlier version of this paper was written while Baye was a Research Fellow at the Center for Economic Research in Tilburg.

REFERENCES

BAYE, M. and KOVENOCK, D. (1993), "How to Sell a Pickup Truck: Beat-or-Pay Advertisements as Facilitating Devices", *International Journal of Industrial Organization* (in press).

BAYE, M., KOVENOCK, D. and DE VRIES, C. G. (1992), "It Takes Two-to-Tango: Equilibria in a Model of Sales", *Games and Economic Behavior*, **4**, 493-510.

BAYE, M. and DE VRIES, C. G. (1992), "Mixed-Strategy Trade Equilibria", *The Canadian Journal of Economics*, **25**, 281-293.

CASE, J. H. (1979) *Economics and the Competitive Process* (New York: New York University Press).

CHANG, S. S. and ZHANG, Y. (1989), "Generalized KKM Theorem and Variational Inequalities" (mimeo).

D'ASPREMONT, C., GABSZEWICZ, J. J. and THISSE, J.-F. (1979), "On Hotelling's 'Stability' in Competition", *Econometrica* **47**, 1145-1150.

DASGUPTA, P. and MASKIN, E. (1986), "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory", *Review of Economic Studies*, **53**, 1-26.

DEBREU, G. (1952), "A Social Equilibrium Existence Theorem", *Proceedings of the National Academy of Sciences of the U.S.A.*, **38**, 386-393.

FAN, K. (1984), "Some Properties of Convex Sets Related to Fixed Points Theorems", *Math. Ann.*, **266**, 519-537.

GROVES, T. (1973), "Incentives in Teams", *Econometrica*, **41**, 617-632.

GROVES, T. and LOEB, M. (1975), "Incentives and Public Inputs", *Journal of Public Economics*, **4**, 211-226.

GROVES, T. and LEDYARD, J. (1987), "Incentive Compatibility Since 1972", Chapter 2 in T. Groves, R. Radner and S. Reiter (eds.), *Information, Incentives, and Economic Mechanisms* (Minneapolis: University of Minnesota Press).

HOTELLING, H. (1929), "Stability in Competition", *Economic Journal*, **39**, 41-57.

HURWICZ, L. (1972), "Informationally Decentralized Systems", in R. Radner and C. B. McGuire (eds.), *Decision and Organization* (*Volume in Honor of J. Marschak*) (Amsterdam: North-Holland), 297-336.

HURWICZ, L. (1986), "Incentive Aspects of Decentralization," in K. J. Arrow and M. D. Intriligator (eds.), *Handbook of Mathematical Economics, Vol. III* (Amsterdam: North-Holland).

McMANUS, M. (1964), "Equilibrium, Numbers and Size in Cournot Oligopoly", *Yorkshire Bulletin of Social and Economic Research*, **16**, 68-75.

MOULIN, H. (1982), *Game Theory for the Social Sciences* (New York: New York University Press).

NASH, J. (1950), "Equilibrium Points in N-Person Games", *Proceedings of the National Academy of Sciences*, **36**, 48-49.

NASH, J. (1951), "Non-Cooperative Games", *Ann. of Math.*, **54**, 286-295.

NIKAIDO, H. and ISODA, K. (1955), "Note on Noncooperative Convex Games", *Pacific J. Math.*, **4**, 65-72.

NISHIMURA, K. and FRIEDMAN, J. (1981), "Existence of Nash Equilibrium in n Person Games Without Quasi-Concavity", *International Economic Review*, **22**, 637-648.

ROBERTS, J. and SONNENSCHEIN, H. (1977), "On the Existence of Cournot Equilibrium Without Concave Profit Functions", *Econometrica*, **45**, 101-113.

TIAN, G. (1992a), "Generalizations of the KKM Theorem and Ky-Fan Minimax Inequality, with Applications to Maximal Elements, Price Equilibrium, and Complementarity", *Journal of Mathematical Analysis and Applications*, **170**, 457-471.

TIAN, G. (1992b), "On the Existence of Optimal Truth-Dominant Truth-Dominant Mechanisms for a Class of Utility Functions" (mimeo).

TIAN, G. (1993), "Necessary and Sufficient Conditions for the Existence of Maximal Elements of Preferences Relations", *Review of Economic Studies*, **60**, 949-958.

TIAN, G. and ZHOU, J. (1992), "The Maximum Theorem and the Existence of Nash Equilibrium of (Generalized) Games Without Lower Semicontinuities", *Journal of Mathematical Analysis and Applications*, **166**, 351-364.

TULLOCK, G. (1980), "Efficient Rent Seeking", in J. Buchanan et al. (eds.), *Toward a Theory of the Rent Seeking Society* (College Station: Texas A & M University Press).

VARIAN, H. (1980), "A Model of Sales", *American Economic Review*, **70**, 651-659.

VICKREY, W. (1961), "Counterspeculation, Auctions, and Competitive Sealed Tenders", *Journal of Finance*, **16**, 8-37.

VIVES, X. (1990), "Nash Equilibrium with Strategic Complementarities", *Journal of Mathematical Economics*, **19**, 305-321.

ZHOU, J. X. and CHEN, G. (1988), "Diagonal Convexity Conditions for Problems in Convex Analysis and Quasi-Variational Inequalities", *Journal of Mathematical Analysis and Applications*, **132**, 213-225.

ZHOU, J. X. and TIAN, G. (1992), "Transfer method for characterizing the existence of maximal elements of binary relations on compact or noncompact sets", *SIAM Journal on Optimization*, **2**, 360-375.