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The unique informational efficiency of the competitive mechanism in economies with production

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Abstract The purpose of this paper is to investigate the informational requirements of resource allocation processes for convex production economies. First, we establish a lower bound of the message space of an informationally decentralized mechanism that realizes Pareto efficient allocations over the class of classical production economies. Then, it is shown that this lower bound is exactly the size of the message space of the competitive (Walrasian) mechanism, and thus the competitive mechanism is informationally efficient for general neoclassical production economies in the sense that it uses the smallest message space among the class of resource allocation processes that are informationally decentralized and realize Pareto optimal allocations. Further, it is shown that the competitive mechanism is the unique informationally efficient decentralized mechanism that realizes Pareto efficient and individually rational allocations. The results obtained in the paper may shed light on the socialist controversy between Mises-Hayek and Lange-Lerner.

1 Introduction

There are many theoretical and real existing economic institutions in the world, such as private ownership, state ownership, collective ownership, and marginal cost pricing institutions to name just a few. Which one is the best? This is a central question raised in the "socialist controversy" of the 1930s, which was the debate between Mises-Hayek and Lange-Lerner on a theoretical feasibility of efficient allocations under socialism (von Hayek (1935, 1945), Lange (1936–1937, 1942), and Lerner (1944)). This paper may shed some light on the answer to the above question.

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What are the general criteria for evaluating an economic institution? There are three basic requirements: allocative efficiency, incentive compatibility, and informational efficiency. They are highly desired properties for an economic mechanism to have. Pareto optimality requires that resources be allocated efficiently. Incentive-compatibility requires the consistence of individual interests and social goals. Informational efficiency requires that an economic system have the least informational cost of operation. These properties are of fundamental importance in evaluating and choosing economic institutions. The poor performance of all the socialist countries in the real world demonstrates the importance of requiring allocative efficiency, incentives, and informational efficiency of a economic system. Studies on the informational requirements and incentives of a system resulted in the realization theory and implementation theory, respectively, which together make up the theory of mechanism design originated by Hurwicz (1960, 1972, 1973, 1979a). This paper mainly considers the problem of informational requirements on resource allocation mechanisms that select Pareto optimal allocations.

The notion of an allocation mechanism was first formalized by Hurwicz (1960). Such a mechanism can be viewed as an abstract planning procedure; it consists of a message space in which communication takes place, rules by which the agents form messages, and an outcome function that translates messages into outcomes (allocations of resources). Mechanisms are imagined to operate iteratively. Attention, however, may be focused on mechanisms that have stationary or equilibrium messages for each possible economic environment. A mechanism realizes a prespecified welfare criterion (called social choice rule, or social choice correspondence) if the outcomes given by the outcome function agree with the welfare criterion at the stationary messages. A mechanism is non-wasteful if it realizes Pareto optimal allocations, and it is privacy-preserving if it is informationally decentralized. The realization theory mainly studies the question of how much communication must be provided to realize a given performance, especially Pareto efficient allocations. More precisely, it studies the minimal informational size in terms of the message space size and determines which economic institution is informationally the most efficient. Such studies can be found in Hurwicz (1972), Mount and Reiter (1974), Calsamiglia (1977), Walker (1977), Osana (1978), Sato (1981), Hurwicz et al. (1985), Williams (1986), Reichelstein and Reiter (1988), Saijo (1988), Calsamiglia and Kirman (1993), Tian (1990, 1994, 2004), Marschak and Reichelstein (1995), Mount (1995), Saari (1995), Ishikida and Marschak (1996) among others.

One of the well-known results in this literature establishes the minimality of the competitive mechanism in using information. Hurwicz (1972), Mount and Reiter (1974), Walker (1977), Osana (1978), Hurwicz (1986) among others proved that, for pure exchange private goods economies, the competitive (Walrasian) allocation process is the most informationally efficient process in the sense that any smooth informationally decentralized allocation mechanism that achieves Pareto optimal allocations must use information as least as large as the competitive mechanism, i.e., the competitive allocation process has a minimal message space size among a certain class of smooth resource allocation processes that are privacy preserving and non-wasteful.[1] Ishikida and Marschak (1996) showed that, even if non-privacy preserving mechanisms are permitted, the competitive mechanism is still informationally efficient in the class of projection mechanisms, wherein each of its messages consists of a proposed action (such as the trade vector) together with a non-action component (such as the price vector), which realize Pareto efficient and individually rational allocations when the number of individuals is greater than the number of commodities. For the class of public goods economies without the presence of by-products, Sato (1981) obtained a similar result showing that the Lindahl allocation process has a minimal message space size among a certain class of resource allocation processes that are privacy preserving and non-wasteful. For the class of public goods economies allowing the presence of by-products, Tian (1994) showed that the Generalized Ratio process is informationally the most efficient process among privacy preserving and non-wasteful resource allocation processes, and the Lindahl allocation process needs more information than the Generalized Ratio process in the presence of by-products. For pure exchange economies with consumption externalities, Tian (2004) obtained a similar result and showed that the distributive Lindahl mechanism is informationally the most efficient allocation process that is informationally decentralized and realizes Pareto efficient allocations over the class of economies that include non-malevolent economies. For brevity, these informational efficiency results have been referred to as the Efficiency Theorem. Jordan (1982) and Calsamiglia and Kirman (1993) further provided the Uniqueness Theorem for private goods pure exchange economies. Jordan (1982) proved that the competitive allocation process is uniquely informationally efficient. Calsamiglia and Kirman (1993) proved the equal income Walrasian mechanism is uniquely informationally efficient among all allocation mechanisms that realize fair allocations. Recently, Tian (2004) showed that the distributive Lindahl mechanism is informationally the most efficient allocation process that is informationally decentralized and realizes Pareto efficient allocations for pure exchange economies with consumption externalities, and further showed that it is the unique informationally efficient decentralized mechanism that realizes Pareto efficient and individually rational allocations over a class of non-malevolent economies.

These efficiency and uniqueness results are of fundamental importance in evaluating and choosing economic institutions from the point of view of political economy. However, it is surprising that after such a thorough exploration of different scenarios the fundamental results concerning the informational efficiency and the uniqueness of the Walrasian process had not been extended yet to cover production economies. Indeed, all of the results on the informational optimality and uniqueness of the competitive mechanism are proved only for pure exchange economies and but not for economies with production. To show the optimality and uniqueness of the capitalism in terms of both allocative efficiency and informational

[1] A mechanism is called *smooth* if the stationary message correspondence is either locally threaded or if the inverse of the stationary message correspondence has a Lipschizian-continuous selection in the subset. Thus, the term "smoothness" used here is not referred as the usual differentiability of a function, but either as local threadedness or the Lipschizian continuity. This terminology was used by Hurwicz (1999). We will give the definition of the local threadedness below.

efficiency, one needs to extend the existing results to include production economic environments that are more general and realistic. The objective of this paper is to fill the gap.

In this paper we establish the unique informational optimality of the private ownership competitive mechanism within general convex production economies. As such, the task of this paper is three-fold. First, we establish a lower bound of information, as measured by the size of the message space, that is required to guarantee an informationally decentralized mechanism to realize Pareto efficient allocations over the class of economies with production. Theorem 1 shows that any smooth informationally decentralized mechanism that realizes Pareto efficient allocations on the class of general neoclassical production economies that include a test family of Cobb–Douglas utility functions and quadratic production functions as a subclass has a message space of dimension no smaller than $(L-1)I+LJ$, where I is the number of consumers, J is the number of firms, and L is the number of commodities.

Second, we establish the informational optimality of the competitive mechanism. Theorem 2 shows that the lower bound is exactly the size of the message space of the competitive, and thus any smooth informationally decentralized mechanism that realizes Pareto efficient allocations over the class of general neoclassical production economies that include Cobb–Douglas utility functions and quadratic production functions has a message space size no smaller than the one for the competitive allocation mechanism defined by the Walrasian process. Thus, the competitive mechanism is informationally the most efficient process among smooth privacy preserving and non-wasteful resource allocation mechanisms for the set of production economies in which competitive equilibria exist.

Third, we show that the competitive mechanism is the unique informationally efficient process that realizes Pareto efficient and individually rational allocations for the class of general neoclassical production economies with Cobb–Douglas utility functions and quadratic production functions. Theorem 3 shows that any informationally decentralized, individually rational, and non-wasteful mechanism with the $(L-1)I+LJ$-dimensional message space and a continuous single-valued stationary message function is essentially the competitive mechanism on the test family. Thus, our Uniqueness Theorems extend Jordan's result for pure exchange economies to economies with production.

This uniqueness result on the competitive mechanism in informational optimality for economies with production is in fact an impossibility theorem: it implies that there exists no other privacy preserving and non-wasteful resource allocation mechanism over whatever ownerships – state ownership, private ownership, or other types of ownership, that uses message space whose informational size is smaller than or the same as that of the Walrasian message space. This result sheds light on the socialist controversy by demonstrating that the competitive mechanism has an informational optimality that is not shared by any other system. From the implementation literature that deals with incentives aspects of an economic institution, we know that an incentive compatible mechanism could be designed to implement Pareto efficient allocations even for state or mixed ownership economic environments (cf. Maskin (1999) and Tian (2000)). Even though there is theoretical results that socialist institutions and capitalist institutions both could reach Pareto efficient allocations by designing suitable incentive compatible mechanisms, capitalism may be a unique optimal mechanism in terms of informational requirements.

Our uniqueness result shows that the competitive market mechanism may be the unique informational efficient mechanism that realizes Pareto efficient allocations. As such, it implies that any type of socialist system is not informationally efficient even if it may yield efficient allocations of resources and its incentive problem may be solved. Consequently, it shows the optimality of capitalism from the perspective of the efficiency of both using information and allocating resources. As a policy implication, an alternative mechanism should be adopted only in the case of market failure such as for non-neoclassical economic environments: externality, non-convexities or continuity of production technology and preferences, etc.

In an unpublished paper, Nayak (1982) attempted to establish the informational efficiency of the competitive mechanism for private ownership production economies. However, he considered an unusual class of production technology sets that results in positive outputs with zero inputs. Furthermore, the proof of the uniqueness of the competitive equilibrium for the class of production economies under consideration is problematic since, when a consumer's wealth contains profit functions of firms, unlike his claim, the demand function derived from Cobb–Douglas utility functions does not necessarily satisfy the gross substitutes. Nevertheless, Nayak only considered the informational efficiency of the competitive mechanism, but did not provide the Uniqueness Theorem. In any case, to the author's knowledge, there is no uniqueness result on informational efficiency of the competitive mechanism for production economies in the literature.

The remainder of this paper is as follows. In Section 2, we provide a formal description of the model. We specify production economic environments, and give notation and definitions on resource allocation, social choice correspondence, outcome function, allocation mechanism, etc. Section 3 establishes a lower bound of the size of the message space which is required to guarantee that a smooth informationally decentralized mechanism realizes Pareto efficient allocations on the class of production economies. Section 4 gives an Efficiency Theorem on the allocative efficiency and informational efficiency of the competitive mechanism for general convex production economies on which the competitive equilibria exist. Section 5 gives Uniqueness Theorems which show that only the competitive mechanism is informationally efficient within a certain class of production economies. Concluding remarks are presented in Section 6.

2 The setup

In this section we will give notation, definitions, and a model that will be used in the paper.

2.1 Economic environments

Consider production economies with L private goods, I consumers (characterized by their consumption sets, preferences, and endowments), and J firms (characterized by their production sets). Throughout this paper, subscripts are used to index consumers or firms, and superscripts are used to index goods unless otherwise stated. By an agent, we will mean either a consumer or a producer, thus there are $N := I + J \geqq 2$ agents.[2] For the ith consumer, his characteristic is denoted by $e_i = (X_i, w_i, R_i)$ is unknown to the designer, where $X_i \subset \mathbb{R}^L$ is his consumption set, w_i is his initial endowment vector, and R_i is a preference ordering that is assumed to be convex,[3] continuous on X_i, and strictly monotone on the set of interior points of X_i. Let P_i be the strict preference (asymmetric part) of R_i. For producer j, his characteristic is denoted by $e_j = (\mathcal{Y}_j)$ where $\mathcal{Y}_j \subset \mathbb{R}^L$ is his production possibility set. We assume that, for $j = I + 1, \ldots, N$, $\mathcal{Y}_j$ is nonempty, closed, convex, contains 0 (possibility of inaction), and $\mathcal{Y}_j - \mathbb{R}^L_+ \subseteq \mathcal{Y}_j$ (free-disposal). We also assume that the economies under consideration have no externalities or public goods.

An economy is the full vector $e = (e_1, \ldots, e_I, e_{I+1}, \ldots, e_N)$ and the set of all such production economies is denoted by E and is called neoclassical production economies. The E is assumed to be endowed with the product topology.

2.2 Allocations

Let x_i denote the net increment in commodity holdings (net trade) by consumer i. x_i is said to be *individually feasible* if $x_i + w_i \in X_i$. Denoted by $x = (x_1, \ldots, x_I)$ which is called a (net) distribution. Similarly, let y_j denote producer j's (net) output vector that has positive components for outputs and negative ones for inputs, and y_i is said to be *individually feasible* if $y_j \in \mathcal{Y}_j$. Denoted by $y = (y_{I+1}, \ldots, y_N)$.

An allocation of the economy e is a vector $z := (x, y) \in \mathbb{R}^{NL}$. An allocation $z = (x, y)$ is said to be *balanced* if $\sum_{i=1}^{I} x_i = \sum_{j=I+1}^{N} y_j$. An allocation $z = (x, y)$ is said to be *feasible* if it is balanced and individually feasible for every individual.

An allocation $z = (x, y)$ is said to be *Pareto efficient* if it is feasible and there does not exist another feasible allocation $z' = (x', y')$ such that $(x_i' + w_i)\, R_i\, (x_i + w_i)$ for all $i = 1, \ldots, I$ and $(x_i' + w_i)\, P_i\, (x_i + w_i)$ for some $i = 1, \ldots, I$. Denote by $P(e)$ the set of all such allocations.

An important characterization of a Pareto optimal allocation is associated with the following concept. Let $\Delta^{L-1} = \{p \in \mathbb{R}^L_{++} : \sum_{l=1}^{L} p^l = 1\}$ be the $L-1$ dimensional unit simplex.

A nonzero vector $p \in \Delta^{L-1}$ is called a vector of *efficiency prices* for a Pareto optimal allocation (x, y) if

(a) $p \cdot x_i \leqq p \cdot x_i'$ for all $i = 1, \ldots, I$ and all x_i' such that $x_i' + w_i \in X_i$ and $(x_i' + w_i)\; R_i(x_i + w_i)$;
(b) $p \cdot y_j \geqq p \cdot y_j'$ for all $y_j' \in \mathcal{Y}_j, \quad j = I + 1, \ldots, N$.

In the terminology of Debreu (1959, p. 93), (x, y) is an equilibrium relative to the price system p. It is well known that under certain regularity conditions such as convexity, local non-satiation, etc, every Pareto optimal allocation (x, y) has an efficiency price associated with it (see Debreu (1959)).

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[2] As usual, vector inequalities, $\geqq$, $\geq$, and $>$, are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \ldots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \ldots, m$.
[3] R_i is convex if for bundles a, b, c with $0 < \lambda \leqq 1$ and $c = \lambda a + (1-\lambda) b$, the relation aP_ib implies $c\, P_ib$. Note that the term "convex" is defined as in see Debreu (1959), not as in some recent textbooks.

As Hurwicz (1979b) pointed out, it is not quite obvious what the appropriate generalization of the individual rationality concept should be for an economy with production. The following definition of individual rationality of an allocation for an economy with production was introduced by Hurwicz (1979b).

An allocation $z = (x, y)$ is said to be *individually rational* with respect to the fixed share guarantee structure $\gamma_i(e; \theta)$ if $(x_i + w_i)\ R_i\ (\gamma_i(e) + w_i)$ for all $i = 1, ..., I$. Here, $\gamma_i(e; \theta)$ is given by

$$\gamma_i(e;\theta) = \frac{p \cdot \sum_{j=I+1}^{N} \theta_{ij} y_j}{p \cdot w_i} w_i,\ i = 1, \ldots, I, \tag{1}$$

where p is an efficiency price vector for e and the θ_{ij} are non-negative fractions such that $\sum_{i=1}^{n} \theta_{ij} = 1$ for $j = I + 1, ..., N$, which can be interpreted as the profit shares of consumer i from producer j. Note that this definition on the individual rationality contains pure exchange as well as constant returns as special cases. Denote by $I_\theta(e)$ the set of all such allocations.

Now we define the competitive equilibria of a private ownership economy in which the i-th consumer owns the share θ_{ij} of the j-th producer, and is, consequently, entitled to the corresponding fraction of its profits. Thus, the ownership structure can be denoted by the matrix $\theta = (\theta_{ij})$. Denoted by Θ the set of all such ownership structures.

An allocation $z = (x, y) = (x_1, x_2, ..., x_I, y_{I+1}, y_{I+2}, ..., y_N) \in \mathbb{R}^{IL} \times \mathcal{Y}$ is a θ-*Walrasian allocation* for an economy e if it is feasible and there is a price vector $p \in \Delta^{L-1}$ such that

(1) $p \cdot x_i = \sum_{j=I+1}^{N} \theta_{ij} p \cdot y_j$ for all $i = 1, ..., I$;
(2) for all $i = 1, ..., I$, $(x_i' + w_i)\ P_i\ (x_i + w_i)$ implies $p \cdot x_i' > \sum_{j=I+1}^{N} \theta_{ij} p \cdot y_j$; and
(3) $p \cdot y_j \geqq p \cdot y_j'$ for all $y_j' \in \mathcal{Y}_j$ and $j = I + 1, ..., N$.

Denoted by $W_\theta(e)$ the set of all such allocations, and by $\mathcal{W}_\theta(e)$ the set of all such price-allocations pair (p, z).

Let E^c*subset* E be the subset of production economies on which $W(e) \neq \emptyset$ for all $e \in E^c$ and call such a subset as the Walrasian production economies.

It may be remarked that, every θ-Walrasian allocation is clearly individually rational with respect to the $\gamma_i(e; \theta)$, and also, by the strict monotonicity of preferences, it is Pareto efficient. Thus we have $W_\theta(e) \subset I_\theta(e) \cap P(e)$ for all $e \in E^c$.

2.3 Allocation mechanisms

Let $Z = \{(x, y) \in \mathbb{R}^{LI+LJ} : \sum_{i=1}^{I} x_i = \sum_{j=I+1}^{N} y_j\}$ and let F be a social choice rule, correspondence from E to Z. Let M be an arbitrary set, possibly but not necessarily of the form $M = M_1 \times \cdots \times M_N$, called the *message space*. M is a set of abstract messages that may consist of proposed allocations of consumption and production, a price vector, economic profile e, or something else reported by individuals. A reported message may depend on economic environment e. Denote by $\mu_i(e)$ the set of all such messages and μ_i is called individual i's equilibrium correspondence that is a mapping from $E = E_1 \times \cdots \times E_N$ to M.

An element $m \in M$ is an equilibrium message for an economy $e \in E$ if and only if $m \in \mu_i(e)$ for all i. Thus, the (joint) equilibrium message correspondence μ : $E \rightarrow\rightarrow M$ is defined by

$$\mu(e) = \bigcap_{i=1}^{N} \mu_i(e)$$

which assigns to every economy e the set of stationary (equilibrium) messages. This is a general formula of the (joint) equilibrium correspondence, which is used in the mechanism design literature (see Hurwicz, p. 243–5, 1986).

Following Mount and Reiter (1974), a *message process* is a pair $\langle M, \mu \rangle$. An equilibrium message $m \in \mu(e)$ is not a final allocation or outcome. That is, individuals do not make a final decision on production and consumption, which is decided by the designer according to the reported message by individuals.

An *allocation mechanism (process)* is then a triple $\langle M, \mu, h \rangle$ defined on E, where $h : M \rightarrow Z$ is the outcome function that assigns to every equilibrium message $m \in \mu(e)$ the corresponding trade $z \in Z$. That is, the actual outcome or allocation $z = (x, y)$ is determined by the outcome function h with $z = h(m)$.

An allocation mechanism $\langle M, \mu, h \rangle$, defined on E, *realizes* the social choice rule F, if for all $e \in E$, $\mu(e) \neq \emptyset$ and $h(m) \in F(e)$ for all $m \in \mu(e)$.

In this paper, the social choice rule is restricted to the one that yields Pareto efficient outcomes. Let $\mathcal{P}(e)$ be a subset of Pareto efficient allocations for $e \in E$. An allocation mechanism $\langle M, \mu, h \rangle$ is said to be *non-wasteful* on E with respect to $\mathcal{P}$ if for all $e \in E$, $\mu(e) \neq \emptyset$ and $h(m) \in \mathcal{P}(e)$ for all $m \in \mu(e)$. If an allocation mechanism $\langle M, \mu, h \rangle$ is non-wasteful on E with respect to P, the set of all Pareto efficient outcomes, then it is said simply to be non-wasteful on E. The concept of non-wastefulness was first introduced by Hurwicz (1960).

In general, an individual equilibrium correspondence μ_i depends on characteristics of all individuals, i.e., μ_i $(e_1, \ldots, e_N)$ as discussed above. However, if the individual equilibrium correspondence μ_i depends only on his own characteristic e_i, μ_i is a mapping from E_i to M and thus the (joint) equilibrium correspondence $\mu(e) = \bigcap_{i=1}^{N} \mu_i(e_i)$ and the resulting mechanism is called a *privacy-preserving mechanism*.[4]

Thus, when a mechanism is privacy-preserving, each individual's response to a message is only based on that person's private information on his/her own characteristic, but not based on characteristics of the other individuals. The privacy-preserving property is an important property for a mechanism to have. This is because, under any type of institution or ownership structure, only the manager or the owner of a firm has better information about her own production set, and only a consumer knows her own preferences and initial endowments.

Remark 1 This important feature of the communication process implies that the so called "crossing condition" has to be satisfied. Mount and Reiter (Lemma 5, 1974)

[4] Notice that, the definition of the privacy-preserving mechanism does not exclude the possibility of the presence of externalities since a message reported by one agent may also include, say, the level of production by other producers.

showed that an allocation mechanism $\langle M, \mu, h\rangle$ is privacy-preserving on E if and only if for every i and every e and e' in E, $\mu(e) \cap \mu(e') = \mu(e_i', e_{-i}) \cap \mu(e_i, e'_{-i})$, where $(e_i', e_{-i}) = (e_1, ..., e_{i-1}, e_i', e_{i+1}, ..., e_N)$, i.e., the ith element of e is replaced by e_i'. Thus, if two economies have the same equilibrium message, then any "crossed economy" in which one agent from one of the two initial economies is "switched" with the agent from the other must have the same equilibrium message. Hence, for a given mechanism, if two economies have the same equilibrium message m, the mechanism leads to the same outcome for both, and further, this outcome must also be the outcome of the mechanism for any of the crossed economies because of the crossing condition.

An allocation mechanism $\langle M, \mu, h\rangle$ is said to be *informationally decentralized* on E if there exist individual message correspondences $\mu_i : E_i \to\to M$, one for each i, such that $\mu(e) = \bigcap_{i=1}^{N} \mu_i(e_i)$ for all $e \in E$, i.e., it is privacy-preserving.

Let $\langle M, \mu, h\rangle$ be an allocation mechanism on E. The stationary message correspondence μ is said to be locally threaded at $e \in E$ if it has locally a continuous, single-valued selection at e. That is, there is a neighborhood $N(e) \subset E$ and a continuous function $f : N(e) \to M$ such that $f(e') \in \mu(e')$ for all $e' \in N(e)$. The stationary message correspondence μ is said to be locally threaded on E if it is locally threaded at every $e \in E$.

The notion of local threadedness was first introduced into the realization literature by Mount and Reiter (1974). This regularity condition is used mainly to exclude the possibility of intuitive smuggling information. Many continuous selection results have been given in the mathematics literature since Michael (1956).

2.4 The Walrasian process

We now define the Walrasian process that is a privacy-preserving process and realizes the Walrasian correspondence W_θ, and in which messages consist of prices and trades of all agents. In defining the Walrasian process, it is assumed that the private ownership structure matrix θ is common knowledge for all the agents.

Define the excess demand correspondence of consumer $i (i = 1, ..., I) D_i : \Delta^{L-1} \times \Theta \times \mathbb{R}_+^J \times E_i \to\to \mathbb{R}^L$ by

$$D_i(p, \theta, \pi_{I+1}, ..., \pi_N, e_i) = \Big\{ x_i : x_i + w_i \in X_i, p \cdot x_i = \sum_{j=I+1}^{N} \theta_{ij}\pi_j \big(x_i' + w_i\big) P_i(x_i + w_i) \text{implies } p \cdot x_i' > \sum_{j=I+1}^{N} \theta_{ij}\pi_j, \Big\} \tag{2}$$

where π_j is the profit of firm $j (j = I+1, ..., N)$.

Define the supply correspondence of firm j $(j = I+1, ..., N)$ $S_j : \Delta^{L-1} \times E_j \to\to \mathbb{R}^L$ by

$$S_i(p, e_j) = \{y_j : y_j \in \mathcal{Y}_j, p \cdot y_j \geqq p \cdot y_j' \ \forall y_j' \in \mathcal{Y}_j.\} \tag{3}$$

Note that (p, x, y) is a θ-Walrasian (competitive) equilibrium for economy e with the private ownership structure θ if $p \in \Delta^{L-1}$, $x_i \in D_i(p, \theta, p \cdot y_{I+1}, ..., p \cdot y_N)$ for $i = 1, ..., I$, $y_j \in S_j(p, e_j)$ for $j = I = 1, ..., N$, and the allocation (x, y) is balanced.

The Walrasian (competitive) process $\langle M_c, \mu_c, h_c\rangle$ is defined as follows.

Define $M_c = \Delta^{L-1} \times Z$.
Define μ_c: $E \to\to M_c$ by

$$\mu_c(e) = \bigcap\nolimits_{i=1}^{N} \mu_{ci}(e_i), \tag{4}$$

where $\mu_{ci} : E_i \to\to M_c$ is defined as follows:

(1) For $i=1, ..., I$, $\mu_{ci}(e_i) = \{(p, x, y) : p \in \Delta^{L-1}, x_i \in D_i(p, \theta, p \cdot y_{I+1}, ..., p \cdot y_N, e_i)$ and $\sum_{i=1}^{I} x_i = \sum_{j=I+1}^{N} y_j\}$.

(2) For $i = I+1, ..., N$, $\mu_{ci}(e_i) = \{(p, x, y) : p \in \Delta^{L-1}, y_i \in S_i(p, e_i)$ and $\sum_{i=1}^{I} x_i = \sum_{j=I+1}^{N} y_j\}$.

Thus, we have $\mu_c(e) = \mathcal{W}_\theta(e)$ for all $e \in E$.

Finally, the Walrasian outcome function $h_c : M_c \to Z$ is defined by

$$h_c(p, x, y) = (x, y), \tag{5}$$

which is an element in $W_\theta(e)$.

The Walrasian process can be viewed as a formalization of resource allocation, simulating the competitive mechanism, which is non-wasteful and individually rational with respect to the fixed share guarantee structure $\gamma_i(e; \theta)$. The competitive message process is privacy-preserving by the construction of the Walrasian process.

Remark 2 For a given private ownership structure matrix θ, since an element, $m = (p, x_i, ..., x_I, y_{I+1}, ..., y_N) \in \mathbb{R}^L_{++} \times \mathbb{R}^{(N)L}$, of the Walrasian message space M_c satisfies the conditions $\sum_{l=1}^{L} p^l = 1$, $\sum_{i=1}^{I} x_i = \sum_{j=I+1}^{N} y_j$ and $p \cdot x_1 = \sum_{j=I+1}^{N} \theta_{ij} p \cdot y_j (i = 1, ..., I)$ one of these equations is not independent by Walras Law, any Walrasian message is contained within a Euclidean space of dimension $(L+ IL+ JL)-(1+L+I)+1 = (L-1)I+LJ$ and thus, an upper bound on the Euclidean dimension of M_c is $(L-1)I+LJ$.

2.5 Informational size of message spaces

The notion of informational size can be considered as a concept that characterizes the relative sizes of topological spaces that are used to convey information in the resource allocation process. It would be natural to consider that a space, say S, has more information than the other space T whenever S is topologically "larger" than T. This suggests the following definition, which was introduced by Walker (1977).

Let S and T be two topological spaces. The space S is said to have as much information as the space T by the Fréchet ordering, denoted by $S \geqq_F T$, if T can be embedded homeomorphically in S, i.e., if there is a subspace of S' of S that is homeomorphic to T.

Let S and T be two topological spaces and let $\psi : T \to S$ be a correspondence. The correspondence ψ is said to be injective if $\psi(t) \cap \psi(t') \neq \emptyset$ implies $t = t'$ for any $t, t' \in T$. That is, the inverse, $(\psi)^{-1}$, of ψ is a single-valued function.

A topological space M is an *n-dimensional manifold* if it is locally homeomorphic to $\mathbb{R}^n$.

An informationally decentralized non-wasteful mechanism $\langle M, \mu, h\rangle$ is said to be *informationally efficient* on E if the size of its message space M is the smallest one among all other informationally decentralized non-wasteful mechanisms.

2.6 Cobb–Douglas-quadratic economies

To establish the informational efficiency of the competitive mechanism, we will adopt a standard approach that is widely used in the realization literature: For a set of admissible economies and a smooth informationally decentralized mechanism realizing a social choice correspondence, if one can find a (parametrized) subset (test family) of the set such that the subset is of dimension n, and the stationary message correspondence is injective, that is, if the inverse of the stationary message correspondence is single-valued, then the size of the message space required for an informationally decentralized mechanism to realize the social choice correspondence cannot be lower than n on the subset. Thus, it cannot be lower than n for any superset of the subset, and in particular, for the entire class of economies. It is this result that was used by Hurwicz (1977), Mount and Reiter (1974), Walker (1977), Osana (1978), Sato (1981), Calsamiglia and Kirman (1993) among others to show the minimal informational size and thus informational efficiency of the competitive mechanism, Lindahl mechanism, and the equal-income Walrasian mechanism over the various classes of economic environments. It is also this result that was used by Calsamiglia (1977) and Hurwicz (1999) to show the non-existence of a smooth finite-dimensional message space mechanism that realize Pareto efficient allocations in certain economies with increasing returns and economies with production externalities that result in non-convex production sets. It is the same result that will be used in the present paper to establish a lower bound of the size of the message space required for an informationally decentralized and non-wasteful smooth mechanism on the test family that we will specify below, and consequently over the entire class of economies with general convex preferences and production sets.

The test family, denoted by $E^{cq}=\prod_{i=1}^{N} E_i^{cq}$, are a special class of economies, where preference orderings are characterized by Cobb–Douglas utility functions, and efficient production technology are characterized by quadratic functions. It will be showed that E^{cq} is a subset of E^c in Lemma 2.

For i = 1, ..., I, consumer i's admissible economic characteristics in E_i^{cq} are given by the set of all e_i = (X_i, w_i, R_i) such that $X_i = \mathbb{R}_+^L$, $w_i > 0$, and R_i is represented by a Cobb–Douglas utility function $u(\cdot,\, a_i)$ with $a_i \in \Delta^{L-1}$ such that $u(x_i + w_i, a_i) = \prod_{l=1}^{L} \left(x_i^l + w_i^l\right)^{a_i^l}$.

For $i = I + 1, ..., N$, producer i's admissible economic characteristics are given by the set of all $e_i = \mathcal{Y}_i = \mathcal{Y}(b_i)$ such that

$$\mathcal{Y}(b_i) = \left\{ y_i \in \mathbb{R}^L : b_i^1 y_i^1 + \sum_{l=2}^{L} \left(y^l + \frac{b_i^l}{2} \left(y^l\right)^2 \right) \leqq 0 - \frac{1}{b_i^l} \leqq y_i^l \leqq 0 \text{ for all } l \neq 1 \right\}, \tag{6}$$

where b_i = (b_i^1, ..., b_i^L) with $b_i^l > \frac{J}{w_i^l}$. It is clear that any economy in E^{cq} is fully specified by the parameters a = (a_1, ..., a_I) and b = (b_{I+1}, ..., b_N). Furthermore, production sets are nonempty, closed, and convex by noting that $0 \in \mathcal{Y}\left(b_j\right)$ and their efficient points are represented by quadratic production functions in which (y_i^2, ..., y_i^L) are inputs and y_i^1 is possibly an output.

Remark 3 When we defined $\mathcal{Y}_i$ as given in Eq. 6, we have assumed that y_i^2, ..., y_i^L are inputs. Then, y_i^1 may be an output (when $y_i^1 \geqq 0$). If we, instead, assume that $0 \leqq y_i^l \leqq \frac{1}{b_i l}$ for 2, ..., L, and y_i^1 is an input, all of the results given in the paper remains true. For instance, we can similarly prove Lemma 2 on the uniqueness of θ-Walrasian equilibrium on E^{cq}.

Remark 4 The Uniqueness Theorem below will be obtained for the class of Cobb–Douglas-Quadratic production economies. In general, the smaller the set of economic environments, the more sensitive the allocation mechanisms would be. This makes the class of allocation mechanisms larger, and the Uniqueness Theorem stronger.

Given an initial endowment $\overline{w} \in \mathbb{R}_{++}^{LI}$, define a subset $\bar{E}^{cq}$ of E^{cq} by $\overline{E}^{cq} = \left\{e \in E^{cd} : w_i = \overline{w}_i \forall = 1, ..., I\right\}$. That is, endowments are constant over $\bar{E}^{cq}$.

A topology is introduced to the class $\bar{E}^{cq}$ as follows. Let $\|\cdot\|$ be the usual Euclidean norm on $\mathbb{R}^L$. For each consumer i, (i = 1, ..., I), define a metric d on $\bar{E}_i^{cq}$ by $d[u(\cdot, a_i), u(\cdot, \bar{a}_i)] = \|a_i - \bar{a}_i\|$. Note that, since endowments are fixed over $\bar{E}_i^{cq}$, this defines a topology on $\bar{E}_i^{cq}$. Similarly, for each producer i, ($i = I+1, ..., N$), define a metric d on $\bar{E}_i^{cq}$ by $d\left[\mathcal{Y}(b_i), \mathcal{Y}\left(\overline{b}_i\right)\right] = \left\|b_i - \overline{b}_i\right\|$. We may endow $\bar{E}^{cq}$ with the product topology of the $\bar{E}_i^{cq}$(i=1, ..., N) and we call this the parameter topology, which will be denoted by $\mathcal{T}_p$. Then it is clear that the topological space $\left(\overline{E}^{cq}, \mathcal{T}_p\right)$ is homeomorphic to the $(L-1)I+LJ$ dimensional Euclidean space $\mathbb{R}^{(L-1)I+LJ}$.

3 The lower bound of informational requirements of mechanisms

In this section we establish a lower bound (the minimal amount) of information, as measured by the size of the message space, that is required to guarantee that an informationally decentralized mechanism realizes Pareto efficient allocations on, E, the class of production economies.

To make the problem nontrivial, as usual, the assumption of interiority has to be made.[5] Indeed, a mechanism that gives everything to a single individual yields Pareto efficient outcomes and no information about prices is needed. Thus, given a class E of economies that includes E^{cq}, we define an optimality correspondence $\mathcal{P} : E \rightarrow\rightarrow Z$ such that the restriction $\mathcal{P}|E^{cq}$ associates with $e \in E^{cq}$ the set $\mathcal{P}(e)$ of all the Pareto efficient allocations that assign strictly positive consumption to every consumer.

The following lemma, which is based on the special class of Cob–Douglas-Quadratic economies $\bar{E}^{cq}$ specified in the above section, is central in finding the lower bound of informational requirements of resource allocation processes.

Lemma 1 *Suppose* $\langle M, \mu, h\rangle$ *is an allocation mechanism on the special class of economies* $\bar{E}^{cq}$ *such that*:

(i) *it is informationally decentralized*;
(ii) *it is non-wasteful with respect to* $\mathcal{P}$.

Then, the stationary message correspondence μ *is injective on* $\bar{E}^{cq}$. *That is, its inverse is a single-valued mapping on* $\mu(\bar{E}^{cq})$.

Proof Suppose that there is a message $m \in \mu(e) \cap \mu(\bar{e})$ for $e, \bar{e} \in \bar{E}^{cq}$. It will be proved that $e = \bar{e}$. Since μ is a privacy-preserving correspondence,

$$\mu(e) \cap \mu(\overline{e}) = \mu(\overline{e}_i, e_{-i}) \cap \mu(e_i, \overline{e}_{-i}) \tag{7}$$

for all $i = 1, ..., N$ by Remark 1, and hence, in particular,

$$m \in \mu(e) \cap \mu(\overline{e}_i, e_{-i}) \tag{8}$$

for all $i = 1, ..., N$. Let $z = (x, y) = h(m)$. Since the process $\langle M, \mu, h\rangle$ is non-wasteful with respect to $\mathcal{P}$, $z = h(m)$, and (8) imply that $z \in \mathcal{P}(e) \cap \mathcal{P}(\overline{e}_i, e_{-i})$. Since Cobb–Douglas utility functions $u_i(x)$ are strictly quasi-concave and production functions defied by efficient points of production sets,

$$y_j^1 = -\frac{1}{b_j^1} \sum\nolimits_{l=2}^{L} \left(y_j^l + \frac{b_j^l}{2} \left(y_j^l \right)^2 \right),$$

are strictly concave, by the usual Lagragian method of constrained maximization, the interior point $z \in \mathcal{P}(e)$ implies

$$\frac{a_i^l \left(x_i^1 + \overline{w}_i^1 \right)}{a_i^1 \left(x_i^l + \overline{w}_i^l \right)} = \frac{1 + b_j^l y_j^l}{b_j^1} \quad l = 2, ..., L, i = 1, ..., I, j = I + 1, ..., N, \tag{9}$$

[5] A stronger condition that can guarantee interior outcomes is that a mechanism is individually rational.

and

$$b_j^1 y_j^1 = -\sum_{l=2}^{L} \left(y_j^l + \frac{b_l^j}{2} \left(y_j^l \right)^2 \right) \quad j = I+1, \ldots, N. \tag{10}$$

Similarly, $z \in \mathcal{P}(\overline{e}_i, e_{-i})$ implies

$$\frac{\overline{a}_i^l \left(x_i^1 + \overline{w}_i^1 \right)}{\overline{a}_i^1 \left(x_i^l + \overline{w}_i^l \right)} = \frac{1 + b_j^l y_j^l}{b_j^1} \quad l = 2, \ldots, L, i = 1, \ldots, I, j = I+1, \ldots, N. \tag{11}$$

From equations (9) and (11), we derive

$$\frac{\overline{a}_i^l}{\overline{a}_i^1} = \frac{a_i^l}{a_i^1} \; l = 2, \ldots, L, i = 1, \ldots, I. \tag{12}$$

As $\sum_{l=1}^{L} a_i^l = 1$ and $\sum_{l=1}^{L} \overline{a}_i^l = 1$, equation (12) implies

$$a_i^l = \overline{a}_i^l \quad l = 1, \ldots, L, i = 1, \ldots, I, \tag{13}$$

and thus $a = \bar{a}$.

As for producers, $z \in \mathcal{P}\left(\overline{e}_j, e_{-j}\right)$ implies

$$\frac{a_i^l \left(x_i^1 + \overline{w}_i^1 \right)}{a_i^1 \left(x_i^l + \overline{w}_i^l \right)} = \frac{1 + \overline{b}_j^l y_j^l}{\overline{b}_j^l} \quad l = 2, \ldots, L, i = 1, \ldots, I, j = I+1, \ldots, N, \tag{14}$$

and

$$\overline{b}_j^1 y_j^1 = -\sum_{l=2}^{L} \left(y_j^l + \overline{b}_j^l 2 \left(y_j^l \right)^2 \right). \tag{15}$$

From equations (9) and (14), we derive

$$\frac{b_j^1}{\overline{b}_j^1} = \frac{1 + b_j^l y_j^l}{1 + \overline{b}_j^l y_j^l} \quad l = 2, \ldots, L, j = I+1, \ldots, N. \tag{16}$$

From equations (10) and (15), we derive

$$\frac{b_j^1}{\overline{b}_j^1} = \frac{\sum_{l=2}^{L} \left(1 + \frac{b_j^l}{2} y_j^l \right) y_j^l}{\sum_{l=2}^{L} \left(1 + \frac{\overline{b}_j^l}{2} y_j^l \right) y_j^l} \quad j = I+1, \ldots, N. \tag{17}$$

From equations (16) and (17), we have

$$\frac{1+b_j^l y_j^l}{\sum_{l=2}^{L}\left(1+\frac{b_j^l}{2}y_j^l\right)y_j^l}=\frac{1+\overline{b}_j^l y_j^l}{\sum_{l=2}^{L}\left(1+\frac{\overline{b}_j^l}{2}y_j^l\right)y_j^l}\quad l=2,\ldots,L, j=I+1,\ldots,N. \tag{18}$$

Multiplying y_j^l on the both sides of equation (18) and making summations, we have

$$\frac{\sum_{l=2}^{L}\left(1+b_j^l y_j^l\right)y_j^l}{\sum_{l=2}^{L}\left(1+\frac{b_j^l}{2}y_j^l\right)y_j^l}=\frac{\sum_{l=2}^{L}\left(1+\overline{b}_j^l y_j^l\right)y_j^l}{\sum_{l=2}^{L}\left(1+\frac{\overline{b}_j^l}{2}y_j^l\right)y_j^l}\quad j=I+1,\ldots,N. \tag{19}$$

Simplifying equation (19), we have

$$\sum_{l=2}^{L} b_j^l\left(y_j^l\right)^2=\sum_{l=2}^{L}\overline{b}_j^l\left(y_j^l\right)^2\quad j=I+1,\ldots,N. \tag{20}$$

Multiplying 1/2 and adding $\sum_{l=2}^{L} y_j^l$ on the both sides of equation (20) and then applying equations (10) and (15), we have

$$b_j^1 y_j^1=\overline{b}_j^1 y_j^1\quad j=I+1,\ldots,N, \tag{21}$$

which implies

$$b_j^1=\overline{b}_j^1\quad j=I+1,\ldots,N. \tag{22}$$

Finally, from equations (16) and (22), we have

$$b_j^l=\overline{b}_j^l\quad l=2,\ldots,L, j=I+1,\ldots,N. \tag{23}$$

Thus, we have proved

$$b_j=\overline{b}_j\quad j=I+1,\ldots,N, \tag{24}$$

which means $b=\overline{b}$. Thus, equations (13) and (24) mean that $e=\bar{e}$. Consequently, the inverse of the stationary message correspondence, $(\mu)^{-1}$, is a single-valued mapping from $\mu(\bar{E}^{cq}$ to $\bar{E}^{cq})$. □

The following theorem establishes a lower bound informational size of messages spaces of any smooth allocation mechanism that is informationally decentralized and non-wasteful over any class of economies that includes E^{cq} such that.

Theorem 1 (Informational Boundedness Theorem) *Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism on any class of production economies E that includes E^{cq} such that:*

(i) *it is informationally decentralized*;
(ii) *it is non-wasteful with respect to* $\wp$;

(iii) *M is a Hausdorff topological space*;
(iv) μ *is locally threaded at some point* $e \in \bar{E}^{cq}$

Then, the size of the message space M is at least as large as $\mathbb{R}^{(L-1)I+LJ}$, *that is*, $M \geqq_F M_c =_F \mathbb{R}^{(L-1)I+LJ}$.

Proof As was noted above, $\bar{E}^{cq}$ is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$. Hence, it suffices to show $M \geqq_F \bar{E}^{cq}$.

By the injectiveness of Lemma 1, we know that the restriction $\mu|_{\bar{E}^{cq}}$ of the stationary message correspondence μ to $\bar{E}^{cq}$ is an injective correspondence. Since μ is locally threaded at $e \in \bar{E}^{cq}$, there exists a neighborhood $N(e)$ of e and a continuous function $f: N(e) \to M$ such that $f(e') \in \mu(e')$ for all $e' \in N(e)$. Then f is a continuous injection from $N(e)$ into M. Since μ is an injective correspondence from $\bar{E}^{cq}$ into M, thus f is a continuous one-to-one function on $N(e)$.

Since $\bar{E}^{cq}$ is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$, there exists a compact set $\overline{N}(e) \subset N(e)$ with nonempty interior point. Also, since f is a continuous one-to-one function on $N(e)$, f is a continuous one-to-one function from the compact space $\overline{N}(e)$ onto a Hausdorff topological space $f\left(\overline{N}(e)\right)$. Hence, it follows that the restriction $f|_{\overline{N}(e)}$ is a homeomorphic imbedding on $\overline{N}(e)$ by Theorem 5.8 in Kelley (1955, p. 141). Choose an open ball $\mathring{N}(e) \subset \overline{N}(e)$. Then $\mathring{N}(e)$ and $f\left(\mathring{N}(e)\right)$ are homeomorphic by a homeomorphism $f\Big|\mathring{N}(e) : \mathring{N}(e) \to f\left(\mathring{N}(e)\right)$. This, together with the fact that $\bar{E}^{cq}$ is homeomorphic to its open ball $\mathring{N}(e)$, implies that $\bar{E}^{cq}$ is homeomorphic to $f\left(\mathring{N}(e)\right) \subset M$. Hence, it follows that $M \geqq_F \overline{E}^{cd} =_F \mathbb{R}^{(L-1)I+LJ}$. □

4 Informational efficiency of competitive mechanism

In the previous section, we found that the lower bound informational size of message spaces for smooth allocation mechanisms that are privacy-preserving and non-wasteful over the class E of production economies that includes $\bar{E}^{cq}$ is the $(L-1)I + LJ$-dimensional Euclidean space $\mathbb{R}^{(L-1)I+LJ}$. In this section we assert that the lower bound is exactly the size of the message space of the Walrasian mechanism, and thus the Walrasian mechanism is informationally efficient among all smooth resource allocation mechanisms that are informationally decentralized and non-wasteful over the set E^c.

By Theorem 1, to show this result, we need to show that M_c is homeomorphic to the $(L-1)I + LJ$-dimensional Euclidean space $R^{(L-1)I+LJ}$. To do so, note that, from Remark 2, we know that the upper bound dimension of the message space of the Walrasian mechanism is $(L-1)I + LJ$. As a result, if we can show that this upper bound can be reached on the restriction of the message space of the competitive mechanism to the test family $\bar{E}^{cq}$ of Cobb–Douglas-Quadratic economies, i.e., if we can show that $\mu_c|_{\overline{E}^{cq}}$ is homeomorphic to the $(L-1)I + LJ$-dimensional Euclidean space $\mathbb{R}^{(L-1)I+LJ}$, then we know that the size of the message space of the Walrasian mechanism is $(L-1)I + LJ$ and thus the competitive mechanism is informationally efficient among all resource allocation mechanisms that are informationally decentralized and non-wasteful over the class of economies in which Walrasian equilibria exist. Hence, to show the informational efficiency of the competitive mechanism, it suffices for us to show that this upper bound can be actually reached on the test family of economies for the Walrasian mechanism.

We will first state the following lemmas that shows that the competitive mechanism is single-valued and continuous so that it is locally threaded on the test family set $\bar{E}^{cq}$ of Cobb–Douglas-Quadratic economies.

Lemma 2 *For any given private ownership structure matrix θ, every economy in $\bar{E}^{cq}$ has a unique θ-Walrasian equilibrium, i.e., $W_\theta(e)$ is a single-valued mapping from $\bar{E}^{cq}$ to Z.*

Proof The existence of a θ-Walrasian equilibrium can be obtained by applying the existence theorem of Debreu (1959, p 83) by noting that $\mathcal{Y}_i$ is closed and convex, $0 \in \mathcal{Y}_i$, $\left(-\mathbb{R}^L_+\right) \in \mathcal{Y}_i$ and $\mathcal{Y}_i \cap (-\mathcal{Y}_i) \subset \{0\}$. To show the θ-Walrasian equilibrium is unique, we first need to derive the supply and demand functions of agents.

Produce i (for $i = I + 1, \ldots, N$) chooses his production plan so as to maximize profit within $\mathcal{Y}(b_i)$. Thus, he solves the following profit maximizing problem:

$$\max_{y_i} p \cdot y_i$$

subject to

$$b_i^1 y_i^1 + \sum_{l=2}^{L} \left(y_i^l + b_i^l 2 \left(y_i^l \right)^2 \right) = 0. \tag{25}$$

and

$$-\frac{1}{b_i^l} \leqq y_i^l \leqq 0 \text{ for all } l \neq 1.$$

An interior solution y must satisfy the following first-order conditions:

$$p^1 = \lambda_i b_i^1 \tag{26}$$

$$p^l = \lambda_i \left(1 + b_i^l y_i^l \right), \quad l = 2, \ldots, L, \tag{27}$$

where λ_i is a Lagrange multiplier. From (26) and (27) and the assumption $y_i^l \leqq 0$, we can obtain the supply functions

$$y_i^l(p) = \begin{cases} \frac{b_i^1 p^l}{b_i^l p^1} - \frac{1}{b_i^l} & \text{if } \frac{p^l}{p^1} < \frac{1}{b_i^1} \\ 0 & \text{otherwise} \end{cases}$$

for $l = 2, ..., L$, and thus, by (25),

$$y_i^1(p) = -\frac{1}{b_i^1}\sum_{l=2}^{L}\left[y_i^l(p) + \frac{b_i^l}{2}\left(y_i^l(p)\right)^2\right]. \tag{28}$$

It may be remarked that $y_i^1(p) \geqq 0$ for all $p \in \Delta^{L-1}$. Indeed, for $l = 2, ..., L$, if $\frac{p^l}{p^1} < \frac{1}{b_i^1}$, then $y_i^l(p) = \frac{b_i^1 p^l}{b_i^l p^1} - \frac{1}{b_i^l}$ and thus

$$\begin{aligned} y_i^l(p) + \frac{b_i^l}{2}\left(y_i^l(p)\right)^2 &= \left[1 + \frac{b_i^l}{2}y_i^l(p)\right]y_i^l(p) \\ &= \left[1 + \frac{b_i^l}{2}\left(\frac{b_i^1 p^l}{b_i^l p^1} - \frac{1}{b_i^l}\right)\right]\left[\frac{b_i^1 p^l}{b_i^l p^1} - \frac{1}{b_i^l}\right] \\ &= \frac{1}{2b_i^l}\left(1 + \frac{b_i^1 p^l}{p^1}\right)\left(\frac{b_i^1 p^l}{p^1} - 1\right) \\ &= \frac{1}{2b_i^l}\left[\left(\frac{b_i^1 p^l}{p^1}\right)^2 - 1\right] < 0. \end{aligned} \tag{29}$$

If $\frac{p^l}{p^1} \geqq \frac{1}{b_i^1}$, then $y_i^l(p) = 0$ by (28). Thus $y_i^l(p) + \frac{b_i^l}{2}\left(y_i^l(p)\right)^2 \leqq 0$ for all $p \in \Delta^{L-1}$, and therefore by (28), we have $y_i^1(p) \geqq 0$ for all $p \in \Delta^{L-1}$.

Consumer i (for $i = 1, ..., I$) chooses his consumption so as to maximize his utility subject to his budget constraint. Since all utility functions are Cobb–Douglas, it is well known that the net demand functions are given by

$$x_i^l(p) = \frac{a_i^l}{p_i^l}\left[p \cdot \overline{w}_i + \sum_{j=I+1}^{N} \theta_{ij} p \cdot y_j(p)\right] - \overline{w}_i^l. \tag{30}$$

Define the aggregate net excess demand function by

$$\widehat{z} = \sum_{i=1}^{I} x_i(p) - \sum_{i=I+1}^{N} y_i(p). \tag{31}$$

Notice that, since every consumer's budget constraint holds with equality, and the demand and supply functions are clearly continuous, the aggregate excess demand function $\widehat{z}(p)$ is continuous and satisfies Walras' Law, i.e., $p \cdot \widehat{z}(p) = 0$ for all $p \in \Delta^{L-1}$. Thus, the existence of a θ-Walrasian equilibrium can be also guaranteed by applying an existence theorem in Varian (1992)), i.e., there exists some $p \in \Delta^{L-1}$ such that $\widehat{z}(p) \leqq 0$, which means a θ-Walrasian equilibrium exists for every economy $e \in \bar{E}^{cq}$.

Now we show that every economy $e \in \bar{E}^{cq}$ has a unique θ-Walrasian equilibrium, and for this, it suffices to show that all goods are gross substitutes at any price $p \in$

Δ^{L-1}, i.e., an increase in price, k, brings about an increase in the excess demand for good l. When $\widehat{z}$ is differentiable, the gross substitutes condition becomes $\frac{\partial \widehat{z}^l(p)}{\partial p^k} > 0$ for $l \neq k$.

For each i=I+1, ..., N, from (28), if $\frac{p^l}{p^1} < \frac{1}{b_i^1}$ we have

$$\frac{\partial y_i^l(p)}{\partial p^l} = \frac{b_i^1}{b_i^l p^1} > 0 \quad l = 2, ..., L, \tag{32}$$

$$\frac{\partial y_i^l(p)}{\partial p^k} = 0 \quad k \neq l, k \neq 1, l \neq 1, \tag{33}$$

$$\frac{\partial y_i^l(p)}{\partial p^1} = -\frac{b_i^1 p^l}{b_i^l (p^1)^2} < 0 \quad l = 2, ..., L, \tag{34}$$

and from (28), we have

$$\frac{\partial y_i^1(p)}{\partial p^1} = -\frac{1}{b_i^1} \sum_{l=2}^{L} \left[1 + b_i^l y_i^l\right] \frac{\partial y_i^l}{\partial p^1} > 0, \tag{35}$$

and

$$\frac{\partial y_i^1(p)}{\partial p^l} = -\frac{1}{b_i^1} \left[1 + b_i^l y_i^l\right] \frac{\partial y_i^l}{\partial p^l} < 0. \tag{36}$$

When $\frac{p^l}{p^1} \geq \frac{1}{b_i^1}$, $y_i^l(p)$ is constant functions for l = 2, ..., L. Thus, $y_i^l(p)$ is a non-increasing function in p^k for any $l \neq k$ and any $p \in \Delta^{L-1}$.

Note that, by Hotelling's Lemma (cf. Varian (1992, p. 43), $\frac{\partial p \cdot y_j(p)}{\partial p^k} = y_j^k(p)$, and $-\frac{1}{b_i^k} > -\frac{\overline{w}_i^k}{J}$ by the assumption that $b_i^k > \frac{J}{\overline{w}_i^k}$. Then, for each i = 1, ..., I, from (28), we have

$$\begin{aligned} \frac{\partial x_i^l(p)}{\partial p^k} &= \frac{a_i^l}{p^l} \left[\overline{w}_i^k + \sum_{j=I+1}^{N} \theta_{ij} y_j^k(p)\right] \\ &\geqq \frac{a_i^l}{p^l} \left[\overline{w}_i^k + \sum_{j=I+1}^{N} \theta_{ij} \left(-\frac{1}{b_i^k}\right)\right] \\ &> \frac{a_i^l}{p^l} \left[\overline{w}_i^k + \sum_{j=I+1}^{N} \left(-\frac{\overline{w}_i^k}{J}\right)\right] = 0 \end{aligned} \tag{37}$$

for $l \neq k$, $k \neq 1$, and

$$\frac{\partial x_i^l(p)}{\partial p^1} = \frac{a_i^l}{p^l} \left[\overline{w}_i^1 + \sum_{j=I+1}^{N} \theta_{ij} y_j^1(p)\right] > 0 \tag{38}$$

by noting that $y_j^1(p) \geqq 0$. Thus, the net demand function $x_i^l(p)$ is an increasing function and the supply function $y_i^l(p)$ is nonincreasing function in price $k \neq l$ and for every $p \in \Delta^{L-1}$. Therefore, an increase in price, k, brings about an increase in the excess demand for good l, and thus all goods are gross substitutes. Hence, the θ-Walrasian equilibrium must be unique (cf. Varian (1992)). □

Lemma 3 *Let μ_{cq} be the Walrasian message correspondence on $\bar{E}^{cq}$. The μ_{cq} is a continuous function.*

Proof By Lemma 2, we know $\mu_{cq} = (p, x, y)$ is a (single-valued) function. Also, from (28), (28) and (30), we know that the demand function $x(p; c)$ and supply function $y(p; c)$ are continuous in p and $c := (a, b)$. So we only need to show the price vector p is a continuous function on $\bar{E}^{cq}$.

Let $\{e(k)\}$ be a sequence in $\bar{E}^{cq}$ and $e(k) \to e \in \bar{E}^{cq}$. Since any economy in $\bar{E}^{cq}$ is fully specified by the parameter vector c, $e(k) \to e$ implies $c(k) \to c$.

Let $\mu_{cq} = (p, x(p; c), y(p; c))$ and $\mu_{cq}(k) = (p(k), x(p(k); c(k)), y(p(k); c(k)))$. Then we have, $\widehat{z}(p; c) = 0$ and $\widehat{z}(p(k); c(k)) = 0$, i.e.,

$$\sum_{i=1}^{I} x_i(p; c) = \sum_{j=I+1}^{N} y_j(p; c)$$

and

$$\sum_{i=1}^{I} x_i(p(k); c(k)) = \sum_{j=I+1}^{N} y_j(p(k); c(k)).$$

Since the sequence $\{p(k)\}$ is contained in the compact set

$$\overline{\Delta}^{L-1} = \left\{p \in \mathbb{R}_+^L : \sum_{l=1}^{L} p^l = 1\right\},$$

there exists a convergent subsequence $\{p(k_t)\}$ that converges, say, $\overline{p} \in \overline{\Delta}^{L-1}$ and $\widehat{z}(p(k_t); c(k_t)) = 0$. Since $x_i(p(k); c(k))$ and $x_i(p(k); c(k))$ are continuous in c, $\widehat{z}(p; c)$ is continuous in c and thus we have $\widehat{z}(p(k_t); c(k_t)) \to \widehat{z}(\overline{p}, c)$ as $k_t \to \infty$ and $c(k_t) \to c$. However, since every $e \in \bar{E}^{cq}$ has the unique θ-Walrasian equilibrium price p that is completely determined by $\widehat{z}(p; c) = 0$, so we must have $\overline{p} = p$. □

Lemma 4 *Let μ_c be the Walrasian message correspondence on E^c. Then $\mu_c(E^c)$ is homeomorphic to $\bar{E}^{cq}$.*

Proof By Lemma 2, we know that μ_{cq} is the restriction of μ_c to $\bar{E}^{cq}$. We first prove that the inverse of μ_{cq}, $(\mu_{cq})^{-1}$ is a function.

Let $m \in \mu_{cq}(\bar{E}^{cq})$ and let $e, e' \in (\mu_{cq})^{-1}(m)$. Then $m \in \mu_{cq}(e) \cap \mu_{cq}(e') = \mu_c(e) \cap \mu_c(e') = \mu_c(e_i', e_{-i}) \cap \mu_c(e_i, e_{-i}')$ for all $i = 1, \dots, N$ by Remark 1. Let $z = h_{cd} \in W(\bar{E}^{cq})$ be the Walrasian outcome function. Since u_i is monotonically increasing, we know z is Pareto efficient by the First Theorem of Welfare Economics. Then, the allocation process $\langle M_c, \mu_{cq}, h_{cq}\rangle$ is privacy-preserving and non-wasteful over $\bar{E}^{cq}$ with respect to $\mathcal{P}$. Then, by Lemma 1, $e = e'$ and thus $(\mu_{cq})^{-1}$ is a function. Also, by Lemma 3, μ_{cq} is a continuous function. Therefore, μ_{cq} is a continuous one-to-one function on $\bar{E}^{cq}$.

Since every e is fully characterized by $(a, b) \in \mathbb{R}^{LI+LJ}_{++}$ with $\sum_{l=1}^{L} a_i^l = 1$ for $i =$ 1, ..., I, $\bar{E}^{cq}$ is homeomorphic to the finite-dimensional Euclidean space $\mathbb{R}^{(L-1)I+LJ}$. Thus, it must be homeomorphic to any open ball centered on any of its points, and also locally compact. It follows that for any $e \in \bar{E}^{cq}$, we can find a neighborhood $N(e)$ of e and a compact set $\overline{N}(e) \subset N(e)$ with a nonempty interior point. Since μ_{cd} is a continuous one-to-one function on $N(e)$, μ_{cd} is a continuous one-to-one function from the compact space $\overline{N}(e)$ onto an Euclidean (and hence Hausdorff topological) space $\mu_{cd}(\overline{N}(e))$. Hence, it follows that the restriction μ_{cd} restricted to $N(e)$ is a homeomorphic imbedding on $\overline{N}(e)$ by Theorem 5.8 in Kelley (1955, p. 141). Choose an open ball $\mathring{N}(e) \subset \overline{N}(e)$. Then $\mathring{N}(\mathrm{e})$ and $\mu_{cd}(\mathring{N}(e))$ are homeomorphic by a homeomorphism $\mu_{cd}|\mathring{N}(e) : \mathring{N}(e) \to \mu_{cd}(\mathring{N}(e))$. This, together with the fact that $\bar{E}^{cq}$ is homeomorphic to its open ball $\mathring{N}(e)$, implies that $\bar{E}^{cq}$ is homeomorphic to $\mu_{cd}(\mathring{N}(e)) \subset M_c$, implying that $\mu_{cd}(E^c)$ can be homeomorphically imbedded in $\mu_c(E^c)$.

Finally, by Remark 2, the Walrasian message space M_c is contained within an Euclidean space of dimension $(L-1)I + LJ$. This necessarily implies that M_c and thus $\mu_c(\bar{E}^{cq})$ is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$ because his restriction $\mu_{cq}(\bar{E}^{cq})$ is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$, and consequently, $\mu_c(E^c)$ is homeomorphic to $\bar{E}^{cq}$. □

From the above lemmas and Theorem 1, we have the following theorem that establish the informational efficiency of the competitive mechanism within the class of all smooth resource allocation mechanisms which are informationally decentralized and non-wasteful over the class of Walrasian production economies E^c.

Theorem 2 (Informational Efficiency Theorem) *Then the Walrasian allocation mechanism $\langle M_c, \mu_c, h_c \rangle$ is informationally efficient among all allocation mechanisms $\langle M, \mu, h \rangle$ defined on E^c that*

(i) *are informationally decentralized*;
(ii) *are non-wasteful with respect to $\mathcal{P}$*;
(iii) *have Hausdorff topological message spaces*;
(iv) *satisfy the local threadedness property at some point $e \in \bar{E}^{cq}$. That is, $M_c =_F \mathbb{R}^{(L-1)I+LJ} \leqq_F M$.*

Proof First note that, since $W(e) \neq \emptyset$ for all $e \in E^c$, the Walrasian mechanism is well defined on E^c. $\langle M_c, \mu_c, h_c \rangle$ is also privacy-preserving as we show in Section 2. Since u_i is strictly monotone on E by assumption, we know x is Pareto efficient by the First Theorem of Welfare Economics. Thus, the competitive process $\langle M_c, \mu_c, h_c \rangle$ is privacy-preserving and non-wasteful over E^c. Furthermore, by Lemmas 2 and 3, we know that μ_c is a single-valued and continuous function on $\bar{E}^{cq}$. Therefore, the Walrasian allocation mechanism $\langle M_c, \mu_c, h_c \rangle$ satisfies Conditions (i)–(iv).

Also, since $\mu_c(E^c)$ is homeomorphic to $\bar{E}^{cq}$ by Lemma 4 and $\bar{E}^{cq}$ is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$ as noted above, then M_c is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$. Thus, by Theorem 1, we have $M \geqq_F M_c =_F \mathbb{R}^{(L-1)I+LJ}$. Hence, the Walrasian allocation mechanism $\langle M_c, \mu_c, h_c \rangle$ is informationally efficient among all allocation mechanisms that satisfy Conditions (i)–(iv). □

5 The uniqueness theorem

In this section we establish that the competitive allocation process is the only most informationally efficient decentralized mechanism for production economies among mechanisms that achieve Pareto optimal and individually rational allocations. We follow the basic strategy of the proof given by Jordan (1982) for the uniqueness of the competitive mechanism in achieving efficiency in general convex production economies, although the proof of our results is more complicated. We first show the following lemmas.

Lemma 5 *Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism on the class of production economies E^{cq}such that*:

(i) *it is informationally decentralized*;
(ii) *it is non-wasteful with respect to $\mathcal{P}$*;
(iii) *it is individually rational with respect to the fixed share guarantee structure $\gamma_i(e;\theta)$*.

Then, there is a function ϕ: $\mu(E^{cq}) \rightarrow \Delta^{L-1} \times Z$ defined by $\phi(m) = (p, x, y)$ where *$(x, y) = h(m)$ and p is an efficiency price vector at the allocation (x, y). In particular, p is proportional to $Du_i(w_i+x_i)$ for consumer $i = 1, ..., I$, and proportional to $DT_i(y)$ for producer $i = I + 1, +..., N$ for each $e \in E^{cq}$ with $m \in \mu(e)$, where $T_i = b_i^1 y_i^1 + \sum_{l=2}^L \left(y^l + \frac{b_i^l}{2}\left(y^l\right)^2\right) = 0$.*

Proof Let $e \in E^{cq}$, let $m \in \mu(e)$, and let $(x, y) = h(m)$. Since (x, y) is Pareto optimal, there exists an efficiency price vector $p \in \Delta^{L-1}$ so that $\phi(m) = (p, x, y)$ is an equilibrium relative to the price system p. Also, since (x, y) individually rational with respect to the fixed share guarantee structure $\gamma_i(e;\theta)$ and utility functions $u_i(x)$ are Cobb–Douglas, we have $w_i + x_i \in \mathbb{R}_{++}^L$ by noting that $p \cdot y_j \geqq 0$ for all $j = I + 1, ..., N$ and thus $(\gamma_i(e;\theta) + w_i) \in \mathbb{R}_{++}^L$ for all $i = 1, ..., I$. Also, since $y_i^1 = -\frac{1}{b_i^1}\sum_{l=2}^L \left(y_i^l + \frac{b_i^l}{2}\left(y_i^l\right)^2\right)$ is strictly concave, y_i is an interior point of $\mathcal{Y}_i$ for all $i = I + 1, ..., N$. Therefore, p must be proportional to $Du_i(w_i + x_i)$ for $i = 1, ..., I$, and to $DT_i(y)$ for $i = I + 1, + ..., N$. Let $e' \in E^{cq}$ be any other environment with $m \in \mu(e')$. Since $\langle M, \mu, h\rangle$ is privacy-preserving, by Remark 1, we have $m \in \mu(e_i', e_{-i})$ for each $i = 1, ..., N$. Therefore, $(x, y) \in h[\mu(e_i', e_{-i})]$ for each $i = 1, ..., N$, and thus p is proportional to $Du_i'(w_i + x_i)$ for $i = 1, ..., I$, and proportional to $DT_i'(y)$ for $i = I+1, + ..., N$. Thus, ϕ is well defined. □

Lemma 6 *Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism on the class of production economies E^{cq} such that*:

(i) *it is informationally decentralized*;
(ii) *it is non-wasteful with respect to $\mathcal{P}$*;
(iii) *it is individually rational with respect to a given fixed share guarantee structure $\gamma_i(e;\theta)$*;
(iv) *M is a $(L-1)I + LJ$ dimensional manifold*;
(v) *μ is a continuous function on E^{cq}*.

Let $\bar{e} \in E^{cq}$ *and let* $(\overline{p}, \overline{x}, \overline{y}) = \phi[\mu(\overline{e})]$, *where* ϕ *is defined as Lemma 5. If* e^* *is any environment such that* $w_i^* + \overline{x}_i > 0$, *and* $Du_i^*\left(w_i^* + \overline{x}_i\right)$ *is proportional to* $\overline{p}$ *for consumer* $i = 1, ..., I$, *and* $DT^*(\overline{y}_i)$ *is proportional to* $\overline{p}$ *for producer* $i = I + 1, + ..., N$, *then* $\mu(e^*) = \mu(\bar{e})$. *In particular,* ϕ *is a one-to-one function.*

Proof For $\bar{e} \in E^{cq}$, define the set $\overline{E}^{cq} = \{(w, a, b) \in E^{cq} : w_i = \overline{w}_i \forall i = 1, ..., I\}$. Note that, by Lemma 1, for each $e, e' \in \bar{E}^{cq}$, if for some $(x, y) \in Z$, $Du_i(\overline{w}_i + x_i)$ is proportional to $Du_i'(\overline{w}_i + x_i)$ for consumer $i = 1, ..., I$, and $DT_i(y)$ is proportional to $DT_i'(y)$ for producer $i = I + 1, ..., N$, then $a = a'$ and $b = b'$. Thus, by Lemma 5, $\phi \cdot \mu$ is one-to-one on $\bar{E}^{cq}$ and so μ is one-to-one on $\bar{E}^{cq}$.

Since $\bar{E}^{cq}$ is homeomorphic to $\mathbb{R}^{(L-1)I+LJ}$, we can consider $\bar{E}_{cq}$ as an open subset of $\mathbb{R}^{(L-1)I+LJ}$. Let $N(\bar{e})$ be an open neighborhood of $\bar{e}$ in $\bar{E}^{cq}$ and let $\overline{m} = \mu(\overline{e})$. Since M is a $(L - 1)I + LJ$ dimensional manifold, M and $\bar{E}^{cd}$ are manifolds of the same dimensional. Also, since μ is a one-to-one function on $N(\bar{e})$, $V(\overline{m}) \equiv \mu(N(\overline{e}))$ is an open neighborhood of $\overline{m}$ and μ is a homeomorphism on $N(\bar{e})$ to $V(\overline{m})$ by Exercise 18.10 in Greenberg (1967, p. 82). Let $n = (L - 1)I + LJ$, and let $H_n(M, M - \overline{m})$ denote the n-th singular homology module of M relative to $M - \overline{m}$, with integral coefficients. Then $H_n(M, M - \overline{m})$ is isomorphic to Z (denoted by $H_n(M, M - \overline{m}) \cong Z$), the set of integers, and the homomorphism i^* : $H_n(V(\overline{m}), V(\overline{m}) - \overline{m}) \rightarrow H_n(M, M - \overline{m})$ induced by the inclusion $i : (V(\overline{m}), V(\overline{m}) - \overline{m}) \rightarrow (M, M - \overline{m})$ is isomorphism (see Greenberg (1967, p. 111)).

Also, by the hypothesis, $D \ln u_i^*\left(w_i^* + \overline{x}_i\right)$ is proportional to $\ln \overline{u}_i(\overline{w}_i + \overline{x}_i)$ for consumer $i = 1, ..., I$, and $DT^*(\overline{y}_i)$ is proportional to $D\overline{T}(\overline{y}_i)$ for producer $i = I + 1, ..., N$ so there is some $\lambda_i^* > 0$, one for each i, such that for consumer $i = 1, ..., I$

$$\frac{a_i^{*l}}{w_i^{*l} + \overline{x}_i^l} = \lambda_i^* \frac{\overline{a}_i^l}{\overline{w}_i^l + \overline{x}_i^l},$$

and for producer $i = I + 1, ..., N$,

$$b_i^{*l} = \begin{cases} \lambda_i^* \overline{b}_i^l & \text{if} \quad l = 1 \\ \frac{\lambda_i^*\left(1 + \overline{b}_i^l y_i^{l-1}\right)}{y_i^l} & \text{if} \quad l = 2, ..., L. \end{cases}$$

Define the function $G : N(\bar{e}) \times [0,1] \rightarrow E^{cq}$ by $G((\overline{w}, a, b), t) = (w', a', b')$, where

$$\begin{aligned} w_i' &= tw_i^* + (1 - t)\overline{w}_i, \\ a_i' &= \frac{\alpha_i'}{\sum_{i=1}^I \alpha_i'} i = I, \ldots, I, \end{aligned}$$

$$b_i' = \begin{cases} \left[t\overline{b}_i^1 + (1 - t)b_i^1\right]\left(t\lambda_i^* + (1 - t)\right) & \text{if } l = 1 \\ \frac{\left[t\lambda_i^* + (1 - t)\right]\left[1 + \left(t\overline{b}_i^l + (1 - t)b_i^l\right)y_i^l\right] - 1}{y_i^l} & l = 2, \ldots, L \end{cases} \quad i = I + 1, \ldots N$$

with

$$\alpha_i^l = \left[t\bar{a}_i^l + (1-t)a_i^l\right]\frac{w_i'^l + \bar{x}_i^l}{\bar{w}_i^l + \bar{x}_i^l}\left[t\lambda_i^* + (1-t)\right].$$

Then, $G(\cdot, 0)$ is the inclusion map by noting that $G((\overline{w}, a, b), 0) = (\overline{w}, a, b)$ for all $(\overline{w}, a, b) \in N(\bar{e})$, and $G(\cdot, 1)$ is the constant map on $N(\bar{e})$ to e^* by noting that $G((\overline{w}, a, b), 1) = (w^*, a^*, b^*)$ for all $(\overline{w}, a, b) \in N(\bar{e})$. Let $t \in [0,1]$, $(\overline{w}, a, b) \in N(\bar{e})$ with $(\overline{w}, a, b) \neq \bar{e}$, and $(w', a', b') = G(\overline{w}, a, b, t)$. Then $\frac{a_i'^l}{w_i'^l + \bar{x}_i^l}$ are proportional to $\frac{t\bar{a}_i^l + (1-t)a_i^l}{\bar{w}_i^l + \bar{x}_i^l}$ for $i = 1, ..., I$, $b_i'^1$ is proportional to $\left[t\bar{b}_i^1 + (1-t)b_i^1\right]$ and $\left[1 + b_i'^l\bar{y}_i^l\right]$ are proportional to $\left[1 + \left(t\bar{b}_i^l + \left(1-t\right)b_i^l\right)\bar{y}_i^l\right]$ for $l = 2, ..., L$ and $i = I+1, ..., N$. Then environment $\left(\overline{w}, t\bar{a} + (1-t)a, t\bar{b} + (1-t)b\right) \in \overline{E}^{cq}$, so by the argument in the first paragraph, for each $t < 1$, $\mu[G(e,t)] = \overline{m}$ only if $e = \bar{e}$.

Now, suppose by way of contradiction that $\mu(e^*) \neq \overline{m}$. Then $\mu[G(e,t)] \neq \overline{m}$ whenever $e \neq \bar{e}$. Define $G' : (V(\overline{m}), V(\overline{m}) - \overline{m}) \times [0,1] \to (M, M - \overline{m})$ by $G'(m,t) = \mu(G[\mu^{-1}(m), t])$. Then G' is a homotopy between the inclusion $i : (V(\overline{m}), V(\overline{m}) - \overline{m}) \to (M, M - \overline{m})$ and the constant map $j : (V(\overline{m}), V(\overline{m}) - \overline{m}) \to (m^*, M - \overline{m})$, where $m^* = \mu(e^*)$. Hence, $i^* = j^*$ and consequently i^* is the zero homomorphism, which contradicts the fact that i^* is isomorphic to Z. □

In the above two lemmas, the individual rationality is only used to guarantee that the allocations are interior so that efficiency prices are normalized utility gradients. We now use the full force of individual rationality to prove the unique informational efficiency of the competitive mechanism.

Theorem 3 *Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism on the class of production economies E^{cq} such that:*

(i) *it is informationally decentralized*;
(ii) *it is non-wasteful with respect to $\mathcal{P}$*;
(iii) *it is individually rational with respect to the fixed share guarantee structure $\gamma_i(e;\theta)$*;
(iv) *M is a $(L-1)I+LJ$ dimensional manifold*;
(v) *μ is a continuous function on E^{cq}.*

Then, there is a homeomorphism ϕ on $\mu(E^{cq})$ to M_c such that

(a) $\mu_c = \phi \cdot \mu$;
(b) $h_c \cdot \phi = h$.

The conclusion of the theorem is summarized in the following commutative homeomorhpism diagram:

$$\begin{array}{ccc} E & \xrightarrow{\mu_c} & M_c \\ \downarrow\mu & \phi \nearrow \swarrow \phi^{-1} & \downarrow h_c \\ \mu(E) & \xrightarrow{h} & Z \end{array}$$

Proof Let $\phi : M \to \Delta^{L-1} \times Z$ be the function defined as Lemma 5. We first show that $\mu_c = \phi \cdot \mu$. Suppose by way of contradiction that $\mu_c(e) \neq \phi[\mu(e)]$ for some $e \in E^{cq}$. Let $(p, x, y) = \phi\ [\mu(e)]$. Then (x,y) is not a θ-Walrasian equilibrium allocation, i.e., $(x, y) \notin W_\theta(e)$. Then, for some i, we have $p \cdot x_i < \sum_{j=I}^{I+I} \theta_{ij} p \cdot y_j$. Let $w_i^* = (1/t)(w_i + x_i) - x_i$, $e_i^* = (w_i^*, a_i, b_i)$, let $e^* = (e_i^*, e^{-i})$, and let $g(t) = u_i[(1 - t)(w_i + x_i) + t(w_i + \gamma_i(e^*;\theta))] - u_i(w_i + x_i)$ where $0 < t < 1$. Then, $g(t) \to 0$ as $t \to 0$, and

$$\begin{aligned} dg/dt &= Du_i[(1-t)(w_i + x_i) + t(w_i + \gamma_i(e^*;\theta))](\gamma_i(e^*;\theta) - x_i) \\ &\to Du_i[w_i + x_i]\left(w_i \sum\nolimits_{j=I+1}^{I+I} \theta_{ij} p \cdot y_j p \cdot w_i - x_i\right) \\ &= \lambda p \cdot \left[w_i \sum\nolimits_{j=I+1}^{I+I} \theta_{ij} p \cdot y_j p \cdot w_i - x_i\right] = \lambda\left(\sum\nolimits_{j=I+1}^{I+I} \theta_{ij} p \cdot y_j - p \cdot x_i\right) > 0 \end{aligned}$$

for some $\lambda > 0$ as $t \to 0$ by noting that $Du_i[w_i + x_i]$ is proportional to p and $p \cdot x_i < \sum_{j=I+1}^{I=I} \theta_{ij} p \cdot y_j$. Thus we have $g(t) = u_i[(1-t)(w_i + x_i) + t\,(w_i + \gamma_i(e^*;\theta))] - u_i(w_i + x_i) > 0$, i.e., $u_i[(1 - t)(w_i + x_i) + t(w_i + \gamma_i(e^*;\theta))] > u_i(w_i + x_i)$ when t is a sufficiently small positive number by noting that $g(\cdot)$ is continuous and $g(t) \to 0$ as $t \to 0$. Then, multiplying $1/t$ on the both sides of $u_i[(1 - t)(w_i + x_i) + t\,(w_i + \gamma_i(e^*;\theta))] > u_i(w_i + x_i)$ and by the linear homogeneity of preferences represented by $u_i(\cdot)$, we have $u_i\ ((1/t)(w_i + x_i)) < u_i\ ((1/t)(w_i + x_i) - x_i + \gamma_i(e^*;\theta))$ when t is a sufficiently small positive number. Thus, since $\langle M, \mu, h\rangle$ is individually rational with respect to the fixed share guarantee structure $\gamma_i(e^*;\theta)$, we must have $\mu(e^*) \neq \mu(e)$ otherwise we have $u_i(w_i^* + x_i) = u_i((1/t)(w_i + x_i)) < u_i((1/t)(w_i + x_i) - x_i + \gamma_i(e^*;\theta)) = u_i(w_i^* + \gamma_i(e^*;\theta)$ that contradicts the hypothesis that $(x, y) = h[\mu(e^*)]$ is individually rational with respect to the fixed share guarantee structure $\gamma_i(e^*;\theta)$. However, on the other hand, (x, y) is Pareto optimal for e^*, and $Du_i(w_i^* + x_i)$ is proportional to p, Lemma 6 implies that $\mu(e^*) = \mu(e)$, a contradiction. So we must have $\mu_c = \phi \cdot \mu$. Furthermore, since h_c is the projection $(p, x, y) \to (x, y)$, it follows that $h_c \cdot \phi = h$.

Now we show that ϕ is a homeomorphism on $\mu(E^{cq})$ to M_c. By Lemma 6, ϕ is a one-to-one mapping. Also, since $h_c \cdot \phi = h$, the range of ϕ is M_c. So it only remains to show that ϕ and ϕ^{-1} are continuous. To show that θ^{-1} is continuous, let $\{m(k)\}$ be a sequence in M_c, which converges to some $m \in M_c$ with $m(k) = \mu(e(k))$ for all k and $m \in \mu(e)$. Since $\phi^{-1}\mu_c = \mu$, $\phi^{-1}(m(k)) = \mu(e(k))$ for all k and $\phi^{-1}(m) = \mu(e)$. Since μ is continuous, $\mu(e(k))$ converges to $\mu(e)$, so ϕ^{-1} is continuous. Since M_c and M are manifolds of the same dimension, ϕ^{-1} is a homeomorphism on M_c to $\phi^{-1}(M_c) = \mu(E^{cq})$ by Exercise 18.10 in Greenberg (1967, p. 82). □

Thus, the above theorem shows that the competitive mechanism is the unique informationally efficient process that realizes Pareto optimal and individually rational allocations over the class of production economies E^c. For a mechanism $\langle M, \mu, h\rangle$, when there does not any homeomorphism ϕ on $\mu(E^{cq})$ to M_c such that (1) $\mu_L = \phi \cdot \mu$ and (2) $h_L \cdot \phi = h$, we may call such a mechanism $\langle M, \mu, h\rangle$ a non-Walrasian allocation mechanism. The Uniqueness Theorem then implies that any non-Walrasian allocation mechanism defined on E^{cq} must use a larger message space.

Remark 5 The above Uniqueness Theorems, unlike the Efficiency Theorem, are based on the assumption of individual rationality. As in Jordan (1982), a similar example can be constructed to show that this assumption cannot be dispensed with for the competitive mechanism to be the unique informationally efficient and non-wasteful mechanism.

Remark 6 The Uniqueness Theorem obtained above is obtained for the class of Cobb–Douglas-Quadratic production economies. As in Jordan (1982), a similar example may be constructed to show that, for the Uniqueness Theorem, it is necessary to require the relation Cobb–Douglas-Quadratic environments and equilibrium messages be single-valued and continuous. The single-valuedness requirement cannot be extended much beyond the Cobb–Douglas-Quadratic class without conflicting with the multiplicity of Walrasian equilibrium. Nevertheless, similar to Jordan (1982), we can extend the above Uniqueness Theorem to more general classes of economic environments if we impose the additional regularity assumptions that the message space is connected and that the set of messages associated with Cobb–Douglas-Quadratic environments is a closed subset of the message space.

6 Conclusion

In this paper, it has been shown that the minimal informational size of message space for privacy preserving and non-wasteful resource allocation processes on the class of general convex production economies is the size of the Walrasian message space, and thus the competitive mechanism is informationally the most efficient decentralized mechanism among the class of mechanisms that realize Pareto efficient allocations. Furthermore, it is shown that the competitive mechanism is the unique informationally efficient and decentralized mechanism that realizes Pareto efficient and individually rational allocations over the class of Cobb–Douglas-Quadratic economies. This optimality and uniqueness result on the competitive mechanism for economies with production is a kind of impossibility theorem: it implies that there exists no other privacy preserving and non-wasteful resource mechanism under whatever ownership – state ownership, private ownership, or other types of ownership, that uses a message space whose informational size is smaller than, or the same as, that of the Walrasian message space. That is, this notable result implies that any economic system except for the competitive market system is not informationally efficient, and thus, as a policy implication, it tells us that one should use the market mechanism as long as it works.

Like most work in the realization literature, informational size we focused on is assumed to reflect information cost. However, this approach may overlook the role and cost of informational quality, complexity, and incomplete information transmission. As a result, an opaque economic environment, which makes very difficult the access to complex information, may be more cheaply served by a recourse allocation mechanism with larger but simpler messages, or messages that represent a finer partition of the set of states of nature. Also, the notion of a stationary message correspondence ignored the issues of its existence, stability, and convergence, but it simply is obtained through ad hoc assumptions. Jordan (1987) showed that, if the properties are considered, the size of the message space of a mechanism has to increase.

Also, like most of the realization literature, we have assumed that agents follow the rules of the mechanism without regard to self-interest. When the incentive aspect of the Walrasian mechanism is also taken into account, Reichelstein and Reiter (1988) have shown that a Nash implementation typically increases the size of the message space of the mechanism. Williams (1986) and Saijo (1988) have provided general lower bounds of the size of the message space required to Nash implement a social choice correspondence.

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