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Implementation of the Walrasian correspondence without continuous, convex, and ordered preferences

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Abstract. This paper consideres the problem of designing "better" mechanisms whose Nash allocations coincide with constrained Walrasian allocations for non-neoclassical economies under the minimal possible assumptions. We show that no assumprions on preferences are needed for feasible and continuous implementation of the constrained Walrasian correspondence. Further, under the monotonicity assumption, we present a mechanism that is completely feasible and continuous. Hence, no continuity and convexity assumptions on preferences are required, and preferences may be nontotal or nontransitive. Thus, this paper gives a somewhat positive answer to the question raised in the literature by showing that, even for non-neoclassical economies, there are "incentive-compatible", "privacy preserving", and "well-behaved" mechanisms which yield Pareto-efficient and individually rational allocations at Nash equilibria.

1. Introduction

In the early literature (e.g., Bergson [2], Lange [17], Debreu [3]), much of the traditional welfare economics takes an economic mechanism (e.g., the competitive mechanism or monopoly mechanism) as given and investigates the properties (e.g. Pareto-efficiency) of its performance correspondence. Subsequently, the reverse problem has come under consideration: instead of regarding the mechanisms as given and seeking a class of environments for which they work, one seeks mechanisms that will implement some social goals (social choice correspondences) for a given class of environments without destroying participants' incentives, and these mechanisms will have some desirable properties. In particular, for a set of "non-neoclassical" economic environments, such as those with non-convexities,

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discontinuities, and increasing returns to scale, etc., where the competitive mechanism is known to fail[1], is there a mechanism which yields Pareto-efficient and individually rational allocations, and if so, does this mechanism have some desirable properties? These questions result in the mechanism design theory originated by the seminal papers of the modern era by Hurwicz [6, 7] (also see Hurwicz [10,p 2; 1986a, p 1443]). However, until Hurwicz [12] generalized Maskin's theorem on Nash-implementation of social choice correspondences to nontotal-nontransitive preferences, all the studies had ignored implementation of social choice correspondences for non-neoclassical economic environments and only considered implementation of social choice correspondences for the neoclassical economic environments in which preferences are explicitly or implicitly assumed to be transitive, total[2], convex, continuous, and monotonically increasing. Besides, the mechanisms which Nash-implement the (constrained) Walrasian correspondence[3] in the literature are unsatisfactory because they do not guarantee complete feasibility (i.e., individual feasibility and balancedness) and/or continuity.

For the neoclassical exchange economies, there have been a number of mechanisms which Nash-implement the (constrained) Walrasian correspondence. Schmeidler [27] proposed a balanced mechanism whose Nash equilibrium allocations are precisely Walrasian allocations. This means that the mechanism fully Nash-implements the Walrasian correspondence. But, his mechanism has two properties one may consider undesirable. One is that his outcome function is not individually feasible: out of equilibria, some outcome allocations may not be in the consumption set; although in equilibrium, they are necessarily in the consumption set. The other is that the outcome function is not continuous: small changes in an agent's strategy choice may lead to large jumps in the resulting allocations. Hurwicz [8] gave a balanced and smooth mechanism whose Nash allocations coincide with Walrasian allocations. But his mechanism is not individually feasible, either. In fact, there does not exist any completely feasible mechanism which Nash-implements the Walrasian correspondence because the correspondence violates Maskin's [20] monotonicity condition when boundary equilibrium allocations occur. However, it is possible to design a completely feasible mechanism whose Nash allocations coincide with a slightly larger set than Walrasian allocations, namely, constrained Walrasian allocations which are Pareto-efficient and individually rational[4].

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[1] Even for the neoclassical economic environments, the competitive (Walrasian) mechanism may fail when the number of agents in an economy is small. This is because, in this case, agents are not likely to be "price takers". Consequently, the efficiency of outcomes may not follow from the competitive mechanism. Thus, the Walrasian mechanism fails to yield Pareto-efficient allocations in the case of a small number of agents. In fact, this is one of the reasons why, even for the neoclassical economies, economists want to seek some alternative mechanisms which yield Pareto-efficient allocations.

[2] A preference relation R is said to be total if for any x and y, $x \neq y$ implies xRy or yRx.

[3] It is important to distinguish the competitive mechanism (the Walrasian mechanism) from the Walrasian correspondence (cf. Hurwicz [13, p 284]). The former is a market economic mechanism, but the latter is a performance correspondence which consists of allocations which can be supported (obtained) by the competitive mechanism and, in fact, can be implemented by non-Walrasian mechanisms, e.g., by the mechanisms presented in the paper.

[4] Note that even though Walrasian equilibria do not exist (so that the competitive mechanism fails) for some non-classical economic environments, the constrained Walrasian equilibria may still exist (cf. Example 1 below).

Economic motivation behind consideration of the constrained Walrasian correspondence is, of course, that considering implementation of the constrained Walrasian correspondence does not lose much generality, compared to implementing any other social choice correspondence which guarantees Pareto-efficiency and individual rationality. Indeed, as Hurwicz et al. (1984) pointed out, a slightly modified varsion of Theorem 1 and 2 of Hurwicz [9] states that for any mechanism, if all of its Nash allocations for a given environment are Pareto-efficient and individually rational, then every constrained Walrasian allocation is a Nash allocation, and every interior Nash allocation is a constrained Walrasian allocation. Thus, considering implementation of the constrained Walrasian correspondence at most loses some possible boundary Pareto-efficient and individually rational allocations.

Hurwicz et al. [14] then proposed a completely feasible mechanism that Nash-implements the constrained Walrasian correspondence. Their mechanism, however, is discontinuous. Postlewaite and Wettstein [25] gave a continuous and feasible (i.e., individually feasible and weakly balanced) mechanism which Nash-implements the constrained Walrasian correspondence. Their mechanism, however, is not balanced and the continuity of utility functions is a critical assumption for their mechanism. Balancedness is highly desirable if we are to seriously consider accepting the mechanisms. For the motivation behind designing balanced mechanisms, see Tian [34, 36].

A similar situation prevailed with regard to Nash-implementation of the Lindahl correspondence for the neoclassical public goods economies (see, e.g., mechanisms in Groves and Ledyard [4], Hurwicz [8], Walker [40], Groves and Ledyard [5], Tian [33]) until Tian [34, 36] and Tian and Li [38] recently gave completely feasible and continuous mechanisms which fully Nash-implement the Lindahl correspondence, and further have message spaces of minimal dimension.

Thus all mechanisms mentioned above only work with the neoclassical preferences which guarantee the existence of Walrasian allocations. There are no answers to the questions (which were stated in the beginning of the paper) whether or not there exist mechanisms which yield Pareto and individually rational allocations at Nash equilibria for the non-neoclassical economies, and if so, whether or not they are continuous and feasible or further if they are continuous, individually feasible, and balanced (not merely weakly balanced). This paper gives somewhat positive answers to these questions by implementing the constrained Walrasian correspondence. It will be noted below that only monotonicity of preferences is assumed in our completely feasible and continuous mechanism and thus no continuity and convexity assumptions on preferences are needed, and further, preferences may be nontotal or nontransitive. In fact, the only assumption, monotonicity, we impose on preferences cannot be dispensed with for balanced implementation (see Remark 7 below). Moreover, the monotonicity assumption can be dropped if we only want to have a weakly balanced, individually feasible, and continuous mechanism. Thus we can have a feasible and continuous mechanism whose Nash allocations coincide with the constrained Walrasian correspondence without imposing any assumptions on preferences. These results show that, even for non-neoclassical economies, there are "incentive-compatible", "privacy preserving", and "well-behaved" mechanisms whose Nash allocations, provided they exist, yield Pareto-efficient and individually rational allocations. These also show that no assumptions on preferences are needed for implementation of the constrained Walrasian correspondence for the class of economies on which it is nonempty-valued. Thus, our mechanisms work at least for the following two cases where the competitive mechanism may fail. One is the case where the number of agents is small. The second case is that constrained Walrasian equilibria exist even though Walrasian equilibria may not exist. So our results add some new dimensions to our understanding of the mechanism design theory.

As can be seen in many cases, preferences of an agent are nontransitive-nontotal. For instance, as Hurwicz [12] pointed out, this is the case when the society whose goals are to be implemented consists of groups whose choices are defined by voting procedures. Also, Mosteller and Nogee [22] noted that in experiments to test transitivity of preferences, one could always find instances in which this postulate was violated. This enables economists to study behavior of agents with nontotal-nontransitive preferences. Many studies in the literature show the existence of maximum elements of preferences and the existence of Walrasian equilibrium such as those in Schmeidler [26], Sonnenschein [29], Mas-Colell [18], Shafer and Sonnenschein [28], and Tian [37], among others.

The plan of this paper is as follows. Section 2 gives notation and definitions used in the mechanism design literature. Section 3 presents a mechanism which is completely feasible and continuous and shows equivalence of Nash allocations of the mechanism and the constrained Walrasian allocations when monotonicity of preferences is assumed. Section 4 drops the monotonicity assumption and gives a feasible and continuous mechanism which implements the constrained Walrasian correspondence without imposing any conditions on preferences. Finally, some concluding remarks on possible extensions of our results are given in Sect. 5.

2. Notation and definitions

2.1. Economic environments

In the economy under consideration, there are n ($n \geqq 3$ agents[5]) who consume L private goods. Denote by $N=\{1,2,\ldots,n\}$ the set of agents. Each agent's characteristic is denoted by $e_i=(w_i, P_i)$, where $w_i \in \mathbb{R}^L_+ \setminus \{0\}$ is the initial endowment of the agent and P_i is the strict (irreflexive) preference defined on $\mathbb{R}^L_+$ which may be nontotal or nontransitive[6].

An economy is the full vector $e=(e_1,\ldots,e_n)$ and the set of all such economies is denoted by E.

[5] This is a necessary condition for the balanced and continuous implementation. Kwan and Nakamura [16] proved that there are no balanced and continuous mechanisms which implement the (constrained) Walrasian correspondence for two-agent economies.

[6] If we define the binary relation P_i^* by $a\ P_i^* b$ if and only if $\neg b P_i a$ where $\neg$ stands for "it is not the case that", then P_i^* is the weak (reflexive) preference and is called the 'canonical conjugate' of P_i (see Kim and Richter [15]). If concepts used in this paper such as Nash equilibrium and the constrained Walrasian allocations are interpreted in terms of the $P^*{}_i$, then the results obtained in this paper for P_i are, inparticular, valid for the P_i^*.

2.2 The constrained Walrasian allocations

An allocation $x^*=(x_1^*, x_2^*, \ldots, x_n^*) \in \mathbb{R}_+^{nL}$ is a *constrained Walrasian allocation* for an economy e if there is a price vector $p^* \in \mathbb{R}_+^L$ such that

(1) $p^* \cdot x_i^* \leqq p^* \cdot w_i$ for all $i \in N$,

(2) for all $i \in N$, there does not exist $x_i \in \mathbb{R}_+^L$ such that

(2.a) $x_i P_i x_i^*$;

(2.b) $p^* \cdot x_i \leqq p^* \cdot w_i$;

(2.c) $x_i \leqq \sum_{j=1}^{n} w_j$,

(3) $\sum_{j=1}^{n} x_j \leqq \sum_{j=1}^{n} w_j$.

Denote by $W_c(e)$ the set of all such allocations.

Remark 1. From the above definition, we can see that every ordinary Walrasian allocation (competitive equilibrium allocation) is a constrained Walrasian allocation and that a constrained Walrasian allocation differs from a Walrasian allocation only in the way that each agent maximizes his preferences not only subject to his budget constraint but also subject to total endowments available to the economy.

Remark 2. Note that even though Walrasian equilibria do not exist (so that the competitive mechanism fails) for some non-classical economic environments, the constrained Walrasian equilibria may still exist. The following example shows that the set of constrained Walrasian allocations can be nonempty but the set of Walrasian allocations is empty for some economies in which preferences of agents are discontinuous and non-convex.

Example 1. Consider an exchange economy e with two goods and two agents whose endowments are given by $w_1=w_2=(1, 1)$. The utility function of agent 1 is given by $u_1(x_{11}, x_{12})=x_{11}$ when $x_{11} \neq 2$ and $u_1(x_{11}, x_{12})=0$ when $x_{11}=2$. The utility function of agent 2 is given by $u_2(x_{21}, x_{22})=x_{22}$. One can see that this economy does not have a competitive equilibrium but the allocation $x=(x_1, x_2)$ with $x_1=(1, 0)$, $x_2=(1, 2)$ is a constrained Walrasian allocation with associated price vector $p^*=(1, 0)$.

An allocation x is *Pareto-efficient* with respect to strict preference profile $P=(P_1, \ldots, P_n)$ if it is feasible $\left(\text{i.e., } x \in \mathbb{R}_+^{nL} \text{ and } \sum_{j=1}^{n} x_j \leqq \sum_{j=1}^{n} w_j\right)$ and there does not exist another feasible allocation x' such that $x_i' P_i x_i$ for all $i \in N$.

An allocation x is *individually rational* with respect to P if $\neg w_i P_i x_i$ for all $i \in N$[7].

[7] This definition coincides with the conventional definition when P_i is the asymmetric part of a reflexive, transitive, and total preference R_i.

It can be easily shown that every constrained Walrasian allocation is Pareto-efficient and individually rational[8].

2.3. Mechanism

Let F be a social choice rule, i.e., a correspondence from E to the set Z of resource allocations. In the rest of the paper, we will use the constrained Walrasian correspondence as the social choice rule.

Let M_i denote the i-th message (strategy) domain. Its element are written as m_i and called messages. Let $M=\prod_{i=1}^{n} M_i$ denote the message (strategy) space. Let $X: M \rightarrow Z$ denote the outcome function, or more explicitly, $X_i(m)$ is the i-th agent's outcome at m. A mechanism consists of $\langle M, X\rangle$ defined on E. A message $m^*=(m_1^*, \ldots, m_n^*) \in M$ is a *Nash equilibrium* (NE) of the mechanism $\langle M, X\rangle$ for an economy e if for any $i \in N$ and for all $m_i \in M_i$,

$$\neg X_i(m^*/m_i, i)\, P_i\, X_i(m^*) \;, \tag{1}$$

where $(m^*/m_i, i)=(m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. The outcome $X(m^*)$ is then called a *Nash (equilibrium) allocation.* Denote by $V_{M,X}(e)$ the set of all such Nash equilibria and by $N_{M,X}(e)$ the set of all such Nash (equilibrium) allocations.

Remark 3. The constrained Walrasian allocations and Nash allocations without ordered preferences which are defined above may not be the same as the usual ones. So Hurwicz [12] called Nash equilibrium defined by (1) as generalized Nash equilibrium. However, when preferences are transitive and total, Nash equilibrium and constrained Walrasian allocations defined above reduce to the conventional ones.

A mechanism $\langle M, X\rangle$ fully *Nash-implements* the social choice correspondence F on E if, for all $e \in E$, $N_{M,X}(e)=F(e)$.

Remark 4. Note that the above definition which was due to Hurwicz [8, p 219] allows the social choice correspondence W_c and the set Nash equilibria to be empty for the main purpose of this paper is to study the equivalence of the constrained Walrasian correspondence and the set of Nash equilibrium allocations under the minimal possible assumptions[9]. A stronger definition of full Nash-implementation used in the literature is that not only $N_{M,X}(e)=F(e)$ but also $N_{M,X}(e) \neq \emptyset$ for all $e \in E$. Thus, if we restrict the domain of W_c to the one on which W_c is nonempty-valued, our results, to be presented below, will be equivalent for both definitions and show that no assumptions on preferences are needed for feasible and continuous implementation of the constrained Walrasian correspondence for the class of economies on which it is nonempty-valued.

A mechanism $\langle M, X\rangle$ is *individually feasible* if $X(m) \in \mathbb{R}_+^{nL}$ for all $m \in M$.

[8] For weak preferences, Thomson [30] showed that a constrained Walrasian allocation may not be (regular) Pareto-efficient (i.e., there is no way of making everyone at least well off and one person better off) even if preferences satisfy local non-satiation. However, when preferences satisfy strict monotonicity, it is (regular) Pareto-efficient by Theorem 2.iv of Tian [32].

[9] Of course, if we impose more assumptions on preferences, by using the results such as in Mas-Colell [18], Shafer and Sonnenschein [28], Tian [35], and Tian and Zhou [39], we can prove the existence of constrained Walrasian equilibria.

A mechanism $\langle M, X\rangle$ is *weakly balanced* if for all $m \in M$

$$\sum_{j=1}^{n} X_j(m) \leqq \sum_{j=1}^{n} w_j \ . \tag{2}$$

A mechanism $\langle M, X\rangle$ is *balanced* if (2) holds with equality for all $m \in M$.

A mechanism $\langle M, X\rangle$ is *completely feasible* (or *feasible*) if it is individually feasible and balanced (or weakly balanced).

Sometimes we say that an outcome function is individually feasible, balanced, or continuous if the mechanism is individually feasible, balanced, or continuous.

3. Completely feasible and continuous implementation

In this Section, we present a simple mechanism which is completely feasible and continuous and fully Nash-implements the constrained Walrasian correspondence for a class of economies $\tilde{E} \subset E$ satisfying monotonicity of preferences. In the following section, we will consider feasible and continuous implementation on E without assuming monotonicity of preferences.

The mechanism can be simply described as follows. The designer first asks each agent to report proposed prices for other agents and proposed contributions that the agent is willing to give to each person including himself. Based on this information, the designer determines prices of goods according to agent's proposed prices. Then a completely feasible choice correspondence B is defined in such a way that for all $m \in M$, allocations in $B(m)$ are completely feasible and satisfy the budget constraints with equality for all agents. The allocation outcome $X(m)$ will be chosen from the completely feasible set $B(m)$ so that it is the closest to the sum of the contributions that each agent is willing to pay.

We now turn to the formal construction of the mechanism. For each $i \in N$, let the message domain of agent i be of the form[10]

$$M_i = \mathbb{R}^L_{++} \times \mathbb{R}^{nL} \ . \tag{3}$$

A generic element of M_i is $m_i = (p_i, x_{i1}, \ldots, x_{in})$, where p_i is interpreted as the price vector proposed by agent i for other agents and x_{ij} is interpreted as the contribution that agent i is willing to make to agent j (a negative x_{ij} means agent i wants to get $-x_{ij}$ amount of goods from agent j).

Define the price vector $p: M \to \mathbb{R}^L_{++}$ by

$$p(m) = \begin{cases} \sum_{i=1}^{n} \frac{a_i}{a} p_i & \text{if } a > 0 \\ \sum_{i=1}^{n} \frac{1}{n} p_i & \text{if } a = 0 \end{cases} , \tag{4}$$

where $a_i = \sum_{j,k \neq i} \|p_j - p_k\|$, and $a = \sum_{i=1}^{n} a_i$, and $\|\cdot\|$ is the Euclidian norm. Then $p(m)$ is continuous on M[11].

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[10] The dimension of the message domain can be reduced by one by normalizing the price of some good to one so that $M_i = \mathbb{R}^{L-1}_{++} \times \mathbb{R}^{nL}$.

[11] $p(m)$ is clearly continuous at m with $a > 0$. When $a = 0$, $p_1 = \ldots = p_n = p$ and thus, for any $\varepsilon > 0$, $\|p(m') - p(m)\| = \left\| \sum_{i=1}^{n} \frac{a_i'}{a'} p_i' - p \right\| = \left\| \sum_{i=1}^{n} \frac{a_i'}{a'} (p_i' - p) \right\| \leqq \sum_{i=1}^{n} \|p_i' - p\| < \varepsilon$ as long as p_i' is sufficiently close to p for all $i \in N$. Hence, $p(m)$ is also continuous at m with $a = 0$.

Note that though $p(\cdot)$ is a function of proposed prices only, for simpli we can write $p(\cdot)$ as a function of m without loss of generality.

Define a completely feasible correspondence $B: M \to\to \mathbb{R}^{nl}_{+}$ by

$$B(m) = \left\{x \in \mathbb{R}^{nl}_{+}: \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} w_i \;\&\; p(m)\cdot x_i = p(m)\cdot w_i \text{ for all } i \in N\right\},$$

which is a continuous correspondence with nonempty, compact, and co values (cf. Tian [31]).

Let $\tilde{x}_i = \sum_{j=1}^{n} x_{ji}$ which is the sum of contributions that agents are willin make to agent i and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$.

The outcome function $X: M \to \mathbb{R}^{nl}_{+}$ is given by

$$X(m) = \left\{y \in \mathbb{R}^{nl}_{+}: \min_{y \in B(m)} \|y - \tilde{x}\|\right\},$$

which is the closet to $\tilde{x}$. Then X is single-valued and continuous on M[12]. A since $X(m) \in \mathbb{R}^{nl}_{+}$ and

$$\sum_{i=1}^{n} X_i(m) = \sum_{i=1}^{n} w_i$$

for all $m \in M$, the mechanism is completely feasible and continuous.

Remark 5. Note that the above mechanism is different from the one give Postlewaite and Wettstein [25] in two ways: *(1)* Our mechanism is bala while their mechanism is merely weakly balanced. *(2)* Each individual in mechanism announces amounts to a whole allocation rather than just his demand. This enables us to get continuous implementation without assum continuous preferences while continuity of preferences is a critical assump for their mechanism.

Now we show equivalence between Nash equilibrium allocations and strained Walrasian allocations. Proposition 1 shows that every Nash equilibr allocation is a constrained Walrasian allocation. Proposition 2 proves that e constrained Walrasian allocation is a Nash equilibrium allocation.

Proposition 1. *If the mechanism $\langle M, X\rangle$ defined above has a Nash equilibrium for $e \in \bar{E}$, then $X(m^*)$ is a constrained Walrasian allocation with $p(m^*)$ as price vector, i.e., $N_{M,X}(e) \subset \bar{W}_c(e)$ for all $e \in \bar{E}$.*

Proof. Let m^* be a Nash equilibrium. Then $X(m^*)$ is a Nash equilibrium a cation. We wish to show that $X(m^*)$ is a constrained Walrasian allocation. the defintion of the mechanism, we know that $p(m^*) \in \mathbb{R}^{l}_{+}$, $X(m^*) \in \mathbb{R}$ $\sum_{i=1}^{n} X_i(m^*) = \sum_{i=1}^{n} w_i$, and $p(m^*)\cdot X_i(m^*) = p(m^*)\cdot w_i$ for all $i \in N$. So we need to show that each individual is maximizing his/her preferences. Supp by way of contradiction, that for some agent, say, agent 1, there exists s $\tilde{x}_1 \in \mathbb{R}^{l}_{+}$ such that $\tilde{x}_1 \leqq \sum_{i=1}^{n} w_i$, $p(m^*)\cdot \tilde{x}_1 \leqq p(m^*)\cdot w_1$, and $\tilde{x}_1 P_1 X_1(m^*)$.

[12] This is because X is an upper semi-continuous correspondence by Berge's Maximum Thec (see Debreu [3, p 19]) and single-valued (see Mas-Colell [19, p 28]).

cause of monotonicity of preferences, it will be enough to confine ourselves to the case of $p(m^*)\cdot\tilde{x}_1 = p(m^*)\cdot w_1$:

Define $\tilde{x}_i\,(i=2,\ldots,n)$ as follows:

$$\tilde{x}_i = \frac{p(m^*)\cdot w_i}{\sum_{j=i}^{n} p(m^*)\cdot w_j}\left[\sum_{j=1}^{n} w_j - \sum_{j=1}^{i-1}\tilde{x}_j\right] . \tag{8}$$

Since $\tilde{x}_1 \leqq \sum_{j=1}^{n} w_j$, it can be easily verified, by reduction, that

$$\sum_{j=1}^{n} w_j - \sum_{j=1}^{i-1}\tilde{x}_j \geqq 0$$

for $i=2,\ldots,n$ and thus $\tilde{x}_k \in \mathbb{R}^L_+$ and

$$p(m^*)\cdot\tilde{x}_k = p(m^*)\cdot w_k$$

for all $k=2,\ldots,n$. Also, by letting $i=n$ in (8), we have

$$\sum_{j=1}^{n}\tilde{x}_j = \sum_{j=1} w_j .$$

Now suppose that agent 1 chooses $x_{1j} = \tilde{x}_j - \sum_{k\neq 1} x^*_{kj}$ for all $j\in N$ and keeps p_1^* unchanged, i.e., $m_1 = (p_1^*, x_{11},\ldots,x_{1n})$. Then $\tilde{x} = (\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n)\in B(m^*/m_1,1)$ and $\tilde{x}_j = x_{1j} + \sum_{k\neq 1} x^*_{kj}$ for all $j\in N$. Hence, we have $X_j(m^*/m_1,1) = \tilde{x}_j$. Therefore, from $\tilde{x}_1 P_1 X_1(m^*)$, we have $X_1(m^*/m_1,1)\,P_1\,X_1(m^*)$. This contradicts the fact that m^* is a Nash equilibrium. So $X(m^*)$ is a constrained Walrasian allocation. Q.E.D.

Proposition 2. *If $x^* = (x_1^*, x_2^*,\ldots,x_n^*)$ is a constrained Walrasian allocation with a price vector $p^*\in\mathbb{R}^L_+$ for $e\in\bar{E}$, then there exists a Nash equilibrium m^* of the mechanism $\langle M, X\rangle$ defined above such that $X_i(m^*) = x_i^*$, $p(m^*) = p^*$, for all $i\in N$, i.e., $W_c(e)\subset N_{M,X}(e)$ for all $e\in\bar{E}$.*

Proof. Since preferences satisfy the monotonicity condition and x^* is a constrained Walrasian allocation, we must have $p^*\in\mathbb{R}^L_{++}$, $\sum_{j=1}^{n} x_j^* = \sum_{j=1}^{n} w_j$ and $p^*\cdot x_i^* = p^*\cdot w_i$ for $i\in N$. Now for each $i\in N$, let $m_i^* = (p^*, x_{i1}^*,\ldots,x_{in}^*)$, where $x_{ii}^* = x_i^*$ and $x_{ij}^* = 0$ for $j\neq i$.

Then x^* is an outcome with p^* as a price vector, i.e., $X_i(m^*) = x_i^*$ for all $i\in N$, and $p(m^*) = p^*$. We show that m^* yields this allocation as a Nash allocation. In fact, agent i cannot change $p(m^*)$ by changing his proposed price since changing p_i yields $a>0$ and $a_i = 0$ so that the new p_i cannot change $p(m^*)$ (i.e., $p(m^*/m_i, i) = p(m^*)$ for all $m_i\in M_i$). Announcing a different message m_i by agent i may yield an allocation $X(m^*/m_i, i)$ such that $X_i(m^*/m_i, i)\in\mathbb{R}^L_+$ and

$$p(m^*)\cdot X_i(m^*/m_i, i) = p(m^*)\cdot w_i . \tag{9}$$

Now suppose, by way of contradiction, that m^* is not a Nash equilibrium. Then

there are $i \in N$ and m_i such that $X_i(m^*/m_i, i) P_i X_i(m^*)$. Since $X_i(m^*/\imath$
$\leq \sum_{i=1}^{n} w_i$, we must have, by the definition of the constrained Walrasian all
tion, $p(m^*) \cdot X_i(m^*/m_i, i) > p(m^*) \cdot w_i$. But this contradicts the budget
straint (9). Thus we have shown that agent i cannot improve his/her utilit
changing his/her own message while the others' messages remain fixed fo
$i \in N$. Hence x^* is a Nash allocation. Q.E.D.

Remark 6. It may be remarked that when we prove that every Nash equilib
allocation is a constrained Walrasian allocation in Proposition 1, we do
require that all participants' price messages be the same at a Nash equilibr
and thus even at a Nash equilibrium a participant can change the outcome
changing his/her own price messages. However, when we prove that every
strained Walrasian allocation is a Nash equilibrium allocation in Propositi
the Nash equilibrium is chosen in a way that all participants' price message
the same so that any participant cannot change the prevailing price vecto
changing his/her own messages (so that every participant takes prices as gi

Summarizing the above discussions, we have the following theorem.

Theorem 1. *For the economies in $\bar{E}$, there exists a completely feasible and contin mechanism which fully Nash-implements the constrained Walrasian correspond on $\bar{E}$.*

Since the constrained Walrasian allocations are Pareto-efficient and ind
ually rational, we then have the following corollary.

Corollary 1. *For the economies in $\bar{E}$, these exists a completely feasible and co uous mechanism whose Nash equilibria, provided they exists, yield Pareto-effi and individually rational allocations.*

Remark 7. Note that the monotonicity assumption we only imposed in this se
cannot be dispensed with for balanced implementation. Indeed, without
assumption, the constrained Walrasian allocations can be weakly balanced
case of free-disposal), and consequently cannot be attained with the bala
mechanisms.

4. Feasible and continuous implementation

In the previous section, we needed to assume that preferences of agent
monotone increasing in order to obtain a completely feasible and contic
mechanism. This assumption is indispensable if one wants to avoid free-disp
This assumption, however, can be dropped if we allow for free-disposal and
want to have a mechanism which is merely feasible (individually feasible
weakly balanced) and continuous and Nash-implements the constrained
rasian correspondence. This section gives such a mechanism which is very si
to the presented in the previous section. We briefly describe it as follows.

The message space is given by

$$M_i = \mathbb{R}^L_+ \times \mathbb{R}^{nL} .$$

The price vector is the same as before and the completely feasible correspondence is modified to the feasible correspondence $B' : M \to\to \mathbb{R}^{nL}_{+}$ defined by

$$B'(m) = \left\{ x \in \mathbb{R}^{nL}_{+} : \sum_{i=1}^{n} x_i \leqq \sum_{i=1}^{n} w_i \,\&\, p(m) \cdot x_i \leqq p(m) \cdot w_i \text{ for all } i \in N \right\} , \quad (11)$$

which is a continuous correspondence with nonempty, compact, and convex values. The outcome function for goods is similarly defined, i.e., $X(m)$ will be chosen from the feasible set $B'(m)$ so that it is the closest to the sum of the contributions that each agent is willing to pay. Thus the mechanism is feasible and continuous.

The proof of equivalence of Nash allocations of the modified mechanism and the constrained Walrasian correspondence remains the same except for replacing "equality" by "inequality" and defining $\tilde{x}_i$ in (8) by $\tilde{x}_i = 0$ for $i = 2, \ldots, n$.

Thus we have the following result.

Theorem 2. *For the class of economies E, there exists a feasible and continuous mechanism which fully Nash-implements the constrained Walrasian correspondence.*

Similarly, we have

Corollary 2. *For the class of economies E, there exists a feasible and continuous mechanism whose Nash equilibria, provided they exist, yield Pareto-efficient and individually rational allocations.*

5. Concluding remarks

In Sect. 3 we gave a simple mechanism which is completely feasible and continuous and fully Nash-implements the constrained Walrasian correspondence for economies without total, transitive, continuous, and convex preferences. We then show in Sect. 4 that if the society allows for free-disposal, monotonicity of preferences can be dropped and gives a mechanism which is feasible and continuous and implements the correspondence. Thus no assumption on preferences are needed for feasible and continuous implementation, and therefore any constrained Walrasian allocation can be attained by our mechanism. In the following, we want to mention some possible extensions of our mechanism.

First, in the mechanisms presented above, the initial endowments are assumed to be known to the designer. However, the mechanism in Sect. 3 can be extended to the case where the initial endowments are private information. Such a mechanism can be obtained by modifying the message space to

$$M_i = (0, w_i] \times \mathbb{R}^{L}_{++} \times \mathbb{R}^{nL} . \quad (12)$$

Here w_i is assumed to be a strictly positive vector. A generic element of M_i is $(v_i, p_i, x_{i1}, \ldots, x_{in})$, where v_i denotes a profession of agent i's endowment, the inequality $0 < v_i < w_i$ means that the agent can understate but not overstate his own endowment, the claimed endowment v_i (like the true endowment w_i) must be positive. Interpretations of other components are the same as before. The completely feasible correspondence is modified to the feasible correspondence $B'' : M \to\to \mathbb{R}^{nL}_{+}$ defined by

$$B''(m) = \left\{ x \in \mathbb{R}^{nL}_{+} : \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} v_i \,\&\, p(m) \cdot x_i = p(m) \cdot v_i \text{ for all } i \in N \right\} , \quad (13)$$

there are $i \in N$ and m_i such that $X_i(m^*/m_i, i)\, P_i X_i(m^*)$. Since $X_i(m^*/m_i, i) \leqq \sum_{i=1}^{n} w_i$, we must have, by the definition of the constrained Walrasian allocation, $p(m^*) \cdot X_i(m^*/m_i, i) > p(m^*) \cdot w_j$. But this contradicts the budget constraint (9). Thus we have shown that agent i cannot improve his/her utility by changing his/her own message while the others' messages remain fixed for all $i \in N$. Hence x^* is a Nash allocation. Q.E.D.

Remark 6. It may be remarked that when we prove that every Nash equilibrium allocation is a constrained Walrasian allocation in Proposition 1, we do not require that all participants' price messages be the same at a Nash equilibrium, and thus even at a Nash equilibrium a participant can change the outcomes by changing his/her own price messages. However, when we prove that every constrained Walrasian allocation is a Nash equilibrium allocation in Proposition 2, the Nash equilibrium is chosen in a way that all participants' price messages are the same so that any participant cannot change the prevailing price vector by changing his/her own messages (so that every participant takes prices as given).

Summarizing the above discussions, we have the following theorem.

Theorem 1. *For the economies in $\tilde{E}$, there exists a completely feasible and continuous mechanism which fully Nash-implements the constrained Walrasian correspondence on $\tilde{E}$.*

Since the constrained Walrasian allocations are Pareto-efficient and individually rational, we then have the following corollary.

Corollary 1. *For the economies in $\tilde{E}$, these exists a completely feasible and continuous mechanism whose Nash equilibria, provided they exists, yield Pareto-efficient and individually rational allocations.*

Remark 7. Note that the monotonicity assumption we only imposed in this section cannot be dispensed with for balanced implementation. Indeed, without this assumption, the constrained Walrasian allocations can be weakly balanced (the case of free-disposal), and consequently cannot be attained with the balanced mechanisms.

4. Feasible and continuous implementation

In the previous section, we needed to assume that preferences of agents are monotone increasing in order to obtain a completely feasible and conticuous mechanism. This assumption is indispensable if one wants to avoid free-disposal. This assumption, however, can be dropped if we allow for free-disposal and only want to have a mechanism which is merely feasible (individually feasible and weakly balanced) and continuous and Nash-implements the constrained Walrasian correspondence. This section gives such a mechanism which is very similar to the presented in the previous section. We briefly describe it as follows.

The message space is given by

$$M_i = \mathbb{R}^L_+ \times \mathbb{R}^{nL} \,. \tag{10}$$

which is a continuous correspondence with nonempty, compact, and convex values. The outcome $X(m)$ will be chosen from the feasible set $B''(m)$ so that it is the closest to the sum of the contributions that each agent is willing to pay. Thus, the mechanisms is feasible and continuous.

Under the assumption of monotonicity of preferences, we can easily show that, at Nash equilibrium, $v_i^* = w_i$ for all $i \in N$. Then the proof of equivalence of Nash allocations of the modified mechanism and the constrained Walrasian correspondence is the same as in Sect. 3. Thus, the mechanism simplifies the mechanism proposed by Postlewaite and Wettstein [25] and only monotonicity of preferences is assumed so that our mechanism works even for non-neoclassical economic environments.

Secondly, the dimension of the message space of the mechanism in this paper, even though it is finite, is higher than that of the mechanism proposed by Postlewaite and Wettstein [24]. So far we do not know whether or not there exists a completely feasible and continuous mechanism which implements the constrained Walrasian correspondence (if so – for what categories of economic environments) and has a message space of lower dimension.

Finally, though this paper only considers implementation of the constrained Walrasian correspondence by using the concept of Nash equilibrium to describe self-interested behavior of individuals, we think some of the techniques developed in the paper can be applied to implement the constrained Walrasian correspondence using other solution concepts such as those of subgame perfect equilibrium and undominated Nash equilibrium. It is well-known that the Nash equilibrium approach may have a problem in the case of multiple equilibria. Some Nash equilibria may be more believable than others. Because of this, some equilibrium concepts may need to be used. Moore and Repullo [21] and Abreu and Sen [1] use subgame perfect equilibrium as the solution concept and gave conditions for subgame perfect implementation of social choice correspondences. Palfrey and Srivastava [23] use undominated Nash equilibrium as the solution concept. This concept, like subgame perfect equilibrium, is a refinement of Nash equilibrium. For our mechanisms, however, the multiple equilibrium problem may not be a serious problem even though these mechanisms have many Nash equilibria for each constrained Walrasian equilibrium. This is because these Nash equilibria correspond to only one outcome when constrained Walrasian equilibrium is unique and thus there is no strong reason to believe that a Nash equilibrium is preferred to others.

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On the transfer and advantageous realloc paradoxes*

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Abstract. It is well known that the transfer and advantage adoxes cannot occur at a Walrasian stable equilibrium in a In an influential recent paper Chichilnisky provided an exa paradox in the context of a globally Walrasian stable three evident, though, that the paradoxes depend on the underlyi erences and net export positions of the economy. This pape generalizes the transfer and advantageous reallocation parad sufficient conditions on the underlying data of a globally Wa omy for the occurence of each of these paradoxes are establ of the third agent in generating these paradoxes is clarified.

1. Introduction

It is clear that in a world in which prices do not change, a the recipient's endowment will raise its welfare (and lower t however, prices can vary and, in fact, change so as to lo recipient's new endowment, then the recipient is actually m gift. This is a form of the transfer paradox[1].

If the gift reduces the welfare of both the recipient and to it as the weak transfer paradox (weak because only reduction is paradoxical). When the gift benefits the donor it will be termed the strong transfer paradox.

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[1] This definition follows Donsimoni and Polemarchakis

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