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Bayesian implementation in exchange economies with state dependent preferences and feasible sets

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Abstract. This paper considers Bayesian and Nash implementation in exchange economic environments with state dependent preferences and feasible sets. We fully characterize Bayesian implementability for both diffuse and non-diffuse information structures. We show that, in exchange economic environments with three or more individuals, a social choice set is Bayesian implementable if and only if closure, non-confiscatority, Bayesian monotonicity, and Bayesian incentive compatibility are satisfied. As such, it improves upon and contains as special cases previously known results about Nash and Bayesian implementation in exchange economic environments. We show that the individual rationality and continuity conditions, imposed in Hurwicz et al. [12], can be weakened to the non-confiscatority and can be dropped, respectively, for Nash implementation. Thus we also give a full characterization for Nash implementation when endowments and preferences are both unknown to the designer.

1 Introduction

It is by now well known that when information is private and direct control is impossible or inappropriate, informationally decentralized processes have to be used to make collective choice decisions or to allocate resources. However, if an institution or organization is not appropriately designed, individuals

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may find it in their interests to distort the information they provide and these distortions may lead to nonoptimal group decisions because such information may strategically advance their own interests. This implies that the basic principle of mechanism design with private information must require the organization to provide individuals with appropriate incentives so that individuals' interests are consistent with the goals of the organization. An organization (social choice) rule which has this consistency property is called incentive compatible. Incentive compatibility then becomes a basic requirement to consider in the design of a social organization in general and an economic organization in particular. One wants to design an incentive compatible mechanism such that the set of equilibrium outcomes of the mechanism coincides with the set of socially desirable alternatives for all environments under consideration. That is, given a social choice goal, one wants to find a mechanism that generates only outcomes which are consistent with the social choice goal under an appropriate solution concept of self-interested behavior. This is a classic problem identified by Hurwicz [11]. A fundamental problem becomes characterizing what various institutions can achieve using incentive compatible mechanisms. Implementation theory studies precisely this problem.

Numerous papers since the seminal work of Maskin [14] have provided characterizations of social choice rules that can be implemented in various solution rules of self-interested behavior. For complete information environments, characterization results were given by Maskin [14], Hurwicz, Maskin, Postlewaite [12], Repullo [28], Sajio [29], Moore and Repullo [19], Dutta and Sen [5], Danilov [4], and others for Nash implementation; Moore and Repullo [18], Abreu and Sen [2] and others for implementation using refinements of Nash equilibrium; Matsushima [15] and Abreu and Sen [3] for virtual Nash implementation. For incomplete information environments, characterization results were given by Postlewaite and Schmeidler [25], Palfrey, and Srivastava [20, 21], Mookherjee and Reichelstein [17], Jackson [13], Hong [9] among many others for Bayesian implementation; by Palfrey and Srivastava [23] and Mookherjee and Reichelstein [17] for implementation in using refinements of Bayesian equilibrium; Abreu and Matsushima [1], Matsushima [16], Duggan [7], and Tian [35] for virtual Bayesian implementation.

However, most of these studies assume that the feasible sets under consideration are common information. This is clearly a very restrictive assumption for the same reason that it is for preferences. The only exceptions are Hurwicz, Maskin, and Postlewaite [12] (hereafter HMP) and Hong [9].[1]

[1] In the "better" mechanism design literature–the design of mechanisms which have desirable properties such as continuity and low dimensionality of message space, Postlewaite and Wettstein [26], Tian [30, 31, 33, 34], Tian and Li [36] considered Nash implementation of some specific social choice rules (such as Walrasian, Lindahl, cost share allocation rules) for complete information environments when endowments and preferences are unknown to the designer. These results are not characterization results for general social choice rules.

HMP [12] considered the problem when endowments and/or preferences are both unknown to the designer. They showed that for exchange economies with at least three agents, monotonically increasing preferences and endowments unknown to the designer, individual rationality is sufficient, and in conjunction with the continuity of preferences, also necessary for a social choice correspondence to be Nash implementable. When endowments and preference are both unknown to the designer, they show that for exchange economic environments with three or more agents and monotonically increasing preferences, individual rationality and Nash monotonicity are sufficient, and in conjunction with the continuity of preferences, also necessary for a social choice correspondence to be Nash implementable. But they assumed that agents have complete information, and the results are not full characterizations since the necessity part requires the continuity of preferences. Hong [9] considered Bayesian implementation for economic environments where endowments are incomplete information, preferences are known to the designer, and information structure is diffuse. She showed that a social choice set is Bayesian implementable if and only if it is Bayesian incentive compatible and Bayesian monotonic.

In this paper we consider Bayesian and Nash implementation when preferences and feasible sets are both state dependent. We complement and extend earlier work on Bayesian and Nash implementation in exchange economic environments. We fully characterize Bayesian implementability by giving necessary and sufficient conditions for both diffuse and non-diffuse information structures. The first theorem in the paper addresses Bayesian implementation in exchange economic environments in which agents have non-diffuse information. We show that, in exchange economic environments where there are three or more individuals and preferences are monotonically increasing, a social choice set is Bayesian implementable if and only if there is an equivalent social choice set which satisfies closure, non-confiscatority, Bayesian monotonicity, and Bayesian incentive compatibility. The second theorem in the paper addresses Bayesian implementation in exchange economic environments in which agents have diffuse information, which shows that, in exchange economic environments where there are three or more individuals and preferences are monotonically increasing, a social choice set is Bayesian implementable if and only if it satisfies closure, non-confiscatority, Bayesian monotonicity, and Bayesian incentive compatibility. Thus, these results generalize and improve upon many of the existing results in the Bayesian implementation literature and include them as special cases such as those in Postlewaite and Schmeidler [25], Palfrey and Srivastava [20], Mookherjee and Reichelstein [17], Jackson [13], and Hong [9] by allowing both preferences and feasible sets to be state dependent.

The third theorem in the paper addresses Nash implementation in complete information exchange economic environments when endowments and/or preferences are unknown to the designer. We fully characterize Nash implementability by giving necessary and sufficient conditions. We show that, in exchange economic environments with three or more individuals, a social choice correspondence is Nash implementable if and only if non-confiscatority and Nash monotonicity conditions are satisfied. As a consequence, we show that the individual rationality condition imposed in Hurwicz, Maskin, and Postlewaite [12] can be weakened to non-confiscatority, and the continuity condition imposed in their paper is redundant when endowments and preferences are both unknown to the designer.

This paper is organized as follows. Section 2 states some notation and definitions about the framework of analysis. In Section 3 we consider Bayesian implementation in incomplete information exchange economic environments and provide the theorems which give necessary and sufficient conditions for both diffuse and non-diffuse information structures. In Section 4, we consider Nash implementation in complete information exchange economic environments, and provide a theorem which gives necessary and sufficient conditions when endowments and/or preferences are unknown to the designer. The concluding remarks are given in Section 5.

2 The framework of analysis

Consider an exchange economy with n individuals ($n > 2$), who consume l goods.[2] The set of individuals is denoted by $N = \{1, 2, \dots, n\}$. Let Q_i be a finite set of possible states of endowment vectors, let T_i be a finite set of possible types for agent i, and let $S_i = Q_i \times T_i$. An element of S_i is denoted by $s_i = (w_i, t_i) \in Q_i \times T_i$ which is called the information set of agent i. An endowment profile is $w = (w_1, \dots, w_n)$ and $Q = \prod_{i \in N} Q_i$ denotes the set of all endowment profiles. We assume that $Q_i \subset \mathbb{R}^l_+$ and $\sum_{i \in N} w_i \geq 0$ for all $w \in Q$. A type profile is $t = (t_1, \dots, t_n)$ and $T = \prod_{i \in N} T_i$ denotes the set of all type profiles.

A state of an economy is a profile vector $s = (s_1, \dots, s_n)$. A state summarizes agents' preferences, endowments, and information. Thus preferences and individual endowments vary across states and are state dependent. The set of states is denoted by $S = \prod_{i \in N} S_i$. For each i, denote by $S_{-i} = \prod_{j \neq i} S_j$ the set of possible state profiles of all agents other than i.

Each agent i has a prior distribution function (probability measure) q_i defined on S. We assume that if $q_i(s) > 0$ for some $i \in N$, then $q_j(s) > 0$ for all $j \neq i$. The set of all such states is denoted by J and is called the support of S, where $J = \{s \in S : q_i(s) > 0 \ \forall i \in N\}$. For each given information set $s_i \in S_i$, let

$$\pi_i(s_i) = \{s' \in S : s'_i = s_i \ \& \ q_i(s') > 0\}$$

[2] As usual, vector inequalities are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \dots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \dots, m$.

which defines the set of states which i believes may be the true state. Denote by Π_i all such sets which defines a partition of J. Let Π be the common knowledge concatenation defined by $\Pi_1, \ldots, \Pi_n$. That is, Π is the finest partition which is coarser than each Π_i. Without loss of generality, we assume that $\pi_i(s_i) \neq \emptyset$ for all $i \in N$ and all $s_i \in S_i$. An information structure is said to be *diffuse* if $J = S$ or to be *non-diffuse* otherwise. We will give characterization results for both structures.

The (ex-post) utility function of agent i, $U_i : \mathbb{R}^l_+ \times T \to \mathbb{R}$, is assumed to be strictly increasing in consumption for each state s . We normalize the utility function so that $U_i(0, t) = 0$ for all $t \in T$.

Let $A(w)$ denote a *net* trade allocation feasible set when endowment is $w \in Q$, which is defined by

$$A(w) = \{z \in \mathbb{R}^{ln} : z_i \geqq -w_i \ \& \ \sum_{i \in N} z_i = 0 \text{ for all } i \in N\} \ .$$

Define $A = \bigcup_{w \in Q} A(w)$.

A social choice function (allocation rule) $x : S \to A$ is a function from states to allocations such that

$$x(s) \in A(w)$$

for all $s = (w, t) \in S$. Thus, a social choice rule x results in feasible allocations for all states $s \in S$. Let

$$X = \{x : S \to A\}$$

be the set of all allocation rules. A social choice set F is a subset of X, which may be thought of as a collection of maps which list desirable allocations as a function of state.

Remark 1 A social choice set is sometimes called a social choice correspondence in the literature. But this may cause confusion between a social choice correspondence setup for complete information and a social choice correspondence setup for incomplete information. A social choice correspondence in the complete information setup is usually defined as a mapping which assigns to each environment a set of desirable allocations. In contrast, a social choice set in incomplete information environments is defined as a set of social choice rules, each of which is a mapping that assigns to each type ("state") in the class of environments a desirable allocation. So to avoid confusion, the incomplete information correspondence is called a social choice set. We will define a social choice correspondence for incomplete information below.

Each agent has preferences over social choice functions which have an interim conditional expected utility representation. Given an allocation rule, $x : S \to A$, the interim (conditional expected) utility of x to agent i at state s is

$$V_i(x, s_i) = \sum_{s \in \pi_i(s_i)} q_i U_i(x_i(s) + w_i, t) \ .$$

We can now define an interim weak state dependent preference relation R_i on X by, for $x, y \in X$,

$$xR_i(s_i)y \;\Leftrightarrow\; V_i(x, s_i) \;\geqq\; V_i(y, s_i) \ .$$

Note that $R_i(s_i)$ are not only dependent on t_i but also on w_i. Preferences are clearly complete and transitive. The strict preference and indifference relations associated with R_i are denoted by P_i and I_i, respectively.

The tuple $e = \langle N, A, S, \{q_i\}, \{U_i\}\rangle$ is called an exchange economic environment. We assume, as is standard, that the structure of e is common knowledge to all agents and that each agent i knows his own type. Denote by E the set of all such environments.

The designer is assumed to know S and q_i, but does not know the true states of agents. To achieve a social goal, the designer needs to construct an informationally decentralized mechanism $\langle M, g\rangle$, where $M = \prod_{i\in N} M_i$ with an element $m = (m_1, \ldots, m_n)$. The set M_i is the set of all possible messages agent i can announce and M is called a message space. $g : M \to A$ is an outcome function. Throughout this paper, we assume that the message space M_i of agent i is of the form

$$M_i = M_i^1 \times M_i^2 \equiv Q_i \times M_i^2 \ ,$$

where M_i^2 is an arbitrary non-empty set, and that, for any $m \in M$ and any $i \in N$,

$$g_i(m) \;\geqq\; -m_i^1 \ .$$

For each $m \in M$, $g(m)$ yields an outcome in A. Given a mechanism $\langle M, g\rangle$, each agent i chooses messages m_i as a function of his types. We call a mapping $\sigma_i : S_i \to M_i$ a strategy for agent i and Σ_i his set of strategies. The notations σ_{-i}, Σ_{-i}, Σ are similarly defined. Given a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $\sigma(s) = (\sigma_1(s_1), \ldots, \sigma_n(s_n))$, $g(\sigma)$ represents the social choice function which results when σ is played.

A mechanism $\langle M, g\rangle$ defined on the domain E is *individually feasible* if $g_i(\sigma(s)) \;\geqq\; -w_i$ for all $i \in N$, $s_i \in S_i$, and all $\sigma \in \Sigma$; it is *balanced* if $\sum_{i\in N} g_i(\sigma) = 0$ (i.e., $\sum_{i\in N} g_i(\sigma(s) = 0$ for all $s \in S$) for all $\sigma \in \Sigma$; [3] it is *feasible* if it is both individually feasible and balanced, i.e., for all $\sigma \in \Sigma$, $g(\sigma) \in A$ with $g(\sigma(s)) \in A(w)$ for all $s \in S$.

A mechanism $\langle M, g\rangle$ is called a *direct* or a *revelation mechanism* if $M = S$. That is, each agent announces a, possibly false, state which is used by $g(s_1, \ldots, s_n)$ to pick up $x \in A$. Such a special strategy is called a deceptive strategy for agent i. Thus a *deception* for agent i is a function from S_i to S_i, say, $\alpha_i : S_i \to S_i$ with $\alpha_i(s_i) = (\alpha_i^1(s_i), \alpha_i^2(s_i)) \in Q_i \times T_i$ for all $s_i \in S_i$. The interpretation is that when i is of type s_i, he acts as if he is of type $\alpha_i(s_i)$. Let τ

[3] In the rest of the paper, whenever we write a function equality (or inequality), it means the equality (or inequality) holds for every point in the domain.

denote the truthful strategy profile, that is, $\tau_i(s_i) = s_i$ for all $i \in N$ and all $s_i \in S_i$. Then truth telling is simply the identity function.

To get a feasible mechanism, we have to assume that $\alpha_i^1(s_i) \in [0, w_i] \cap Q_i$ for all $w_i \in Q_i$. That is, agents are allowed to under report but not over report their endowments. This assumption is necessary, as was proved in Hurwicz, Maskin, and Postlewaite [12], to insure feasible outcomes when endowments are state dependent and unknown to the designer. Note that although the true endowment is the upper bound of the announced endowment, the designer does not need to know this upper bound. This is because whenever an agent claims an endowment of a certain amount, the designer can ask him to *exhibit* it (we may, for instance, imagine that the rules of the game require that the agent 'put on the table' the reported amount $\alpha_i^1(s_i)$). Note that, since $\alpha_i^1(s_i) \leqq w_i$ is permitted, the agent is able to withhold a part of the true endowment. For the problem to be non-trivial, complete withholding by all agents is ruled out by assuming that $\sum_{i \in N} \alpha_i^1(s_i) \geq 0$ for all $w \in Q$. Also note that, by assumption, $g_i(\sigma) \geqq -\alpha_i^1 \geqq -\tau_i^1$ for all $i \in N$ and $\sigma \in \Sigma$. That is, the outcome function will never deprive the agent of goods in excess of this claimed endowment and thus it is individually feasible.

The self-interested behavior of agents is assumed to be described by the Bayesian equilibrium solution notion developed by Harsanyi [8].

Definition 1 *A strategy profile σ is a* **Bayesian equilibrium** *of a mechanism $\langle M, g \rangle$ defined on E if for all $i \in N$ and $s_i \in S_i$,*

$$g(\sigma)\ R_i(s_i)\ g(\hat{\sigma}_i, \sigma_{-i})$$

for all $\hat{\sigma} \in \Sigma$. Denote by $B_{\langle M,g \rangle}(e)$ the set of all such allocations for environment e. When σ is a Bayesian equilibrium, then $g(\sigma)$ is called a Bayesian equilibrium outcome.

Let the notation $x \circ \alpha$ represent the social choice function which results in $x(\alpha(s))$ for all $s \in S$. For any equilibrium σ of a given mechanism $\langle M, g' \rangle$, we can define a direct mechanism $\langle S, g \rangle$ by $g = g' \circ \sigma$ (i.e, $g(s) = g'(\sigma(s))$ for all $s \in S$). Then, by the well-known revelation principle[4], we know that the truthful strategy τ is a Bayesian equilibrium for this direct mechanism.

Definition 2 *For an economic environment e, a mechanism $\langle M, g \rangle$ is said to* **(fully) Bayesian implement** *a social choice set F if*

(i) *For any $x \in F$, there is a Bayesian equilibrium σ to the mechanism $\langle M, g \rangle$ such that $g(\sigma(s)) = x(s)$ for all $s \in J$.*

[4] The revelation principle roughly says that a social choice rule attainable by an abstract incentive mechanism can also be attained by a truthtelling mechanism when solution concepts are given by a dominant strategy equilibrium, Bayesian equilibrium, or maximin equilibrium. The revelation principle provides a simple method for representing the constraints imposed by private information and strategic behavior. The revelation principle has played a prominent role in the study of incentive compatibility of economic mechanisms.

(ii) *For any Bayesian equilibrium* σ *to the mechanism* $\langle M, g\rangle$, *there exists some* $x \in F$ *such that* $g(\sigma(s)) = x(s)$ *for all* $s \in J$.

If there is a mechanism $\langle M, g\rangle$ *which Bayesian implements a social choice set* F, *then* F *is said to be* **Bayesian implementable**.

Remark 2 Note that, by the definition of implementation, the concern for implementation is only on the set J. Since allocations in states occurring with probability zero are irrelevant in this finite state model considered in the paper, it enables us to concentrate only on the set J even though social choice functions are defined on all of S. Hence any two social choice functions which agree on J can be regarded to be equivalent.

Definition 3 *Two social choice functions* $x, y \in F$ *are said to be equivalent if* $x(s) = y(s)$ *for all* $s \in J$. *Two social choice sets* F *and* $\hat{F}$ *are said to be* **equivalent** *if for each* $x \in F$, *there exists* $y \in \hat{F}$ *which is equivalent to* x, *and for each* $y \in \hat{F}$, *there exists* $x \in F$ *which is equivalent to* y.

We can also consider global Bayesian-implementation. Define a social choice correspondence $\mathscr{F} : S \to 2^X$ on the domain E as a set-valued function which assigns to every economy $e \in E$ a social choice set $\mathscr{F}(e) \subset X$.

Definition 4 *A social choice correspondence* $\mathscr{F} : S \to 2^X$ *is said to be* **globally Bayesian implementable** *relative to* E *if, for all* $e \in E$, $\mathscr{F}(e)$ *is Bayesian implementable*.

The following closure property of the social choice set is a basic requirement for an implementable social choice set.

Definition 5 *Let* B *and* D *be any two disjoint sets such that* $B \cup D = J$ *and for any* $\pi \in \Pi$ *either* $\pi \in B$ *or* $\pi \in D$. *A social choice set* F *is said to satisfy* **closure** *if for any* $x, y \in F$, *there exists* $z \in F$ *such that* $z(s) = x(s)$ *for all* $s \in B$ *and* $z(s) = y(s)$ *for all* $s \in D$.

Postlewaite and Schmeidler [25] showed that the closure condition is always necessary for implementation of a social choice set. This is because, for any two equilibrium strategies of a mechanism, a third strategy, in which agents act according to the first strategy on the states of $B \in J$ and the second strategy on the states of $D \in J$, must also be an equilibrium of the mechanism. Since F is implementable, the allocation resulting from the third strategy must lie in F. This means that closure must be satisfied.

We now define the non-confiscatority, Bayesian incentive compatibility and monotonicity of a social choice set F, which are key conditions to characterize Bayesian implementability of F. It may be remarked that our notations and formulations may be easier to understand and more transparent than those provided in the literature.

Definition 6 *A social choice set* F *is* **non-confiscatory** *if for any* $x \in F$ *and* $i \in N$, *we have* $x_i(s) + w_i \geq 0$ *for all* $s \in J$.

Definition 7 *A social choice set* F *is said to be* **individually rational** *if for all* $x \in F$ *and* $i \in N$, $xR_i(s_i)0$ *for all* $s \in J$.

Even though the non-confiscatority condition is not needed for implementation of F when feasible sets are known to the designer and agents, it is necessary for implementation of F when feasible set is state dependent. Note that, by non-confiscatority, we have $x_i(s)P_i(s_i) - w_i$ for any $x \in F$ and $i \in N$. Also note that individual rationality of F implies non-confiscatority when $w \geq 0$ and utility functions are strictly increasing. Indeed, suppose this were not true. Then $x_i(s) + w_i \leqq 0 \leq w_i$ for some $s \in J$ and some $i \in N$, and thus $0P_i(s_i)x_i(s)$ by strict monotonicity of preferences, which contradicts the supposition that x is an individual rational allocation.

Definition 8 *A social choice set* F *is* **Bayesian incentive-compatible** *if for all* $x \in F$, $i \in N$, $s_i \in S_i$, *we have* $x\, R_i(s_i)\, x \circ (\alpha_i, \tau_{-i})$ *for all deceptions* α_i.

The concept of Bayesian incentive-compatibility means that every agent will report his type truthfully provided *all* other agents are employing their truthful strategies and thus every truthful strategy profile is a Bayesian equilibrium of the direct mechanism $\langle S, x \rangle$. Notice that Bayesian incentive compatibility does not say what is the best response of an agent when other agents are not using truthful strategies. So it may also contain some undesirable equilibrium outcomes when a mechanism has multiple equilibria. The goal for designing a mechanism is to reach a desirable equilibrium outcome, but it may also result in an undesirable equilibrium outcome. Thus, while the incentive compatibility requirement is central, it may not be sufficient for a mechanism to give all of desirable outcomes. The severity of this multiple equilibrium problem has been exemplified by Demski and Sappington [6], Postlewaite and Schmeidler [25], Repullo [27], and others. The implementation problem involves designing mechanisms to ensure that all equilibria result in desirable outcomes which are captured by the social choice set.

Note that, by the definition of π_i, α_i is constant on $\pi_i(s_i)$ (i.e., $\alpha_i(s_i') = \alpha_i(s_i)$ for all $s' \in \pi_i(s_i)$). Thus the above definition is the same as the one in, say, Jackson [13], but in our formulation it is easier to understand the meaning of Bayesian incentive compatibility.

Definition 9 *A social choice set* F *is said to satisfy* **Bayesian monotonicity** *if for every* $x \in F$ *and every deception* α *such that* $x \circ \alpha$ *is not equivalent to any social choice function in* F, *there exist* $i \in N$, $s_i \in S_i$, *and* $y \in X$ *such that*

(1) $y \circ \alpha\, P_i(s_i)\, x \circ \alpha$;
(2) $x\, R_i(s_i')\, y \circ (\bar{\alpha}_{s_i}, \tau_{-i})$ *for all* $s_i' \in S_i$, *where* $\bar{\alpha}_{s_i}$ *is a constant function defined by* $\bar{\alpha}_{s_i}(s_i') = \alpha_i(s_i)$ *for all* $s_i' \in S_i$.

Bayesian monotonicity means that for every $x \in F$ and for every deception α, either $x \circ \alpha$ is equivalent to F or $x \circ \alpha$ is not. If $x \circ \alpha$ is not equivalent to F, then there exists some agent i and s_i such that $x \circ \alpha$ is upset by $y \circ \alpha$ for some $y \in X$ (so $x \circ \alpha$ is not an equilibrium) but x is not upset by $y \circ (\bar{\alpha}_{s_i}, \tau_{-i})$. In essence, the Bayesian monotonicity condition assures the "selective elimination" of undesirable equilibrium. Consider a mechanism which may implement F and has an equilibrium such that $g \circ \sigma = x$. Suppose that agents use a deception α, so that the strategies $\sigma \circ \alpha$ are played leading to the outcome $x \circ \alpha$. If there is no social choice function in F which is equivalent to $x \circ \alpha$, then $\sigma \circ \alpha$ must not be a Bayesian equilibrium. The Bayesian monotonicity condition assures that in this case strategy $\sigma \circ \alpha$ can be ruled out as an equilibrium, thus there exists an agent i who is better off at $y \circ \alpha$ for some allocation y at some state s_i, and assures that agent i cannot gain by falsely accusing the other agents of deceiving.

3 Implementation in incomplete information

In this section, we will characterize Bayesian implementability by giving necessary and sufficient conditions for both non-diffuse and diffuse information structures in the exchange economic environments specified in Section 2. The first theorem provides a characterization of Bayesian implementable social choice set in non-diffuse information structures. The characterization is that there exists an equivalent social choice set which satisfies closure, non-confiscatority, Bayesian incentive-compatibility, and Bayesian monotonicity. The second theorem provides a characterization of a Bayesian implementable social choice set in diffuse information structures, which shows that a social choice set is implementable if and only if it satisfies closure, non-confiscatority, Bayesian incentive-compatibility, and Bayesian monotonicity.

Theorem 1 *In any exchange economic environment e in which $n \geqq 3$, information structure is non-diffuse, and utility functions $U_i : \mathbb{R}^l_+ \times T \to \mathbb{R}_+$ are strictly increasing in consumption for each type $t \in T$, a social choice set F is Bayesian implementable if and only if there exists an equivalent social choice set $\hat{F}$ which satisfies closure, non-confiscatority, Bayesian incentive compatibility, and Bayesian monotonicity.*

Proof. Necessity. Let $\langle M, g \rangle$ implement F and define $\hat{F} = \{x \in X : x = g(\sigma)$ for some equilibrium σ to $\langle M, g \rangle\}$. By the definition of implementation it follows that $\hat{F}$ is equivalent to F. We show that $\hat{F}$ is non-confiscatory, Bayesian incentive compatible, and Bayesian monotonic.

Suppose, by way of contradiction, that F is confiscatory. Then there exist $x \in F$, $s \in J$ and $i \in N$ such that $x_i(s) = -w_i$. Then, by strict monotonicity of U_i and $w_i \geq 0$, we have $-w_i/2P_i(s_i)x$. Since $\langle M, g \rangle$ implements F, there exists a Bayesian equilibrium $\sigma \in \Sigma$ such that $h(\sigma) = x$. Hence we have $-w_i/2P_i(s_i)h(\sigma)$. But, for any $\hat{\sigma}_i$ with $\hat{\sigma}_i = (\hat{\sigma}^1_i, \hat{\sigma}^2_i) = (w_i/2, \hat{\sigma}^2_i)$, we have $h_i(\hat{\sigma}_i, \sigma_{-i}) \geqq -\hat{\sigma}^1_i = -w_i/2$ by assumption. Hence, $h(\hat{\sigma}_i, \sigma_{-i})R_i(s_i)$ $-w_i/2P_i(s_i)h(\sigma)$, which contradicts the supposition that σ is a Bayesian equilibrium. So F must be non-confiscatory.

Take any $x \in \hat{F}$ and an equilibrium σ such that $g(\sigma(s)) = x(s)$ for all $s \in S$. Note that for any deception α_i, we have $\sigma_i \circ \alpha_i \in \Sigma_i$. Since σ is an equilibrium strategy, then for any $i \in N$,

$$g(\sigma)\ R_i(s_i)\ g(\sigma) \circ (\alpha_i, \tau_{-i}) \ .$$

Since $g(\sigma) = x$ and $g(\sigma) \circ (\alpha_i, \tau_{-i}) = x \circ (\alpha_i, \tau_{-i})$, then for any $s_i \in S_i$ we have

$$x\ R_i(s_i)\ x \circ (\alpha_i, \tau_{-i})$$

for all deceptions α_i which establishes Bayesian incentive compatibility.

Take any $x \in \hat{F}$ and an equilibrium σ such that $g(\sigma(s)) = x(s)$ for all $s \in S$. Suppose that there is some deception α such that $x \circ \alpha$ is not equivalent to $\hat{F}$. Then there exists no $z \in F$ such that $z(s) = x \circ \alpha(s)$ for all $s \in S$ and thus it must be that $\sigma \circ \alpha$ is not equilibrium at some $s \in S$. Therefore, there exist $i \in N$, $\hat{m}_i \in M_i$, and $\hat{\sigma}_i \in \Sigma_i$ such that $g(\hat{\sigma}_i, \sigma_{-i}) \circ (\tau_i, \alpha_{-i})\ P_i(s_i)\ g(\sigma) \circ \alpha$, where $\hat{\sigma}_i(s_i') = \hat{m}_i$ for all $s_i' \in S_i$. Then $\hat{\sigma}_i \circ \alpha_i = \hat{\sigma}_i$ and thus

$$g(\hat{\sigma}_i, \sigma_{-i}) \circ \alpha\ P_i(s_i)\ g(\sigma) \circ \alpha \ .$$

Let $y = g(\hat{\sigma}_i, \sigma_{-i})$. From above, we have

$$y \circ \alpha\ P_i(s_i)\ x \circ \alpha \ .$$

Since σ is an equilibrium, it follows that $g(\sigma)\ R_i(s_i')\ g(\hat{\sigma}_i, \sigma_{-i}) \circ (\bar{\alpha}_{s_i}, \tau_{-i})$ for all $s_i' \in S_i$. Since $g(\sigma) = x$ and $y = g(\hat{\sigma}_i, \sigma_{-i})$, we have

$$x\ R_i(s_i')\ y \circ (\bar{\alpha}_{s_i}, \tau_{-i})$$

for all $s_i' \in S_i$. Thus F is Bayesian monotonic.

Now we prove the sufficiency portion of the theorem. We want to construct a mechanism which implements a social choice set $\hat{F}$ if it satisfies the conditions of the theorem. Note that, when $\hat{F}$ is equivalent to F, the mechanism also implements F by definition. Thus the proof of the sufficiency portion of the theorem can be done by considering implementation of $\hat{F}$.

For every $i \in N$, let

$$M_i = Q_i \times T_i \times X \times [0, 1) \ .$$

An element in $[0, 1)$ could be interpreted as the desirability to agent i of deviating from an agreement on a social choice rule. Then $M = M_1 \times M_2 \times \ldots \times M_n$. Partition M into the following sets:

$$D_1 = \{m \in M : \exists x \in F\ s.t.\ m_j = (\cdot, \cdot, x, \cdot)\ \forall j \in N\} \ . \tag{1}$$

$$D_2(i) = \{m \in M : m \notin D_1,\ \exists x \in F\ s.t.\ m_j = (\cdot, \cdot, x, \cdot)\ \forall j \neq i \tag{2}$$

$$m_i = (\cdot, \cdot, y, m_i^4) \text{ with } m_i^4 \geqq \max_{j \in N}\{m_j^4\} \text{ and } y \in X\} \ . \tag{3}$$

$$D_3 = \{m \in M : m \notin D_1 \cup D_2\} \ . \tag{4}$$

Here $D_2 = \bigcup_{i \in N} D_2(i)$. Define $\bar{\alpha}_i$ by $\bar{\alpha}_i(s_i) = (m_i^1, m_i^2)$ for all $s_i \in S_i$. We can further subdivide $D_2(i)$ as follows

$$D_{2a}(i) = \{m \in D_2(i) : x\, R_i(s_i')\, y \circ (\bar{\alpha}_i, \tau_{-i}) \text{ for all } s_i' \in S_i\} \tag{5}$$

$$D_{2b}(i) = \{m \in D_2(i) : y \circ (\bar{\alpha}_i, \tau_{-i})\, P_i(s_i')\, x \text{ for some } s_i' \in S_i\}\ . \tag{6}$$

Define $D_{2a} = \bigcup_{i \in N} D_{2a}(i)$ and $D_{2b} = \bigcup_{i \in N} D_{2b}(i)$.

Let $N'(m) = \{i \in N : m_i^4 \geqq m_j^4 \text{ for all } j \in N\}$, let

$$N(m) = \begin{cases} N'(m) & \text{if } N'(m) \neq N \\ \{1\} & \text{otherwise} \end{cases},$$

and let $n(m) = \#N(m)$. That is, $N(m)$ is a subset of N, which either consists of agent 1 if all agents choose the same number in $[0, 1)$, or the agents who bid the highest value in $[0, 1)$ otherwise.

The outcome function $g : M \to A$ is defined by

$$g(m) = \begin{cases} x(m^1, m^2) & \text{if } \ m \in D_1 \\ y(m^1, m^2) & \text{if } \ m \in D_{2a} \\ x(m^1, m^2) & \text{if } \ m \in D_{2b}\ . \end{cases} \tag{8}$$

and for $m \in D_3$

$$g_i(m) = \begin{cases} \frac{1}{n(m)} \sum_{j \notin N(m)} m_j^1 & \text{if } \ i \in N(m) \\ -m_i^1 & \text{otherwise.} \end{cases} \tag{9}$$

Note that when a message is given by $m = \sigma(s)$, the outcome can be denoted as

$$g(\sigma(s)) = \begin{cases} x \circ \alpha(s) & \text{if } \ \sigma(s) \in D_1 \\ y \circ \alpha(s) & \text{if } \ \sigma(s) \in D_{2a} \\ x \circ \alpha(s) & \text{if } \ \sigma(s) \in D_{2b}\ . \end{cases} \tag{10}$$

and for $\sigma(s) \in D_3$

$$g_i(\sigma(s)) = \begin{cases} \frac{1}{n(\sigma(s))} \sum_{j \notin N(\sigma(s))} \alpha_j^1(s_j) & \text{if } \ i \in N(\sigma(s)) \\ -\alpha_i^1(s_i) & \text{otherwise}\ . \end{cases} \tag{11}$$

Also note that, by the construction of the mechanism, $g_i(\sigma) \geqq -\alpha_i^1 \geqq -\tau_i^1$ for all $i \in N$ and $\sigma \in \Sigma$, and $\sum_{i \in N} g_i(\sigma) = 0$ for all $\sigma \in \Sigma$ so that it is balanced (not merely weakly balanced). Thus the mechanism is feasible.

Remark 3 The structure of the mechanism is similar to many of the mechanisms used in the Nash and Bayesian implementation literature, which has the following natural explanations. The message space is portioned into three basic regions, $D_1 - D_3$. D_1 is a region in which all agents request the same allocation rule, say, $x \in F$. In this region, the outcome is given by $x(m^1, m^2)$. $D_2(i)$ is a region in which agent i deviates unilaterally from D_1 by signaling the highest desirability in $[0, 1)$ and asks for some $y \in X$. In this region, the outcome depends on whether the message m lies in $D_{2a}(i)$ or $D_{2b}(i)$. In $D_{2a}(i)$, agent i appears no better off with $y \circ (\bar{\alpha}_i, \tau_{-i})$ than with x for all $s_i' \in S_i$. In this case, the outcome is $y(m^1, m^2)$. In $D_{2b}(i)$, agent i is better off with $y \circ (\bar{\alpha}_i, \tau_{-i})$

than with x for some $s_i' \in S_i$. The outcome is $x(m^1, m^2)$. Finally, the remaining region is called D_3, in which the agent who has the highest request for deviation receives all the reported net endowments of other agents. Since any $j \neq i$ can move unilaterally from $D_2(i)$ to D_3, there is no equilibrium in $D_2(i)$. There is clearly no equilibrium in D_3. Thus, all equilibria lie in D_1. The above intuition will be formalized and proved in three lemmas below.

The above mechanism is in the class of so-called augmented revelation mechanisms which was studied by Mookherjee and Reichelstein [17]. In these mechanisms, agents can either report their private information, or send some auxiliary "non-state" messages so that a social choice set F can be implemented by an augmented revelation mechanism. This approach is standard in characterizing implementable social choice rules in the literature.

The following lemmas establish the sufficiency portion of the theorem.

Lemma 1 *If a social choice set F satisfies the Bayesian incentive compatibility condition, then for every $x \in F$ the strategy σ with $\sigma_i = (\tau_i^1, \tau_i^2, x, 0)$ is a Bayesian equilibrium to the mechanism $\langle M, g \rangle$ constructed above.*

Proof. We first note that $\sigma(s) \in D_1$ and $\alpha = (\tau^1, \tau^2)$ so that $g(\sigma(s)) = x(s)$ for all $s \in S$. We verify that σ is an equilibrium by showing that there are no improving deviations.

For $i \in N$ and $s \in S$, suppose $\hat{\sigma}_i(s_i) \neq \sigma_i(s_i)$. There are two cases to consider: (1) $\hat{\sigma}_i(s_i) = (\hat{\alpha}_i^1(s_i), \hat{\alpha}_i^2(s_i), x, \hat{\sigma}_i^4(s_i))$ and (2) $\hat{\sigma}_i(s_i) = (\hat{\alpha}_i^1(s_i), \hat{\alpha}_i^2(s_i), y, \hat{\sigma}_i^4(s_i))$ for some $y \neq x$.

In case (1), it is clear $(\hat{\sigma}_i, \sigma_{-i})(s') \in D_1$ for all $s' \in \pi_i(s_i)$ and thus the resulting allocation is $g(\hat{\sigma}_i, \sigma_{-i}) = g(\sigma) \circ (\hat{\alpha}_i, \tau_{-i})$ on $\pi_i(s_i)$. By the Bayesian incentive compatibility, we have $x \; R_i(s_i) \; x \circ (\hat{\alpha}_i, \tau_{-i})$ for all $\hat{\alpha}_i$.

In case (2), we have $(\hat{\sigma}_i, \sigma_{-i})(s) \in D_2(i)$ since $\hat{\sigma}_i^4(s_i) \geqq \max_{j \neq i} \sigma_j^4(s_j) = 0$. If $x \; R_i(s_i') \; y \circ (\hat{\alpha}_i, \tau_{-i})$ for all $s_i' \in S_i$, then the resulting allocation outcome is $y \circ (\hat{\alpha}_i, \tau_{-i})$ on $\pi_i(s_i)$ (by noting $\alpha_{-i} = \tau_{-i}$) and thus we have $g(\sigma_i, \sigma_{-i}) \; R_i(s_i) \; g(\hat{\sigma}_i, \sigma_{-i})$, which is not improving. Otherwise, the resulting outcome $g(\hat{\sigma}_i, \sigma_{-i}) = x(\hat{\alpha}_i, \tau_{-i}) = x \circ (\hat{\alpha}_i, \tau_{-i})$ on $\pi_i(s_i)$. By the Bayesian incentive compatibility, we have $x \; R_i(s_i) \; x \circ (\hat{\alpha}_i, \tau_{-i})$.

Thus in either case we have shown that for all $i \in N$ and $s \in S$, $g(\sigma_i, \sigma_{-i}) \; R_i(s_i) \; g(\hat{\sigma}_i, \sigma_{-i})$ for all $\hat{\sigma}_i \in \Sigma_i$. So σ is an equilibrium. Q.E.D.

Lemma 2 *If σ is a Bayesian equilibrium to the mechanism $\langle M, g \rangle$ constructed above, then $\sigma(s) \in D_1$ for all $s \in S$.*

Proof. Suppose σ is a Bayesian equilibrium, but there is some $s \in S$ such that $\sigma(s) \notin D_1$. There are two cases to consider: (1) $\sigma(s) \in D_2(i)$ for some i and (2) $\sigma(s) \in D_3$.

Case 1. $\sigma(s) \in D_2(i)$ for some i. Since $n \geqq 3$, there exists some $k \neq i$ such that

$$g_k(\sigma(s)) \leq \sum_{j \neq k} \sigma_j^1(s_j) \quad . \tag{12}$$

Indeed, suppose not. Then $g_k(\sigma(s)) \geqq \sum_{j\neq k}\sigma_j^1(s_j)$ for all $k \neq i$. Also, $g_i(\sigma(s)) \geqq -\sigma_i^1(s_i)$ by the construction of the mechanism. Summing these inequalities over all individuals, we have $0 = \sum_{j\in N} g_j(\sigma(s)) \geqq (n-2) \sum_{j\in N}\sigma_j^1(s_j) \geq 0$, a contradiction.

Let $\hat{\sigma}_k = (\sigma_k^1, \sigma_k^2, \sigma_k^3, \hat{\sigma}_k^4)$, where $\hat{\sigma}_k^4(s_k') = \hat{m}^4 > \sigma_j^4(s_k')$ for all $j \in N$ and $s' \in S$. (Note that the first three components of $\hat{\sigma}_k$ are the same as σ_k and only the fourth component is different.) Then $(\hat{\sigma}_k, \sigma_{-k})(s) \in D_3$, and $N(\hat{\sigma}_k, \sigma_{-k})(s) = \{k\}$ so that $n(\hat{\sigma}_k, \sigma_{-k})(s) = 1$. Therefore, we have

$$g_k(\hat{\sigma}_k, \sigma_{-k})(s) = \sum_{j\neq k}\sigma_j^1(s_j) \ .$$

Since U_k is strictly increasing, we have by (12)

$$U_k(g(\hat{\sigma}_k, \sigma_{-k})(s) + w_k, t) > U_k(g(\sigma)(s) + w_k, t) \ .$$

For any $s' \in \pi_k(s_k)$, there are five subcases to consider. For each subcase, we show that

$$U_k(g(\hat{\sigma}_k, \sigma_{-k})(s') + w_k, t) \ \geqq \ U_k(g(\sigma)(s') + w_k, t) \ .$$

Subcase 1.a. $\sigma(s') \in D_1$. In this case, $(\hat{\sigma}_k, \sigma_{-k})(s') \in D_1$. Since the resulting outcome is independent of the fourth component of a message m in D_1 by the definition of the mechanism, we have $g(\hat{\sigma}_k, \sigma_{-k})(s') = g(\sigma)(s')$.

Subcase 1.b. $\sigma(s') \in D_2(k)$. Since the first three components are the same as σ_k and $\hat{\sigma}_k^4(s_k') > \max_{j\neq k}\{\sigma_j^4(s_j')\}$, we also have $(\hat{\sigma}_k, \sigma_{-k})(s') \in D_2(k)$ and thus $g(\hat{\sigma}_k, \sigma_{-k})(s') = g(\sigma)(s')$.

Subcase 1.c. $\sigma(s') \in D_2(j)$ for some $j \neq k$. In this case, $g_k(\sigma)(s') = y_k(\sigma(s'))$ if $\sigma(s') \in D_{2a}(j)$ or $g_k(\sigma)(s') = x_k(\sigma(s'))$ if $\sigma(s') \in D_{2b}(j)$, and $(\hat{\sigma}_k, \sigma_{-k})(s') \in D_3$ by also noting that $\hat{\sigma}_k^4(s_k') > \max_{j\neq k}\{\sigma_j^4(s_j')\}$. So we also have $g_k(\hat{\sigma}_k, \sigma_{-k})(s') = \sum_{j\neq k}\sigma_j^1(s_j') \ \geqq \ g_k(\sigma)(s')$ by the feasibility of x and y.

Subcase 1.d. $\sigma(s') \in D_3$, $\sigma_k^3(s_k') = x \neq y = \sigma_j^3(s_j')$ for all $j \neq k$ and $\sigma_k^4(s_k') < \sigma_j^4(s_j')$ for some $j \neq k$.

Then $(\hat{\sigma}_k, \sigma_{-k})(s') \in D_2$ and thus $g_k(\hat{\sigma}_k, \sigma_{-k})(s') \ \geqq \ -\sigma_k^1(s_k) = g_k(\sigma)(s')$.

Subcase 1.e. $\sigma(s') \in D_3$ and $(\hat{\sigma}_k, \sigma_{-k})(s') \in D_3$. In this, it is clear that $g_k(\hat{\sigma}_k, \sigma_{-k})(s') = \sum_{j\neq k}\sigma_j^1(s_j') \ \geqq \ g_k(\sigma)(s')$.
Hence we have

$$g(\hat{\sigma}_k, \sigma_{-k}) \ P_k(s_k) \ g(\sigma) \ ,$$

which contradicts the fact that σ is a Bayesian equilibrium.

Case 2. $\sigma(s) \in D_3$. Since $n \geqq 3$, there exists $k \in N$ such that $g_k(\sigma(s)) = -\sigma_k^1(s_k)$ and not the case $\sigma^3 = x \in F$ for all $j \neq k$.

Let $\hat{\sigma}_k = (\sigma_k^1, \sigma_k^2, \sigma_k^3, \hat{\sigma}_k^4)$, where $\hat{\sigma}_k^4(s') = \hat{m}^4 > \max_{j\neq k}\{\sigma_j^4(s_j')\}$ for all $s_j' \in S_j$. Then $(\hat{\sigma}_k, \sigma_{-k})(s) \in D_3$, and $N(\hat{\sigma}_k, \sigma_{-k}) = \{k\}$ so that $n(\hat{\sigma}_k, \sigma_{-k}) = 1$. Thus we have

$$g_k(\hat{\sigma}_k, \sigma_{-k})(s) = \sum_{j \neq k} \sigma_j^1(s_j) \geq -\sigma_k^1 = g_k(\sigma)(s)$$

by noting that $\sum_{i \in N} \sigma_i(s_i) \geq 0$ by assumption. Hence, by strict monotonicity of U_k, we have

$$U_k(g(\hat{\sigma}_k, \sigma_{-k})(s) + w_k, t) > U_k(g(\sigma)(s) + w_k, t) \ .$$

As in Case 1, we can similarly show, by discussing five subcases, that for any $s' \in \pi_k(s_k)$,

$$U_k(g(\hat{\sigma}_k, \sigma_{-k})(s') + w'_k, t') \ \geqq \ U_k(g(\sigma)(s') + w'_k, t') \ .$$

Thus we have

$$g(\hat{\sigma}_k, \sigma_{-k}) \ P_k(s_k) \ g(\sigma_k, \sigma_{-k})$$

for some $s_k \in N$. But this contradicts the fact that σ is a Bayesian equilibrium. Q.E.D.

Lemma 3 *If a social choice set F satisfies closure and Bayesian monotonicity, then for each Bayesian equilibrium σ to the mechanism $\langle M, g \rangle$ constructed above, there exists $z \in F$ which is equivalent to $g(\sigma)$.*

Proof. Since σ is a Bayesian equilibrium, $\sigma(s) \in D_1$ for all $s \in S$ by Lemma 2. Then there exists some $x \in F$ such that $\sigma_i^3(s) = x$ for all $i \in N$ and $s \in S$. Thus, $g(\sigma) = x \circ \alpha$ (by noting that $\alpha = (\sigma^1, \sigma^2)$). Suppose, by way of contradiction, that there does not exist a social choice function in F which is equivalent to $x \circ \alpha$ ($= g(\sigma)$). By the Bayesian monotonicity of F, there exist $i \in N$, $s_i \in S_i$, and $y \in F$ such that

$$y \circ \alpha \ P_i(s_i) \ x \circ \alpha$$

and

$$x \ R_i(s'_i) \ y \circ (\bar{\alpha}_{s_i}, \tau_{-i})$$

for all $s'_i \in S_i$.

Let $\hat{\sigma}_i = (\bar{\alpha}^1_{s_i}, \bar{\alpha}^2_{s_i}, y, \hat{\sigma}_i^4)$ with $\hat{\sigma}_i^4(s'_i) = \hat{m}^4 > \sigma_j^4(s'_j)$ for all $j \in N$ and $s' \in S$. Then $(\hat{\sigma}_i, \sigma_{-i})(s') \in D_{2a}$ for all $s' \in S$ and thus we have

$$g(\hat{\sigma}_i, \sigma_{-i}) = y \circ (\bar{\alpha}_{s_i}, \alpha_{-i}) \ .$$

Hence,

$$g(\hat{\sigma}_i, \sigma_{-i}) \ P_i(s_i) \ g_i(\sigma_i, \sigma_{-i}) \ ,$$

but this contradicts the fact that σ is a Bayesian equilibrium. Thus $x \circ \alpha = g(\sigma) \in F$. Q.E.D.

From Lemmas 1–3, we know that F is implementable and thus the proof of the sufficiency of Theorem 1 is completed.

Summarizing our discussion, we can conclude that, for the set of exchange economic environments E in which $n \geqq 3$ and utility functions $U_i : \mathbb{R}^l_+ \times T \to \mathbb{R}_+$ are strictly increasing for each type $t \in T$, a social choice set correspondence $\mathscr{F} : E \to 2^X$ is globally Bayesian implementable if and only if for any $F \in \mathscr{F}(e)$, there exists an equivalent social choice set $\hat{F}$ which satisfies closure, Bayesian incentive compatibility, and Bayesian monotonicity.

For this non-diffuse information structure, Theorem 1 depends on the existence of a social choice set which is equivalent to the social choice set F which one wants to implement. The question then is if one can actually find such a social choice set which is relatively easy to check if it is equivalent to F. The following corollary answers this question.

For any social choice function $x \in F$, define a social choice function $x^0 \in X$ by

$$x^0(s) = \begin{cases} x(s) & \text{if } s \in J \\ -w & \text{otherwise} \end{cases} . \tag{13}$$

Let F^0 be the set of all such social choice functions satisfy (13). It is clear that F^0 is equivalent to F.

Corollary 1 *In an exchange economic environment e in which $n \geqq 3$ and utility functions $U_i : \mathbb{R}^l_+ \times T \to \mathbb{R}_+$ are strictly increasing in consumption for each type $t \in T$, a social choice set F is Bayesian implementable if and only if F^0 satisfies closure, non-confiscatority, Bayesian incentive compatibility, and Bayesian monotonicity.*

Proof. The sufficiency portion of Corollary 1 follows from Theorem 1. We only need to show necessity.

Let $\langle M, g\rangle$ implement F and define $\hat{F} = \{x \in X : x = g(\sigma)$ for some equilibrium σ to $\langle M, g\rangle\}$. F^0 satisfies closure and non-confiscatority since closure and non-confiscatority depend only on the allocations resulting on J. We now show that F^0 are Bayesian incentive compatible and Bayesian monotonic.

Take any $x \in \hat{F}$ and an equilibrium σ such that $g(\sigma(s)) = x(s)$ for all $s \in S$. Since $\hat{F}$ is Bayesian incentive compatible from the proof in Theorem 1, we have $x\, R_i(s_i)\, x \circ (\alpha_i, \tau_{-i})$ for all deceptions α_i. Since $U_i(x_i^0(s) + w_i, t) = U_i(0, t) = 0$ for all $s \in S \setminus J$, then x remains an equilibrium to $\langle M, x^0\rangle$ and $x\, R_i(s_i)\, x \circ (\alpha_i, \tau_{-i})$ implies that $x^0\, R_i(s_i)\, x^0 \circ (\alpha_i, \tau_{-i})$. Thus F^0 is Bayesian incentive compatible.

Take any $x \in \hat{F}$. Suppose that there is some deception α such that $x \circ \alpha$ is not equivalent to F^0. Let $\hat{x} \in \hat{F}$ be a social choice function which is equivalent to x^0. If $\hat{x} \circ \alpha$ is equivalent to $x^0 \circ \alpha$, then the existence of such i, s_i, and y follows directly from the fact $\hat{F}$ satisfies Bayesian monotonicity. So consider the case $\hat{x} \circ \alpha$ is not equivalent to $x^0 \circ \alpha$. From the definitions of $\hat{x}$ and x^0 it follows that $x^0(\alpha(s)) + w = 0 \neq \hat{x}(\alpha(s)) + w$ for some $s \in J$. Furthermore $x^0(\alpha(s')) + w = 0$ whenever $x^0(\alpha(s')) \neq \hat{x}(\alpha(s'))$ for some $s' \in S$. Hence by strict monotonicity of utility functions, there is $i \in N$ such that $\hat{x} \circ \alpha\, P_i(s_i)\, x^0 \circ \alpha$. Since σ is an equilibrium, it follows that $\hat{x}\, R_i\ (s_i')\ x^0 \circ (\bar{\alpha}_{s_i}, \tau_{-i})$ for all $s_i' \in S_i$. This implies that $x^0\, R_i(s_i')\, x^0 \circ (\bar{\alpha}_{s_i}, \tau_{-i})$ for all $s_i' \in S_i$. Thus, we establish the Bayesian monotonicity of F^0. Q.E.D.

When $J = S$, a non-diffuse information structure reduces to a diffuse information structure and thus any social choice set which is equivalent to F consists simply of F itself. So we have the following theorem which characterizes implementability of a social choice set with diffuse information structures.

Theorem 2 *In an exchange economic environment e in which $n \geqq 3$, information structure is diffuse, and utility functions $U_i : \mathbb{R}^l_+ \times T \to \mathbb{R}_+$ are strictly increasing in consumption for each type $t \in T$, a social choice set F is Bayesian implementable if and only if it satisfies closure, non-confiscatority, Bayesian incentive compatibility, and Bayesian monotonicity.*

4 Implementation in complete information

The results in the last section generalize and improve upon the existing results for Bayesian implementation in exchange economic environments and include them as special cases such as those of Postlewaite and Schmeidler [25], Palfrey and Srivastava [20], Mookherjee and Reichelstein [17], Jackson [13], and Hong [9] by allowing both preferences and feasible sets to be state dependent. Since complete information is a special case of incomplete information, we can use the above results to fully characterize Nash implementability in complete information economic environments for an arbitrary (finite or infinite) set A.[5]

As we remarked above, the concept of a social choice set in general differs from that of a social choice correspondence. However, when closure is satisfied and information is complete, the two definitions are equivalent. This is because, under complete information, every $\pi_i(s_i)$ is a singleton element set and thus closure yields that every selection from F is an element of F, so F can be written as a correspondence from S into A. In this case, Bayesian monotonicity reduces to Nash monotonicity for both preferences and endowments unknown to the designer and Bayesian implementation reduces to Nash implementation.

Definition 10 *A social choice correspondence $F : S \to 2^A$ is said to satisfy* **Nash monotonicity** *if for any $s \in S$ and $s' \in S$ with $w' \geqq w$, $x \in F(s)$ and $x \notin F(s')$, there exist $i \in N$ and $y \in A(w')$ such that*

(1) $U_i(y_i + w'_i, t') > U_i(x_i + w'_i), t'_i)$;
(2) $U_i(x_i + w_i, t) \geqq U_i(y_i + w_i, t)$ whenever $y \in A(w)$.

[5] Under complete information, every $\pi_i(s_i)$ consists of *singleton* element s, and thus one can easily see that, even for an infinite set of A, the conclusion of Theorem 1 is also true by checking the proof of Theorem 1.

Note that when endowments are known to the designer, $w' = w$ and thus the above definition reduces to the conventional Nash-Maskin monotonicity defined by Maskin [14].

Definition 11 *A social choice correspondence $F : S \to 2^A$ is said to be Nash implementable if there is a mechanism $\langle M, g\rangle$ such that $N_{M,g}(s) \neq \emptyset$ and $N_{M,g}(s) = F(s)$ for all $s \in S$. Here $N_{M,g}(s)$ is the set of all Nash equilibrium outcomes of the $\langle M, g\rangle$.*

Incentive compatibility has played a major role in the theory of implementation with incomplete information, but not when information is complete. The reason is that with complete information and three or more agents incentive compatibility is automatically satisfied (cf. Palfrey and Srivastava [24]).

HMP [12] considered Nash implementation of a social choice correspondence for complete information exchange economic environments when endowments and/or preference are unknown to the designer. When preferences are known but endowments are unknown to the designer, they show that for exchange economic environments with three or more agents and monotonically increasing preferences, individual rationality is sufficient, and in conjunction with the continuity of preferences, also necessary for a social choice correspondence to be Nash implementable. When endowments and preference are both unknown to the designer, they show that for exchange economic environments with three or more agents and monotonically increasing preferences, individual rationality and Nash-Maskin monotonicity are sufficient, and in conjunction with the continuity of preferences, also necessary for a social choice correspondence to be Nash implementable.

As in Theorem 1, we can similarly prove the following theorem which improves upon the result of HMP [12] by relaxing the continuity of utility functions and the individual rationality of a social choice correspondence.

Theorem 3 *In complete information exchange economic environments in which $n \geqq 3$ and utility functions $U_i : \mathbb{R}^l_+ \times T \to \mathbb{R}_+$ are strictly increasing in consumption for each $t \in T$, a social choice correspondence $F :\to 2^A$ is Nash implementable if and only if F is non-confiscatory and Nash monotonic.*

Thus, when endowments and preference are both unknown to the designer, the individual rationality condition imposed in Theorem 2.B.B of HMP [12] can be weakened to non-confiscatority, and the continuity condition becomes redundant. Also, when preferences are known but endowments are unknown to the designer, we have dropped the continuity condition and weakened individual rationality to non-confiscatority and Nash monotonicity. Indeed, from Theorem 1 of HMP [12] and Theorem 3 above, we know that, for exchange economic environments considered in Theorem 3 above, a social choice correspondence F is Nash monotonic if it is individually rational. This is because, by Theorem 1 of HMP [12], if individual rationality is satisfied, F is implementable. Then, by the necessity of Theorem 3, we know it is Nash monotonic.

5 Conclusion

In this paper we have extended the theory of Bayesian implementation to cover exchange economic environments in which both preferences and feasible sets are incomplete information. Thus our results stand in sharp contrast to previous results in the literature by allowing both preferences and feasible sets to be state dependent. We fully characterize Bayesian implementability by giving necessary and sufficient conditions for both diffuse and non-diffuse information structures when there are at least three individuals. We show that, in exchange economic environments where there are three or more individuals and preferences are monotonically increasing, a social choice set is Bayesian implementable if and only if closure, non-confiscatority, Bayesian monotonicity, and Bayesian incentive compatibility are satisfied. As a consequence, we also fully characterize Nash implementability by giving necessary and sufficient conditions. We show that the individual rationality condition imposed in Hurwicz, Maskin, and Postlewaite [12] can be weakened to non-confiscatority and the continuity condition becomes redundant for Nash implementation when endowments and preferences are both unknown to the designer. Thus our results generalize and improve upon the existing results about Nash and Bayesian implementation in exchange economic environments and include them as special cases.

However, there are a number of issues which have been ignored in the paper. One issue not addressed in the paper is to consider implementation in more general non-economic environments which have general feasible sets. This extension is not trivial since the general feasible sets have less mathematical structure. One has to impose some mathematical structure in the model so that feasible sets are comparable with each other. Recently, Hong [10] developed a feasible Bayesian implementation framework which is used to deal with such non-economic environments. She give necessary and sufficient conditions for a social choice set to be feasibly Bayesian implementable for environments with conflicting interests and a best element.

The second issue is how to extend our results to consider Bayesian implementability by not only a feasible but also continuous mechanism. Most characterization results in the Bayesian implementation literature have ignored this requirement. The characterization results just show what is possible for implementation of a general social choice set, but not what is realistic. Wettstein [37] and Tian [34] gave the feasible and continuous mechanisms for Bayesian implementation when individual endowments are complete information. These mechanisms may be modified to consider Bayesian implementation in incomplete information environments with both state dependent preferences and endowments.

The third issue not addressed in the paper is two-person Bayesian implementation. However, by using similar techniques in the paper, one may extend the results of Dutta and Sen [5] to the case where preferences and endowments are both unknown to the designer.

References

1. Abreu R, Matsushima H (1990) Virtual implementation in iteratively undominated strategies, mimeo
2. Abreu R, Sen A (1990) Subgame perfect implementation: a necessary and almost sufficient condition. Journal of Economic Theory 50: 285–299
3. Abreu R, Sen A (1991) Virtual implementation in Nash equilibrium. Econometrica 59: 997–1021
4. Danilov V (1992) Implementation via Nash equilibria. Econometrica 60: 43–56
5. Dutta B, Sen A (1994) 2-person Bayesian implementation. Economic Design 1: 41–54
6. Demski J, Sappington D (1984) Optimal incentive contracts with multiple agents. Journal of Economic Theory 33: 152–171
7. Duggan J (1993) Virtual implementation in Bayesian equilibrium with infinite types, Parts I and II, mimeo
8. Harsanyi J (1967, 1968) Games with incomplete information played by Bayesian players. Management Science 14: 159–182, 320–334, 486–502
9. Hong L (1994) Bayesian implementation in exchange economies with state dependent feasible sets, mimeo
10. Hong L (1994) Feasible Bayesian implementation with state dependent feasible sets. Social Choice and Welfare 13: 433–444
11. Hurwicz L (1972) On informationally decentralized systems. In: Radner R, Mcguire CS (eds) Decision and Organization, Volume in Honor of J. Marschak. North-Holland, Amsterdam, pp 297–336
12. Hurwicz L, Maskin E, Postlewaite A (1995) Feasible Nash implementation of social choice rules when the designer does not know endowments or production sets, In: J. Ledyard (ed) Essays in the Honor of S. Reiter. pp 367–433
13. Jackson MO (1991) Bayesian implementation. Econometrica 59: 461–477
14. Maskin E (1977) Nash Equilibrium and Welfare Optimality. Working Paper, October, (M.I.T.)
15. Matsushima H (1988) A new approach to the implementation problem. Journal of Economic Theory 45: 128–144
16. Matsushima H (1993) Bayesian monotonicity with side payments. Journal of Economic Theory 59: 107–121
17. Mookherjee D, Reichelstein S (1990) Implementation via augmented revelation mechanisms. Review of Economic Studies 57: 453–475
18. Moore J, Repullo R (1988) Subgame perfect implementation. Econometrica 56: 1191–1220
19. Moore J, Repullo R (1990) Nash implementation: a full characterization. Econometrica 56: 1083–1099
20. Palfrey T, Srivastava S (1987) On Bayesian implementable allocations. Review of Economic Studies 54: 193–208
21. Palfrey T, Srivastava S (1989a) Mechanism design with incomplete information: a solution to the implementation problem. Journal of Political Economy 97: 668–691
22. Palfrey T, Srivastava S (1989b) Implementation with incomplete information in exchange economies. Econometrica 57: 115–134
23. Palfrey T, Srivastava S (1991a) Nash implementation using undominated strategy. Econometrica 59: 479–502
24. Palfrey T, Srivastava S (1991b) Bayesian implementation, monograph, to appear in the series. Fundamentals of Pure and Applied Economics. Harwood, Yverdon
25. Postlewaite A, Schmeidler D (1986) Implementation in differential information economies. Journal of Economic Theory 39: 14–33

26. Postlewaite A, Wettstein D (1989) Continuous and feasible implementation. Review of Economic Studies 56: 603-611
27. Repullo R (1986) The revelation principle under complete and incomplete information, In: Binmore K, Dasgupta P (eds.) Economic Organization as Games. Basil Blackwell, Oxford
28. Repullo R (1987) A simple proof of Maskin's theorem on Nash implementation. Social Choice and Welfare 4: 39-41
29. Sajio T (1988) Strategy space reductions in Maskin's theorem: sufficient conditions for Nash implementation. Econometrica 56: 693–700
30. Tian G (1989) Implementation of the Lindahl correspondence by a single-valued, feasible, and continuous mechanism. Review of Economic Studies 56: 613–621
31. Tian G (1991) Implementation of Lindahl allocations with nontotal-nontransitive preferences. Journal of Public Economics 46: 247–259
32. Tian G (1993) Implementing Lindahl allocations by a withholding mechanism. Journal of Mathematical Economics 22: 169–179
33. Tian G (1994) Implementation of Linear cost share equilibrium allocations. Journal of Economic Theory 64: 568–584
34. Tian G (1996) Continuous and Feasible Implementation of Rational Expectation Lindahl Allocations. Games and Economic Behavior 16: 135–151
35. Tian G (1997) Characterizations of Virtual Implementation in Incomplete Information Environments with General Sets of Alternatives and Types. Journal of Mathematical Economics 28: 313–339
36. Tian G, Li Q (1995) On Nash-Implementation in the Presence of Withholding. Games and Economic Behavior 9: 222–233
37. Wettstein D (1992) Continuous Implementation in Economies with Incomplete Information. Games and Economic Behavior 4: 463–483