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Implementation in economies with non-convex production technologies unknown to the designer

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Abstract

This paper deals with the problem of incentive mechanism design in non-convex production economies when production sets and preferences both are unknown to the designer. We consider Nash-implementation of loss-free, average cost, marginal cost, voluntary trading, and quantity-taking pricing equilibrium allocations in economies involving increasing returns to scale or more general types of non-convexities. The mechanisms presented in the paper are well-behaved. They are feasible, continuous, and use finite dimensional message spaces. Moreover, the mechanisms work not only for three or more agents, but also for two-agent economies.

JEL classification: C72; D51; D61; D71

Keywords: Incentive mechanism design; Implementation; Various pricing equilibrium principles; Increasing returns; Well-behaved mechanism design

1. Introduction

This paper studies the problem of incentive mechanism design in economies with increasing returns to scale (IRS) or more general types of non-convexities. Since Walrasian equilibrium precludes the existence of an equilibrium in the presence of increasing returns to scale and assumes price-taking and profit-maximizing behavior, some other alternative equilibrium principles must be used.

The analysis of the strategic behavior of producers in the presence of increasing returns to scale is a fundamental question as indicated by Cornet (1988, p. 107). In fact, the incentive issue in the presence of non-convexities in production is more severe than in convex economies (because of the failure of markets). However, the results on non-convex production economies obtained so far in the literature are mainly negative. Calsamiglia (1977) showed that, for a certain class of economic environments involving non-convex production technologies, there is no mechanism that results in Pareto efficient allocations using a finite-dimensional message space. Because of this impossibility result,

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a general belief is that it is impossible to have an incentive mechanism with a finite-dimensional message space that leads to Pareto efficient allocations on a relatively rich class of economies involving increasing returns to scale.

Hurwicz et al. (1995) considered incentive mechanism design for general production economic environments that include non-convex production technologies. Like Calsamiglia's result, their mechanism also has an infinite-dimensional message space. Moreover, due to the general nature of social choice rules, their mechanism turns out to be quite complex. Characterization results only show what is possible for implementation of a social choice rule but not what is realistic. Like most characterization results in the literature, Hurwicz–Maskin–Postlewaite's mechanism is not continuous. The "better" mechanism design, which requires some desired properties such as continuity, feasibility, and lower dimensionality, has mainly been done only for exchange economies or convex production economies such as those in Groves and Ledyard (1977), Hurwicz (1979), Schmeidler (1980), Hurwicz et al. (1995), Postlewaite and Wettstein (1989), Tian (1989, 1992a, 1992b, 1996, 1999), Hong (1995), Peleg (1996), and Duggan (2003), among others. Tian (2005) considered this mechanism design for non-convex production economies. However, the mechanism is obtained by assuming that production technologies are known to the designer.

This paper investigates implementation of the often considered pricing equilibrium principles for general variable returns when both preferences and production technologies are unknown to the designer. We present mechanisms that implement loss-free, average cost, marginal (cost), and voluntary trading pricing equilibrium allocations in Nash equilibrium for general economies involving non-convex production technologies. Since a voluntary trading pricing equilibrium reduces to competitive equilibrium with quantity-taking producers under strict monotonicity of preferences, the mechanism that fully implements voluntary trading pricing equilibrium allocations also fully implements competitive equilibrium allocations with quantity-taking producers.

The mechanisms have a number of desired properties. They are feasible and continuous (except in the case of the voluntary trading pricing principle). In addition, they are price-quantity market type mechanisms, and use finite-dimensional message spaces. In fact, they use smaller message spaces than the existing mechanisms that implement Walrasian allocations for economies with more than two agents. Also, our mechanisms work not only for three or more agents but also for two-agent economies. Thus, they are unified mechanisms that are irrespective to the number of agents.

It is well known by now that in order to guarantee feasibility even at disequilibrium, the designer must have some information about production sets, but does not necessarily need to know firms' true production sets.[1] The existing mechanisms, such as those in Hong (1995) and Tian (1999), require that all individual agents report price vectors and production plans of firms. Not only does this requirement result in a high-dimensional message space, but it is also highly unrealistic, since not everyone in the real world knows the prices of all commodities and the production technologies of all firms.

On the contrary, the mechanisms constructed in this paper require only two agents to announce prices of commodities and production plans for each firm, one of whom knows the firm's true production set. The person who is asked to announce production plans from the firm's production set can be regarded as the manager or CEO of the firm, and the other may be regarded as the owner or a group of investors who may not be involved directly in production, but may oversee or govern the manager in her production activities. Thus, our mechanisms not only use smaller message spaces than those in Hurwicz et al. (1995), Hong (1995), and Tian (1999), but they may be more realistic as well.

The pricing rules under consideration do not allow profit maximization for non-convex production technologies. In fact, there is no production optimization problem involved for loss-free or average cost pricing rules. In light of this, the implementation problem here is more difficult than the one for the conventional Walrasian solution. For instance, the managers should have special incentives to produce efficiently in equilibrium, but at the same time, they do not have incentives to pursue the goal of profit maximization, which may result in the non-existence of pricing equilibrium in the presence of increasing returns.

In addition, different incentive schemes should be used so that firms follow different pricing rules. For instance, when dealing with implementation of loss-free or average cost pricing equilibrium allocations, we need to construct a mechanism such that each individual should have no incentives to move from one efficient production plan to another. When dealing with implementation of voluntary trading or marginal cost pricing equilibrium allocations, we need a mechanism in which a firm's manager chooses an optimal production plan at equilibrium for a given pricing rule. As

[1] It is required only one agent knows the true production set for each firm.

such, some of the techniques used in implementing market-like-equilibrium solutions, such as Walrasian equilibrium for private goods economies or Lindahl equilibrium for public goods economies, may not be used here. As a result, we develop some new techniques for constructing a system of incentives for firms to follow a pre-specified pricing rule. The method of constructing the incentive schemes may shed some light on how to design incentive schemes for other problems, even a real world problem.

To see the main forces why our mechanisms work even for non-convex production economies without imposing profit-maximizing and price-taking behavior assumptions, consider a simple economy with one firm and several consumers. Consumer 1 is the CEO of the firm who has full information about the production set, while consumer 2 is the owner of the firm who has limited information about production. Since the designer does not know consumers' preferences, she must present consumers with a system of incentives to maximize their preferences, subject to the budget constraints. Furthermore, since the designer does not know the true production set, she must present the owner and CEO of the firm with a system of incentives to pursue the pricing rule that may not maximize the profit or any optimal goal, except efficiency of production.

The intuition behind the construction of such an incentive mechanism can be briefly described as follows. Each consumer is asked to announce a consumption bundle and a shrinking index that is used to shrink the consumption of other agents in order to get feasible allocations. In addition, consumer 1, as the CEO of the firm, and consumer 2, as the owner of the firm, are required to report the information about pricing and production. Consumer 1 is asked to set up the prices of products and report two or three production plans (depending on pricing rules to be implemented) that will be used to determine feasible allocations and efficient production plans. Consumer 2 is asked to announce a price vector and a production plan that will be used to match the prices and production plans announced by the manager, and to induce the efficient production plan and pricing rule. According to the messages announced by all the individuals, the price vector and the production plan used by the manager is determined by the owner, while the prices and production plans of all others are determined by the manager. Thus, each individual takes the prices of goods and her income as given and cannot change her budget set by changing her own messages. Define a feasible consumption set for each consumer, and a set of production plans that either satisfy a pre-specified pricing rule or is used to construct the pricing rule, depending on which pricing rule is to be implemented. Each consumer's affordable consumption bundle is chosen from her feasible consumption set that is closest to her announced consumption bundle.

To give the manager an incentive to produce efficiently and follow a pre-specified pricing rule, two specific compensation schemes are provided respectively to her, by way of a positive amount of compensation if she can produce more efficiently or more profitably than the proposed production plan by the owner. To give the owner an incentive to agree with the production plan announced by the manager, the owner's affordable consumption outcome from her budget constraint is discounted by the difference of the two production plans announced by them. In addition, to give the owner an incentive to match the price vector announced by the CEO, the owner's affordable consumption outcome is also discounted by the difference of the price vectors announced by them. Finally, to obtain the feasible consumption outcome, we shrink the discounted preliminary consumption bundles in a certain way. The mechanism constructed in such a way has the following desired properties: all individuals take the prices as given and maximize their preferences subject their budget constraints, and the firm follows the pricing rule at equilibrium. Thus, the mechanism implements the pre-specified pricing equilibrium allocations.

From the above description of the mechanism, one can see that the mechanism is more realistic to some extent and may be used in the real world for economies with non-convex production. This may provide a partial response to the criticism that most mechanisms in the implementation literature are highly unrealistic or only for convex production economies. The manner of constructing the incentive schemes to induce firms to follow a pre-specified pricing rule may shed some light on how to design incentive schemes for other economic problems, and may even be used to solve real incentive problems in business firms.

The remainder of this paper is structured as follows. Section 2 presents the general setup, introduces often considered pricing principles, and gives related definitions on economic mechanism design. Section 3 presents specific mechanisms for these pricing principles. Section 4 proves that the mechanisms fully Nash-implement these social choice rules. Section 5 concludes. The appendix provides all lemmas and their proofs.

2. The setup

2.1. Economic environments

We consider production economies with L commodities, $n \geqq 2$ consumers, and J firms.[2] Let $N = \{1, 2, \ldots, n\}$ denote the set of consumers. Each agent's characteristic is denoted by $e_i = (C_i, w_i, R_i)$, where $C_i = \mathbb{R}^L_+$ is the consumption set, $w_i \in \mathbb{R}^L_{++}$ is the initial endowment vector of commodities, and R_i is the preference ordering defined on $\mathbb{R}^L_+$. Let P_i denote the asymmetric part of R_i (i.e., $a\ P_i\ b$ if and only if $a\ R_i\ b$, but not $b\ R_i\ a$). We assume that R_i is continuous and convex on $\mathbb{R}^L_+$, and strictly monotonically increasing on $\mathbb{R}^L_{++}$.[3]

Production technologies of firms are denoted by $\mathcal{Y}_1, \ldots, \mathcal{Y}_j, \ldots, \mathcal{Y}_J$. We assume that, for $j = 1, \ldots, J$, $\mathcal{Y}_j$ is closed, contains 0 (possibility of inaction), and $\{\mathcal{Y}_j - \mathbb{R}^L_+\} \subseteq \mathcal{Y}_j$ (free-disposal). It is important to note that, under these assumptions, $\partial\mathcal{Y}_j$, the boundary of the production set $\mathcal{Y}_j$, is exactly the set of (weakly) efficient production plans of the jth producer, that is,

$$\partial\mathcal{Y}_j = \{y_j \in \mathcal{Y}_j\colon\ \nexists z_j \in \mathcal{Y}_j,\ z_j > y_j\}.$$

An economy is the full vector $e = (e_1, \ldots, e_n, \mathcal{Y}_1, \ldots, \mathcal{Y}_J)$ and the set of all such economies is denoted by E which is assumed to be endowed with the product topology.

An allocation of the economy e is a vector $(x_1, \ldots, x_n, y_1, \ldots, y_J) \in \mathbb{R}^{L(n+J)}$. An allocation x is *individually feasible* if (1) $x_i \in \mathbb{R}^L_+$ for all $i = 1, \ldots, n$, and (2) $y_j \in \mathcal{Y}_j$ for $j = 1, 2, \ldots, J$. Denote by $y = (y_1, \ldots, y_J)$ the profile of production plans of firms.

An allocation (x, y) is *feasible* if it is individually feasible and satisfies

$$\sum_{i=1}^{n} x_i \leqq \sum_{i=1}^{n} w_i + \sum_{j=1}^{J} y_j. \tag{1}$$

Denote the aggregate endowment, consumption, and production by $\hat{w} = \sum_{i=1}^n w_i$, $\hat{x} = \sum_{i=1}^n x_i$, and $\hat{y} = \sum_{j=1}^J y_j$, respectively.[4] Then the feasibility condition can be written as

$$\hat{x} \leqq \hat{w} + \hat{y}.$$

An allocation (x, y) is *Pareto-optimal* for an economy e if it is feasible and there is no other feasible allocation (x', y') such that $x'_i\ R_i x_i$ for all $i \in N$ and $x'_i\ P_i x_i$ for some $i \in N$.

2.2. Various pricing principles

It is generally recognized that the standard behavioral assumption of profit maximization may be inapplicable in the presence of increasing returns to scale, and thus one needs to adopt alternative rules of firms' behavior.

The behavior of the jth producer is described by its supply correspondence, η_j, which associates with each normalized price vector in $p \in \mathbb{R}^L_+$ a subset of $\eta_j(p)$. The production equilibrium condition of the jth producer at the pair $(y_j, p) \in \partial\mathcal{Y}_j \times \mathbb{R}^L_+$ is defined by $y \in \eta_j(p)$. Note that this assumes the behavior of a firm embodies the normative requirement that its production is efficient. This condition may not be satisfied in models of imperfect competition, as presented in Arrow and Hahn (1971). Since $\eta_j(p)$ is in general not convex in the presence of increasing returns, there is an alterative approach to describing the behavior of production with the notion of a pricing rule which is the inverse correspondence of η_j. In the absence of convexity assumptions, a pricing rule correspondence may satisfy continuity and convex valuedness properties even when the supply correspondence does not. This is the main reason for which most papers on the existence of equilibrium for economies with non-convex production technologies use the notion of a pricing rule, although they are equivalent.

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[2] As usual, vector inequalities, $\geqq$, $\geq$, and $>$, are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \ldots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \ldots, m$.

[3] R_i is convex if, for any pair of consumption bundles a and b, $a\ P_i\ b$ implies $\lambda a + (1-\lambda) b\ P_i\ b$ for all $0 < \lambda \leqq 1$. Note that the term "convex" is defined as in Debreu (1995), not as in some recent textbooks.

[4] For notational convenience, "$\hat{a}$" will be used throughout the paper to denote the sum of vectors a_i, i.e., $\hat{a} = \sum a_l$.

A pricing rule for firm j is thus a correspondence $\phi_j : \partial\mathcal{Y}_j \to\to \mathbb{R}^L_+$ defined by $\phi_j(y_j) = \{p \in \mathbb{R}^L_+\colon\ y_j \in \eta_j(p)\}$, satisfying the following homogeneity assumption: $\phi_j(y_j)$ is a cone with vertex 0. The jth firm is said to follow the pricing rule ϕ_j at the pair of a price system $p \in \mathbb{R}^L_+$ and an efficient production plan $y_j \in \partial\mathcal{Y}_j$ if $p \in \phi_j(y_j)$. The pair (p, y_j) is then called a production equilibrium.

An advantage of the above notion of production equilibrium is that it is flexible enough to allow prices or quantities as exogenous variables or endogenous variables for producers, and includes the class of price-taking behaviors, as in the Walrasian model. It also allows price-setting behavior such as the models of Dicker et al. (1985), and Dehez and Dréze (1988). The most often considered pricing rules in the literature include the following:

(1) The profit maximizing rule: $PM_j(p) = \{y_j \in \mathcal{Y}_j\colon\ p \cdot y_j \geqq p \cdot y'_j \text{ for all } y'_j \in \mathcal{Y}_j\}$.
(2) The marginal (cost) pricing rule: $MC_j(y_j) = N_{\mathcal{Y}_j}(y_j)$, where $N_{\mathcal{Y}_j}(y_j)$ is Clarke's normal cone to $\mathcal{Y}_j$ which is a generalization of the notion of the marginal rate of transformation for the producer in the absence of smoothness and convexity assumptions (cf. Clarke, 1975).[5]
(3) The average cost pricing rule: $AC_j(y_j) = \{p \in \mathbb{R}^L_+\colon\ p \cdot y_j = 0\}$.
(4) The loss-free pricing rule: $LF_j(y_j) = \{p \in \mathbb{R}^L_+\colon\ p \in \phi_j(y_j) \text{ implies } p \cdot y_j \geqq 0\}$, i.e., the loss-free pricing rule of the producers precludes negative profits.
(5) The voluntary trading pricing rule: $VT_j(y_j) = \{p \in \mathbb{R}^L_+\colon\ p \cdot y_j \geqq p \cdot y'_j \text{ for all } y'_j \in \mathcal{Y}_j \text{ such that } y'_j \leqq y^+_j\}$, where y^+_j denotes the vector in $\mathbb{R}^L$ with coordinates $\max\{0, y^l_j\}$ for $l = 1, \ldots, L$. Note that the voluntary trading rule implies, in particular, cost minimization.
(6) The quantity-taking pricing rule: $QT_j(y_j) = \{p \in VT_j(y_j)\colon\ \nexists p' \in VT_j(y_j),\ p' \leq p,\ p'^l = p^l \text{ for all } l \in I(y_j)\}$, where $I(y_j) = \{l\colon\ y^l_j < 0 \text{ or } y'_j \leqq 0 \text{ for all } y'_j \in \mathcal{Y}_j\}$. That is, a quantity-taking pricing rule satisfies two conditions: (1) it is a voluntary trading pricing rule, and (2) minimality of the output prices: lower output prices will not sustain the same output quantities. As shown by Dehez and Dréze (1988), the quantity-taking pricing rule reduces to the profit maximizing pricing rule when the production set is convex.

It can be verified that the above listed pricing rules have closed graphs and that, for all $y_j \in \partial\mathcal{Y}_j$, they are closed convex cones with vertex 0.

The ith consumer's wealth function is a function $(w_i, p, \pi) \to r_i(w_i, p, \pi)$ on $\mathbb{R}^L_+ \times \mathbb{R}^L_+ \times \mathbb{R}^J$, where $\pi = (\pi_1, \ldots, \pi_J)$ is the profile of profit/loss functions with $\pi_j\colon \mathbb{R}^L_+ \times \mathcal{Y}_j \to \mathbb{R}$ which can be a linear pricing function or a nonlinear pricing function. This abstract wealth structure clearly encompasses the case of a private ownership economy so that $r_i(w, p, p \cdot y_1, \ldots, p \cdot y_J) = p \cdot w_i + \sum_{i=1}^n \theta_{ij} p \cdot y_j$, where $\theta_{ij} \in \mathbb{R}_+$ are the profit shares of private firms j, $j = 1, \ldots, J$, satisfying $\sum_{i=1}^n \theta_{ij} = 1$. It is assumed that $r_i(\cdot)$ is increasing in w_i, continuous, $\sum_{i=1}^n r_i(w_i, p, \pi) = \sum_{i=1}^n p \cdot w_i + \sum_{j=1}^J \pi_j(p, y_j)$, $r_i(w_i, tp, t\pi) = t r_i(w_i, p, \pi)$ for all $t > 0$, and $\sum_{i=1}^n p \cdot w_i + \sum_{j=1}^J \pi_j(p, y_j) > 0$ implies that $r_i(w_i, p, \pi) > 0$.

A pricing economy then is referred to

$$e = \left(e_1, e_2, \ldots, e_n, \{Y_j\}_{j=1}^J, \left\{\phi_j(y_j)\right\}_{j=1}^J, \{r_i\}_{i=1}^n\right). \tag{2}$$

The set of all such pricing economies are denoted by E.[6]

A pricing equilibrium of a pricing economy e is then a list of consumption bundles (x^*_i), a list of production plans (y^*_j), and a price system p^* such that (a) every consumer maximizes his preferences subject to his budget constraint, (b) every producer follows her pricing rule, i.e., $p^* \in \phi_j(y^*_j)$ for $j = 1, \ldots, J$, and (c) the aggregate demand over supply is non-positive. The nature of the equilibrium condition (b) is the main difference from the Walrasian model. Formally, we have the following definition.

An allocation $z^* = (x^*, y^*) = (x^*_1, x^*_2, \ldots, x^*_n, y^*_1, y^*_2, \ldots, y^*_J) \in \mathbb{R}^{nL}_+ \times \mathcal{Y}$ is a *pricing equilibrium allocation* for a pricing economy e if it is feasible and there is a price system $p^* \in \mathbb{R}^L_+$ such that

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[5] The formal definition of the Clarke normal cone is defined as follows. First, we need the notion of the Clarke tangent cone. For a non-empty set $Y \subseteq \mathbb{R}^L$ and $y \in Y$, the tangent cone of Y is given by $T_Y(y) = \{x \in \mathbb{R}^L\colon$ for every sequence $y^k \in Y$ with $y^k \to y$ and every sequence $t^k \in (0, \infty)$ with $t^k \to 0$, there exists a sequence $x^k \in \mathbb{R}^L$ with $x^k \to x$ such that $y^k + t^k x^k \in Y$ for all $k\}$. The Clarke normal cone is then given by $N_Y(y) = \{x \in \mathbb{R}^L\colon\ (z, x) \leqq 0\ \forall z \in T_Y(y)\}$.

[6] Here we use the same e to denote a pricing economy if there is no confusion.

(1) $p^* \cdot x_i^* \leqq p^* \cdot r_i(w_i, p^*, \pi(p^*, y^*))$ for all $i \in N$;
(2) for all $i \in N$, $x_i \; P_i \; x_i^*$ implies $p^* \cdot x_i > r_i(w_i, p^*, \pi(p^*, y^*))$; and
(3) $y_j^* \in \partial \mathcal{Y}_j$ and $p^* \in \phi_j(y_j^*)$ for $j = 1, \dots, J$.

The detailed discussions on the settings of the model in economies with increasing returns and the existence of pricing equilibrium for the general setting can be found in Beato (1982), Brown and Heal (1982), Cornet (1988, 1989), Bonnisseau (1988), Bonnisseau and Cornet (1988, 1990), Kamiya (1988), Vohra (1988), Brown (1990), and Brown et al. (1992).

From the above homogeneity assumption of r_i and ϕ_j and the monotonicity of preferences, we can assume that equilibrium price systems p^* belong to the $L-1$ dimensional unit simplex $\Delta_+^{L-1} = \{t \in \mathbb{R}_+^n \colon \sum_{i=1}^L t_i = 1\}$.

When a pricing rule is given by the profit maximizing, marginal cost, average cost, loss-free, voluntary, or quantity-taking pricing rules, the resulting pricing equilibrium is called the competitive equilibrium (Walrasian equilibrium), the marginal cost pricing equilibrium, the average cost pricing equilibrium, the loss-free pricing equilibrium, the voluntary trading pricing equilibrium, or the quantity-taking pricing equilibrium, respectively. Denote by $W(e)$, $MCP(e)$, $ACP(e)$, $LFP(e)$, $VTP(e)$ and $QTP(e)$, the set of all such pricing equilibrium allocations, respectively. To consider incentive mechanism design for these equilibrium principles, we need to make the following indispensable assumption.

Assumption 1 *(Interiority of Preferences).* For all $i \in N$, $x_i \; P_i \; x_i'$ for all $x_i \in \mathbb{R}_{++}^L$ and $x_i' \in \partial \mathbb{R}_+^L$, where $\partial \mathbb{R}_+^L$ is the boundary of $\mathbb{R}_+^L$.

Remark 1. Assumption 1 cannot be relaxed. Like the Walrasian correspondence, one can show that a pricing equilibrium correspondence violates Maskin's (1999) monotonicity condition on the boundary of the consumption space, and thus cannot be Nash implemented by a feasible mechanism. However, if we modify the pricing equilibrium principle to the constrained pricing equilibrium principle that is obtained by changing Condition 2 in the pricing equilibrium allocation to the following:

Condition (2′): for all $i \in N$, $x_i \; P_i \; x_i^*$ implies either $p^* \cdot x_i > r_i(w_i, p^*, \pi(p^*, y^*))$ or $x_i > \sum_{i=1}^n w_i + \sum_{j=1}^n y_j^*$,

then the constrained pricing equilibrium correspondence satisfies Maskin's monotonicity condition, and it can be shown that the mechanisms presented below implement constrained loss-free, average cost, marginal cost, and voluntary trade pricing equilibrium allocations, respectively, without Assumption 1. It is clear that every pricing equilibrium allocation is a constrained pricing equilibrium allocation, and the converse may not be true. However, it can be shown that the constrained pricing equilibrium correspondence coincides with the pricing equilibrium correspondence for interior allocations.

Remark 2. The family of Cobb–Douglas utility functions satisfies Assumption 1.

2.3. Notation and definitions for mechanism design

Let $F : E \to\to \mathbb{R}_+^{L(n+J)}$ be a social choice correspondence. Let M_i denote the ith agent's message space. Its elements are written as m_i and called messages. Let $M = \prod_{i=1}^n M_i$ denote the message space which is endowed with the product topology. Denote by $h : M \to \mathbb{R}_+^{L(n+J)}$ the outcome function, or more explicitly, $h(m) = (X_1(m), \dots, X_n(m), Y_1(m), \dots, Y_J(m))$. Then a mechanism, which is defined on E, consists of a message space M and an outcome function. It is denoted by $\langle M, h \rangle$.

A message $m^* = (m_1^*, \dots, m_n^*) \in M$ is said to be a *Nash equilibrium* of a mechanism $\langle M, h \rangle$ for an economy e if, for all $i \in N$ and $m_i \in M_i$,

$$X_i(m^*) \; R_i \; X_i\bigl(m_i, m_{-i}^*\bigr), \tag{3}$$

where $(m_i, m_{-i}^*) = (m_1^*, \dots, m_{i-1}^*, m_i, m_{i+1}^*, \dots, m_n^*)$. The outcome $h(m^*)$ is then called a *Nash (equilibrium) allocation* of the mechanism for economy e. Denote by $V_{M,h}(e)$ the set of all such Nash equilibria and by $N_{M,h}(e)$ the set of all such Nash equilibrium allocations.

A mechanism $\langle M, h\rangle$ is said to *Nash-implement* a social choice correspondence F on E, if, for all $e \in E$, $N_{M,h}(e) \subseteq F(e)$. It is said to *fully Nash-implement* a social choice correspondence F on E, if, for all $e \in E$, $N_{M,h}(e) = F(e)$.

A mechanism $\langle M, h\rangle$ is said to be *continuous*, if the outcome function h is continuous on M.

A mechanism $\langle M, h\rangle$ is said to be *feasible*, if for all $m \in M$, (1) $X(m) \in \mathbb{R}^{nL}_{+}$, (2) $Y_j(m) \in \mathcal{Y}_j$ for $j = 1, \ldots, J$, and (3) $\sum_{i=1}^{n} X_i(m) \leqq \sum_{i=1}^{n} w_i + \sum_{j=1}^{J} Y_j(m)$.

In the remainder of the paper, we assume that social choice correspondences are given by loss-free, average cost, marginal cost, and voluntary trading/quantity-taking pricing equilibrium allocations.[7]

3. Mechanisms

In this section we present specific mechanisms that have the desired properties and fully implement loss-free, average cost, marginal cost, and voluntary trading/quantity-taking pricing equilibrium allocations for general non-convex production economies.

Although we need to construct different incentive schemes for firms to follow different pricing rules, all the mechanisms constructed below are basically the same in structure and differ only in the award to the agents, who are the managers of firms. Thus we provide a detailed presentation for just one mechanism. We then outline the modifications for each of the other equilibrium principles. We will incorporate the implementation results on these mechanisms into one theorem and prove them in the next section.

In the mechanisms constructed below, we only require two agents to announce production plans for each firm j. The designer can identify one agent, denoted by $\mu(j)$, as the manager of the firm. The other person, denoted by $\kappa(j)$, can be regarded as the owner of the firm, who may not be involved directly in production, but may oversee or govern the manager in her production activities. Agent $\mu(j)$, as the manager, knows the production set and is asked to announce production plans from firm j's production set while agent $\kappa(j)$, as the owner, may or may not know the firm's production. The mechanisms also only require two agents, denoted by ζ and ι respectively, to announce price vectors, denoted by p_ζ and p_ι.

To see this, consider the following example of a simple economy with three consumers and two firms. Suppose consumer 2 and consumer 3 are the manager and owner of firm 1, and consumer 3 and consumer 1 are the manager and owner of firm 2, respectively. We then have $\mu(1) = 2$ and $\kappa(1) = 3$ and $\mu(2) = 3$ and $\kappa(2) = 1$. Suppose the prices of goods are determined by consumer 1 and 3 so that $p_\zeta = p_1$ and $p_\iota = p_3$. We want the mechanism to implement marginal cost pricing allocations.

Then, the price vector $p_i(m)$ and production profile $Y_i(m) = (Y_{i1}(m), Y_{i2}(m))$ used in agent i's budget set is proposed by another agent. For instance, for agent 3, we can let $p_3(m)$ be determined by agent 1, and production profile $Y_3(m) = (Y_{31}(m), Y_{32}(m))$ be determined by agent 2. Thus, each agent takes the prices and her income as given and cannot change the budget set by changing her own messages. The feasible production plan for each firm is announced by the manager of the firm.

Let each agent's consumption outcome $X_i(m)$ be positively proportional to her preliminary consumption bundle $x_i(m)$ (to be specified in Section 3.4), which in turn is proportional to her affordable consumption bundle $x_i'(m)$. In the economy specified in this example, $x_1(m)$ is negatively related to $\|p_1 - p_3\|$ and $\|y_{12} - y_{32}\|$, by noting that agent 1 announces price vector p_1 and is the owner of firm 2; $x_2(m)$ is positively related to the compensation (denoted by ν_{21}) that induces firm 1 to produce efficiently and the compensation (denoted by τ_{21}) that induces firm 1 to follow the marginal pricing rule at equilibrium, by noting that agent 2 is the manager of firm 1. Similarly, $x_3(m)$ is negatively related to $\|y_{31} - y_{21}\|$ and positively related to ν_{32} and τ_{32}, by noting that agent 3 announces a price vector p_3 and is the owner of firm 1 and the manager of firm 2.

From the above example, one can see that any agent can be the owner or manager of a firm, and different firms can have different agents as their owners or managers. However, the analysis on the mechanism becomes complicated in expression when the number of agents and firms become large. Thus, to simplify the exposition, without loss of generality, it is assumed that $\mu(j) = 1$ and $\kappa(j) = 2$ for all j, i.e., agent 1 is the manager of all firms and consumer 2 is the owner of all firms, and further agents 1 and 2 also announce the prices of commodities.

[7] Under assumptions imposed in the paper, the set of voluntary trading pricing equilibrium allocations and competitive equilibrium allocations with quantity-taking producers coincide when prices are normalized in the simplex price space.

3.1. Mechanism for loss-free pricing principle

3.1.1. Loss-free pricing equilibrium

The notion of the loss-free pricing rule is proposed to eliminate a disadvantage of marginal cost pricing rule by which a firm may suffer losses at equilibrium. A loss-free pricing equilibrium is a family of consumption bundles, production plans, and prices such that households are maximizing utility subject to their budget constraints, firms' production plans guarantee nonnegative profits, and all markets clear.

For the loss-free pricing rule, one typically considers a private ownership economy so that consumer i's wealth function is $r_i(w, p, p \cdot y_1, \dots, p \cdot y_J) = p \cdot w_i + \sum_{i=1}^n \theta_{ij} p \cdot y_j$, which includes the profit maximizing rule for production sets containing the null production and the average cost pricing rule as special cases. But our implementation result holds for general forms of wealth functions.

3.1.2. The description of the mechanism

The design for an incentive mechanism that (fully) implements loss-free pricing equilibrium allocations is different from those that implement Walrasian allocations or more general marginal cost pricing equilibrium allocations since the loss-free pricing rule does not involve any type of profit maximization or cost minimization in production. We then need to require that the wealth functions $r_i(\cdot)$ of consumers be independent of their own announced production plans of firms. On the other hand, we should give the consumers who are the managers of firms certain incentives so that efficient productions are realized at equilibrium.

The mechanism is defined as follows:

For each $i \in N$, the message space of agent i is

$$M_i = \begin{cases} G_1 \times \mathbb{R}^{JL} \times \mathbb{R}^L_+ \times \mathbb{R}_{++} \times (0,1] & \text{if } i = 1, \\ \Delta^{L-1}_{++} \times \mathbb{R}^{JL} \times \mathbb{R}^L_+ \times \mathbb{R}_{++} & \text{if } i = 2, \\ \mathbb{R}^L_+ \times \mathbb{R}_{++} & \text{if } i \geqq 3, \end{cases}$$

where

$$G_1 = \left\{ (p, y) \in \Delta^{L-1}_{++} \times \mathcal{Y}\colon \sum_{i=1}^n w_i + \sum_{j=1}^J y_j > 0 \ \&\ p \cdot y_j \geqq 0\ \forall j \right\} \tag{4}$$

is the set of price vectors and production profiles such that the aggregate supplies for all goods are positive and the proposed price vectors satisfy the loss-free pricing rule. G_1 is clearly non-empty since $(p, 0) \in G_1$ for any $p \in \Delta^{L-1}_{++}$.

A generic element of M_i is denoted by $m_i = (p_i, y_i, q_i, z_i, \gamma_i, \eta_i)$ for $i = 1$, $m_i = (p_i, y_i, z_i, \gamma_i)$ for $i = 2$, and $m_i = (z_i, \gamma_i)$ for $i = 3, \dots, n$, whose components have the following interpretations. The component p_i is the price vector proposed by agent i ($i = 1, 2$) and is used as a price vector by the other agents $k \neq i$. The components $y_1 = (y_{11}, \dots, y_{1J})$ and $q_1 = (q_{11}, \dots, q_{1J})$ are production profiles announced by agent 1 who works as the manager (CEO) of firms, and will be used to determine feasible allocations and efficient production profiles, respectively. The component $y_2 = (y_{21}, \dots, y_{2J})$ is a production profile announced by agent 2, and will be used to induce the efficient production profile. The component z_i is a consumption bundle proposed by agent i. The component γ_i is a shrinking index of agent i used to shrink the consumption of other agents so that the feasibility of allocation can be guaranteed.

The feasible production profile outcome function $Y : M \to \mathcal{Y}$ is defined by

$$Y(m) = y_1, \tag{5}$$

which is proposed by agent 1, the manager of firms.

The price and production profile outcome functions $(p_i, Y_i) : \to \Delta^{L-1}_{++} \times \mathbb{R}^{JL}$, which are used to define agent i's budget constraint and wealth function, are respectively determined by

$$p_i(m) = \begin{cases} p_2 & \text{if } i = 1, \\ p_1 & \text{otherwise,} \end{cases}$$

and

$$Y_i(m) = \begin{cases} y_2 & \text{if } i = 1, \\ y_1 & \text{otherwise.} \end{cases}$$

Thus, the manager of firms sets the prices and production profile for all other consumers except for herself while the owner of firms sets the prices and production profile for the manager. These will be used to determined consumers' budget constraints. This construction ensures each agent's wealth is independent of his own proposed production profiles and price vectors so that every individual takes prices and production profiles as given. Note that, by the construction of the message space, $(p_1, y_1) \in G_1$ so that we have $\hat{w} + \sum_{j=1}^{J} Y_j(m) > 0$ and $p_i(m) \cdot Y(m) \geq 0$ for $i \neq 1$ so that the loss-free pricing rule is satisfied.

Agent i's outcome function for consumption bundle $X_i : M \to \mathbb{R}^L_+$ is given by

$$X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m). \tag{6}$$

Here $\bar{\gamma}(m)$ is the largest element of $A(m)$ defined by

$$A(m) = \left\{ \gamma \in \mathbb{R}_+ \colon \gamma\gamma_i \leq 1 \ \forall i \in N \ \& \ \gamma \sum_{i=1}^{n} \gamma_i x_i(m) \leqq \sum_{i=1}^{n} w_i + \sum_{j=1}^{J} Y_j(m) \right\}, \tag{7}$$

i.e., $\bar{\gamma}(m) \in A(m)$ and $\bar{\gamma}(m) \geq \gamma$ for all $\gamma \in A(m)$, and $x_i(m)$ is agent i's preliminary consumption defined by

$$x_i(m) = \begin{cases} [\nu_1(m) + \frac{1}{1+\eta_1\|q_1 - y_2\|}]x'_1(m) & \text{if } i = 1, \\ \frac{1}{1+\|p_1 - p_2\| + \|y_1 - y_2\|} x'_2(m) & \text{if } i = 2, \\ x'_i(m) & \text{if } i \geqq 3, \end{cases}$$

where

$$\nu_1(m) = \sum_{j=1}^{J} \nu_{1j}(m) \tag{8}$$

with

$$\nu_{1j}(m) = \prod_{l=1}^{L} \max\{0, q^l_{1j} - y^l_{2j}\}, \tag{9}$$

and

$$x'_i(m) = \{x_i \colon \min_{x_i \in B_i(m)} \|x_i - z_i\|\} \tag{10}$$

with

$$B_i(m) = \left\{ x_i \in \mathbb{R}^L_+ \colon \ p_i(m) \cdot x_i \leqq r_i\big(w_i, p_i(m), \pi\big(p_i(m), Y_i(m)\big)\big) \ \& \ x_i \leqq \sum_{i=1}^{n} w_i + \sum_{j=1}^{J} Y_j(m) \right\}. \tag{11}$$

Thus, $x'_i(m) \in B_i(m)$ is an affordable consumption that is the closest to z_i. Note that, since the feasible consumption correspondence $B_i : M \to\to \mathbb{R}^L_+$ is a continuous correspondence with non-empty, compact, and convex values, the affordable consumption $x'_i \colon M \to B_i$ is a well-defined single-valued continuous function on M by the Maximum Theorem.

Thus the outcome function $(X(m), Y(m))$ resulting from the strategic configuration m is continuous and feasible since, by the construction of the mechanism, $(X(m), Y(m)) \in \mathbb{R}^{nL}_+ \times \mathcal{Y}$, and

$$\sum_{i=1}^{n} X_i(m) \leqq \sum_{i=1}^{n} w_i + \sum_{j=1}^{J} Y_j(m) \tag{12}$$

for all $m \in M$.

Remark 3. $\nu_1(m)$ can be regarded as a compensation or reward scheme. In order for the manager to produce efficiently at equilibrium, some incentive scheme should be provided to the manager, and, at the same time, the manager should not have incentives to pursue an unbounded profit (since profit maximizing production may not exist for non-convex production technologies). Agent 1, as the manager of firm j, will then receive a positive amount of compensation $\nu_{1j}(m)$ if she can propose a more efficient production plan q_{1j} than y_{2j} proposed by agent 2 for firm j. In other words, $\nu_{1j}(m) > 0$ if and only if $q_{1j} > y_{2j}$.

Remark 4. The feasible consumption sets $B_i(m)$ are constructed in a way that each agent's wealth function r_i is independent of his own proposed production profile.[8] Thus, in defining the feasible consumption sets $B_i(m)$, we used the production profile $Y_i(m)$ proposed by another agent to determine agent i's wealth. This ensures each consumer i's wealth is independent of the production profile proposed by himself, and in turn is independent of profits $\pi(p_i(m), Y_i(m))$ at Nash equilibrium.

Remark 5. Unlike the existing mechanisms, when $n > 2$, we do not require all consumers, except for the first two consumers, to propose price vectors and production profiles of production plans. This may be more realistic since in general not all individuals have the same information about production technologies of firms. Thus, the dimension of the message space is lower than those of the existing mechanisms that implement Walrasian allocations for convex production economies with more than two agents.

3.2. Mechanism for average cost pricing principle

Although the average cost pricing rule is a special case of the loss-free pricing rule, the mechanism for the loss-free pricing equilibrium principle does not implement average cost pricing equilibrium allocations. This is because $ACP(e)$ is only a subset of $LFP(e)$ and $N_{M,h}(e) = LFP(e)$, and thus a Nash equilibrium allocation of the mechanism constructed above may not be an average cost pricing equilibrium allocation. Nevertheless, this mechanism can be slightly modified to fully implement average cost pricing equilibrium allocations.

The modified mechanism is almost identical, differing only in determining the pricing rule. The loss-free pricing and constrained production set G_1 given by (4) is modified to be

$$A_1 = \left\{ (p, y) \in \Delta_{++}^{L-1} \times \mathcal{Y}\colon \sum_{i=1}^{n} w_i + \sum_{j=1}^{J} y_j > 0 \ \&\ p \cdot y_j = 0\ \forall j \right\}. \tag{13}$$

Remark 6. Since the set of average cost pricing equilibrium allocations are a subset of loss-free pricing equilibrium allocations, this modified mechanism also implements loss-free pricing equilibrium allocations in Nash equilibrium.

3.3. Mechanism for marginal cost pricing principle

The marginal cost pricing equilibrium principle is a natural generalization of the Walrasian equilibrium principle to economies involving non-convex production technologies.

The most common justification of marginal cost pricing is that a suitable price system and distribution of income allow the decentralization of every Pareto optimal allocation when each producer is instructed to follow the marginal cost pricing rule. Brown and Heal (1979), Dicker (1986), Cornet (1990), and Quinzii (1992) provide the conditions that guarantee Pareto efficiency at marginal cost pricing equilibrium.

3.3.1. Marginal cost pricing equilibrium

For the marginal cost pricing rule, one usually considers a private ownership economy so that consumer i's wealth function is $r_i(w, p, p \cdot y_1, \dots, p \cdot y_J) = p \cdot w_i + \sum_{i=1}^{n} \theta_{ij} p \cdot y_j$, in which case the marginal cost pricing rule includes the profit maximizing rule for convex production sets containing the null production as a special case, or more typically, consumer i's wealth function is given by $r_i(p, y) = p \cdot w_i + \theta_i p \cdot y$ with $w_i = \theta_i w$. This implies Guesnerie's fixed structure of revenues condition holds, i.e. $r_i(p, y) = \theta_i p \cdot (w + y)$. In this case, the lump sum taxation to cover

[8] If we only want to implement, but not fully implement loss-free pricing allocations, we neither need to distinguish $Y(m)$ from $Y_i(m)$ that is used to determine agents' wealth, nor to introduce the compensation scheme. Then the mechanism would be simpler and the dimension of the message space can be reduced by JL.

the losses of the firm is implicit in the formation of budget constraints, i.e., $r_i(p, y) = \theta_i p \cdot (w + y)$ should be interpreted as "after-tax" income. However, again, the mechanism constructed in this subsection works for a general form of the wealth function. At a marginal cost pricing equilibrium, firms' production plans satisfy the first-order necessary conditions for profit maximization, i.e., at the given production plans the market prices lie in the Clarke normal cone.

The design for an incentive mechanism that implements marginal cost pricing equilibrium allocations is more complicated than the one for the loss-free or average cost pricing principles. To have a feasible and continuous mechanism, two difficulties are involved. One is that, since production sets $\mathcal{Y}_j$ are assumed to be unknown to the designer, the designer cannot form the Clarke normal cone $N_{\mathcal{Y}_j}(y_j)$ for a reported production plan. For any subset $\tilde{\mathcal{Y}}_j$ of $\mathcal{Y}_j$, which is formed by reported production plans, its Clarke normal cone is larger than $N_{\mathcal{Y}_j}(y_j)$ so that a price vector in $N_{\tilde{\mathcal{Y}}_j}(y_j)$ may not be in $N_{\mathcal{Y}_j}(y_j)$. The second difficulty is that the marginal cost pricing rule does not coincide with the profit maximization rule when production sets is non-convex, and thus we cannot use the profit maximization approach used in Hong (1995) and Tian (1999) to study the problem of implementation of marginal cost pricing equilibrium allocations.

These difficulties, however, can be resolved using the following approach. As shown by Cornet (1990), even though the MCP rule does not coincide with the profit maximizing rule for the linear profit function $\pi_j = p \cdot y_j$, it is equivalent to a profit maximizing rule under which every MCP equilibrium production plan maximizes a quadratic profit function such that the first order conditions for maximizing the linear profit function and the quadratic profit function coincide at the MCP equilibrium production plan. Thus, to show that a price system satisfies the MCP rule under a proposed mechanism, it is sufficient to show that the proposed production plan maximizes some quadratic profit function for every firm at Nash equilibrium.

To do so, define the perpendicular set

$$\perp_{\mathcal{Y}_j}\left(y_j^*\right) = \left\{p \in \Delta_+^{L-1}\colon \exists \rho \geqq 0,\ \forall y_j \in \mathcal{Y}_j,\ p \cdot y_j^* \geqq p \cdot y_j - \rho \left\| y_j - y_j^* \right\|^2\right\}. \tag{14}$$

When $p \in \perp_{\mathcal{Y}_j}(y_j^*)$, p is said to be a perpendicular vector (or proximal normal vector) to $\mathcal{Y}_j$ at y_j^*. It can be verified that $\perp_{\mathcal{Y}_j}(y_j^*) \subseteq N_{\mathcal{Y}_j}(y_j^*)$. The condition that p is a perpendicular vector is a necessary condition for profit maximization for $p \cdot y_j$ over $\mathcal{Y}_j$. Indeed, if y_j^* maximizes the profit $p \cdot y_j$ over $\mathcal{Y}_j$, it also maximizes the quadratic function $p \cdot y_j - \rho \| y_j - y_j^* \|^2$ over the production set $\mathcal{Y}_j$ for every $\rho \geqq 0$ so that $p \cdot y_j^* \geqq p \cdot y_j - \rho \| y_j - y_j^* \|^2$ for all $y_j \in \mathcal{Y}_j$, and thus $p \in \perp_{\mathcal{Y}_j}(y_j^*)$. One can also see that if $\mathcal{Y}_j$ is convex, this necessary condition for profit maximization is also sufficient by taking $\rho = 0$.

This condition can also be interpreted in terms of "nonlinear prices" since it is equivalent to saying that y_j^* maximizes the quadratic function

$$\pi_j(y_j) = \left[p - \rho\left(y_j - y_j^*\right)\right]\left(y_j - y_j^*\right)$$

over the production set $\mathcal{Y}_j$.

Since $\perp_{\mathcal{Y}_j}(y_j^*) \neq N_{\mathcal{Y}_j}(y_j^*)$ in general, $\perp_{\mathcal{Y}_j}(y_j^*)$ only implements, but not fully, implements marginal cost pricing equilibrium allocations. To have a full implementation, we need to impose an additional condition on production sets. It is assumed that $\mathcal{Y}_j$ has a twice continuously differentiable hypersurface, i.e., there exists a twice continuously differentiable function, f_j, from $\mathbb{R}^L$ into $\mathbb{R}$, such that $\mathcal{Y}_j = \{x \in \mathbb{R}^L\colon\ f_j(x) \leqq 0\}$, 0 is a regular value of f_j, and $\partial \mathcal{Y}_j = f_j^{-1}(0)$. Let $\nabla f_j(y_j)$ denote the gradient of f_j at y_j, which defines the marginal cost pricing rule for a firm with a smooth technology. Then, if $f_j(y_j) = 0$ and $\nabla f_j(y_j) \neq 0$, by Proposition 2 in Cornet (1990), we have the following set equivalences.

$$\perp_{\mathcal{Y}_j}\left(y_j^*\right) = N_{\mathcal{Y}_j}\left(y_j^*\right) = \left\{\lambda \nabla f_j\left(y_j^*\right)\colon\ \lambda \geqq 0\right\}$$

for all $y \in \partial \mathcal{Y}$. Thus, the marginal cost pricing rule, profit maximizing prices for the quadratic profit function (or for the profit function with nonlinear prices), and the first order condition for the linear profit function are all equivalent.[9]

[9] Under the convexity of production sets, all the three conditions are also equivalent.

3.3.2. The description of the mechanism

The feasible and continuous mechanism for implementing the marginal cost pricing correspondence is basically the same in structure as those for implementing the loss-free and average cost pricing principles. We only describe the modifications.

The message space for agent i is modified to be

$$M_i = \begin{cases} \Delta_{++}^{L-1} \times \tilde{\mathcal{Y}} \times \mathcal{Y} \times \mathcal{Y} \times \mathbb{R}_+ \times \mathbb{R}_+^L \times \mathbb{R}_{++} \times (0,1] & \text{if } i = 1, \\ \Delta_{++}^{L-1} \times \mathbb{R}^{JL} \times \mathbb{R}_+ \times \mathbb{R}_+^L \times \mathbb{R}_{++} & \text{if } i = 2, \\ \mathbb{R}_+^L \times \mathbb{R}_{++} & \text{if } i \geqq 3, \end{cases}$$

where

$$\tilde{\mathcal{Y}} = \{y \in \mathcal{Y}\colon\ \hat{w} + \hat{y} > 0\}. \tag{15}$$

A generic element of M_i is written as $m_i = (p_i, y_i, q_i, s_i, \rho_i, z_i, \gamma_i, \eta_i)$ for $i = 1$, $m_i = (p_i, y_i, \rho_i, z_i, \gamma_i)$ for $i = 2$, and $m_i = (z_i, \gamma_i)$ for $i = 3, \dots, n$.

Remark 7. Note that, agent 1 is required to announce one more production profile here. This additional announcement is necessary for firms to follow the marginal cost pricing rule at equilibrium. While y_1, q_1, and s_1 are required to be production profiles in $\mathcal{Y}$, the production profile y_2 is not necessarily in $\mathcal{Y}$.

The construction for outcome functions $p_i(m)$, $Y(m)$, and $Y_i(m)$ are the same as before, but the term $x(m)$ in the outcome function for consumption bundles $X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m)$ is modified to be:

$$x_i(m) = \begin{cases} [\nu_1(m) + \tau_1(m) + \frac{1}{1+\eta_1(\|q_1 - y_2\| + \|s_1 - y_2\|)}] x_1'(m) & \text{if } i = 1, \\ \frac{1}{1+\|p_1 - p_2\| + \|\rho_1 - \rho_2\| + \|y_1 - y_2\|^2} x_2'(m) & \text{if } i = 2, \\ x_i'(m) & \text{if } i \geqq 3, \end{cases}$$

where $x'(m)$ and $\nu_1(m)$ are defined as the same as before, and $\tau_1(m)$ is given by

$$\tau_1(m) = \sum_{j=1}^{J} \tau_j(m) \tag{16}$$

with

$$\tau_{1j}(m) = \max\bigl\{0, p_1(m) \cdot (s_{1j} - y_{2j}) - \rho_2 \|s_{1j} - y_{2j}\|^2\bigr\}. \tag{17}$$

The compensation formula τ_1 means that agent 1 will receive a positive amount of compensation $\tau_{1j}(m)$ if the manager can announce a more profitable production plan s_{1j} than the production plan y_{2j} proposed by agent 2, i.e., $p_1(m) \cdot s_{1j} - \rho_2 \|s_{1j} - y_{2j}\|^2 > p_1(m) \cdot y_{2j}$, or receive zero compensation.

Remark 8. The reason for using two additional production profiles q_1 and s_1 to construct two compensations $\nu_1(m)$ and $\tau_1(m)$ is the following. While ν_1 is used to induce the manager of firms to produce efficiently, τ_1 is used to induce the manager of firms to follow the marginal cost pricing rule at equilibrium. In determining the marginal cost pricing rule, if the production plan y_{2j}, announced by agent 2, does not maximize $p_1(m) \cdot y_j - \rho_2 \|y_j - y_{2j}\|^2$ over production set $\mathcal{Y}_j$, then there is a 'test' production plan $y_j \in \mathcal{Y}_j$ such that $p_1(m) \cdot y_j - \rho_2 \|y_j - y_{2j}\|^2 > p_1(m) \cdot y_{2j}$. Such a test production plan y_j may not be in the constrained production set $\tilde{\mathcal{Y}}_j$, and thus we cannot use a production plan in $\tilde{\mathcal{Y}}_j$.

3.4. Mechanism for VTP/QTP principle

In this subsection we assume that the behavioral pricing rule for firms is given by the voluntary trading pricing rule, which includes the profit maximizing rule for convex production sets containing the null production, the loss-free rule pricing rule, and the average cost pricing rule as special cases.

3.4.1. Voluntary trading pricing and quantity-taking pricing equilibrium

The notion of the voluntary trading rule was introduced and studied by Dehez and Dréze (1988). The profit maximization rule may lead to unbounded outputs with a non-convex production. Besides this problem, even in the convex case, this behavior assumption often lacks in realism. Many firms announce prices and satisfy the demand that materializes at these prices, instead of choosing optimal quantities in reaction to prices. The notion of the voluntary trading pricing rule can deal with these problems. It means that, at the prevailing prices, profit cannot be increased by reducing output or by choosing a different input combination, although the producer may be eager to sell more at these prices. That is, it is a price system p at which it is profitable for the producer to meet the demand as given by y_j^+, instead of producing less. Thus, at equilibrium, producers maximize profit subject to a sales constraint. The voluntary trading pricing rule clearly implies that production costs are minimized. Furthermore, because inactivity is feasible, voluntary trading implies non-negative profits. Also, when output prices are minimized for given output, a voluntary trading pricing rule reduces to a quantity-taking pricing rule.

The voluntary trading/quantity-taking pricing rule studied by Dehez and Dréze (1988) assumes the economy under consideration is a private ownership economy so that consumer i's wealth function is $r_i(w, p, p \cdot y_1, \ldots, p \cdot y_J) = p \cdot w_i + \sum_{i=1}^{n} \theta_{ij} p \cdot y_j$. However, again, our implementation result holds for a general form of the wealth function.

Remark 9. Under monotonicity of preferences, voluntary trading equilibrium prices must be positive for all commodities. If prices are normalized in the space of Δ_+^{L-1}, a voluntary trading pricing equilibrium coincides with the quantity-taking pricing equilibrium. Thus, the mechanism proposed below also fully Nash-implements competitive equilibrium allocations with quantity-taking producers, and consequently, it fully Nash-implements Walrasian allocations for convex production economies.

3.4.2. The description of the mechanism

The mechanism for implementing the voluntary trading/quantity-taking pricing correspondence is also the same in structure as the previous mechanisms. We only describe the modifications.

The message space for agent i is modified to be

$$M_i = \begin{cases} \Delta_{++}^{L-1} \times \tilde{\mathcal{Y}} \times \mathcal{Y} \times \mathcal{Y} \times \mathbb{R}_+^L \times \mathbb{R}_{++} \times (0, 1] & \text{if } i = 1, \\ \Delta_{++}^{L-1} \times \mathbb{R}^{JL} \times \mathbb{R}_+^L \times \mathbb{R}_{++} & \text{if } i = 2, \\ \mathbb{R}_+^L \times \mathbb{R}_{++} & \text{if } i \geqq 3, \end{cases}$$

where $\tilde{\mathcal{Y}}$ is defined as the same as in the above subsection.

A generic element of M_i is $m_i = (p_i, y_i, q_i, t_i, z_i, \gamma_i, \eta_1)$ for $i = 1$, $m_i = (p_i, y_i, z_i, \gamma_i)$ for $i = 2$, and $m_i = (z_i, \gamma_i)$ for $i = 3, \ldots, n$. The explanations for these components are similar to those in the mechanisms constructed in previous subsections.

The outcome functions for personalized price vectors $p_i(m)$, proposed production profiles $Y_i(m)$, the feasible production profile $Y(m)$, the incentive scheme to induce firms to produce efficiently $\nu_1(m)$, the feasible consumption correspondence $B_i(m)$, and $x_i'(m)$ are the same as those for the marginal cost pricing equilibrium principle. The term $x(m)$ in the outcome function for consumption bundles $X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m)$ is modified to be:

$$x_i(m) = \begin{cases} [\nu_1(m) + \mu_1(m) + \frac{1}{1+\eta_1(\|q_1-y_2\|+\|t_1-y_2\|)}]x_1'(m) & \text{if } i = 1, \\ \frac{1}{1+\|p_1-p_2\|+\|y_1-y_2\|}x_2'(m) & \text{if } i = 2, \\ x_i'(m) & \text{if } i \geqq 3, \end{cases}$$

where the compensation $\mu_1(m)$, used to induce the voluntary trading pricing rule at equilibrium, is defined by

$$\mu_1(m) = \sum_{j=1}^{J} \delta_{1j}(m)\alpha_{1j}(m) \tag{18}$$

with

$$\alpha_{1j}(m) = \max\{0, p_1(m) \cdot (t_{1j} - y_{2j})\}, \tag{19}$$

and

$$\delta_{1j}(m) = \begin{cases} 1 & \text{if } t_{1j} \leqq y_{2j}^{+}, \\ 0 & \text{otherwise,} \end{cases}$$

which means that agent 1 will be punished if she proposes a production plan t_{1j} that exceeds the target production plan y_{2j}^{+} proposed by agent 2, i.e., if it does not satisfy $t_{1j} \leqq y_{2j}^{+}$.

Remark 10. The compensation formula μ_1 means that agent 1 will receive a positive amount of compensation $\alpha_{1j}(m)$ if the manager can propose a more profitable production plan t_{1j} with $t_{1j} \leqq y_2^{+}$ than the target production, y_{2j}, proposed by agent 2, i.e., $\alpha_{1j}(m) > 0$ if and only if $p_1(m) \cdot t_{1j} > p_1(m) \cdot y_{2j}$ for $t_{1j} \leqq y_2^{+}$, otherwise she receives a zero compensation.

It may be remarked that the mechanism is feasible on M, but it is not continuous because $\delta_{1j}(m)$ is not continuous.

4. Implementation result

In this section we prove that the mechanisms constructed in the previous section (fully) Nash-implement loss-free, average cost, marginal cost, and voluntary trading/quantity-taking pricing equilibrium allocations, respectively. We do so by showing the equivalence between a pricing equilibrium allocation and a Nash equilibrium allocation of a corresponding mechanism.

Proposition 1 below proves that every Nash equilibrium allocation of the mechanism for the loss-free, average cost, marginal cost, or voluntary trading/quantity-taking pricing equilibrium principle is a corresponding loss-free, average cost, marginal cost, or voluntary trading/quantity-taking pricing equilibrium allocation. To show this, we need to show that, at every Nash equilibrium, every consumer maximizes his preference subject to his budget constraint, every producer follows the pre-specified pricing rule, and the aggregate demand over supply is non-positive. Proposition 2 below proves that every pricing equilibrium allocation is a Nash equilibrium allocation of the corresponding mechanism. That is, for every pricing equilibrium, we need to show there is a Nash equilibrium of the corresponding mechanism such that the allocation result from the Nash equilibrium is the pricing equilibrium allocation. The proofs of these claims need use some lemmas given in Appendix A.

Proposition 1. *Suppose m^* is a Nash equilibrium allocation of the mechanism for the loss-free, average cost, marginal cost, or voluntary trading/quantity-taking pricing equilibrium principle. Then the Nash equilibrium allocation $(X(m^*), Y(m^*))$ is a corresponding loss-free, average cost, marginal cost, or voluntary trading/quantity-taking pricing equilibrium allocation with $p(m^*)$ as a price system.*

Proof. Let m^* be a Nash equilibrium of the mechanism for the loss-free, average cost, marginal cost, or voluntary trading/quantity-taking pricing equilibrium principle. We need to prove that $(X(m^*), Y(m^*))$ is a corresponding loss-free, average cost, marginal cost, or voluntary trading/quantity-taking pricing equilibrium allocation with $p(m^*)$ as a price system. Note that the mechanism is feasible by construction, the budget constraint holds with equality by Lemma 5, $Y(m^*) \in \partial\mathcal{Y}$ by Lemma 7, and $p(m^*)$ satisfies the loss-free, average cost, marginal cost, or voluntary trading/quantity-taking pricing rule by the constructions of the mechanisms and Lemmas 8 and 9. So we only need to show that each individual maximizes her preferences subject to her budget constraint.

Suppose, by way of contradiction, that there is some $x_i \in \mathbb{R}_+^L$ such that $x_i \; P_i \; X_i(m^*)$ and $p(m^*) \cdot x_i \leqq r_i(w_i, p(m^*), \pi(p(m^*), Y(m^*)))$. Since $X(m^*) \in \mathbb{R}_{++}^{nL}$ by Lemma 2, there is some $m_i \in M$ such that $X_i(m_i, m_{-i}^*) \; P_i \; X_i(m^*)$ by Lemma 4. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Thus, $(X(m^*), Y(m^*))$ is a corresponding loss-free, average cost, marginal cost, or voluntary trading/quantity-taking pricing equilibrium allocation. □

Proposition 2. *Let (x^*, y^*) be a loss-free, average cost, marginal cost, or voluntary trading/quantity-taking pricing equilibrium allocation with $p^* \in \Delta_+^{L-1}$ as a price system. Then we have*

(1) *For the mechanisms for the loss-free and average cost pricing principles, there is a Nash equilibrium m^* such that $Y(m^*) = y^*$, $p(m^*) = p^*$, and $X_i(m^*) = x_i^*$ for $i \in N$, i.e., LFP$(e) \subseteq N_{M,h}(e)$ and ACP$(e) \subseteq N_{M,h}(e)$.*

(2) *For the mechanism for the marginal cost pricing principle, if $\mathcal{Y}_j$ has a twice continuously differentiable hypersurface, then there is a Nash equilibrium m^* such that $Y(m^*) = y^*$, $p(m^*) = p^*$, and $X_i(m^*) = x_i^*$ for $i \in N$, i.e., $MCP(e) \subseteq N_{M,h}(e)$.*
(3) *For the mechanism for the voluntary trading/quantity-taking pricing principle, there is a Nash equilibrium m^* such that $Y(m^*) = y^*$, $p(m^*) = p^*$, and $X_i(m^*) = x_i^*$ for $i \in N$, i.e., $VTP(e) \subseteq N_{M,h}(e)$.*

Proof.

1. We first note that $x^* \in \mathbb{R}^L_{++}$ by interiority of preferences, and thus $\sum_{i=1}^n w_i + \sum_{j=1}^J y_j^* > 0$. Also, by the strict monotonicity of preference orderings, the normalized price vector p^* must be in Δ^{L-1}_{++}. We need to show that there is a message m^* such that (x^*, y^*) is a Nash equilibrium allocation. For each $i \in N$, define m_i^* by $p_i^* = p^*$ (for $i = 1, 2$), $y_1^* = q_1^* = y^*$, $y_2^* = y^*$, $z_i^* = x_i^*$, $\gamma_i^* = 1$. Then, it can be easily verified that $Y(m^*) = y^*$, $p_i(m^*) = p^*$, and $X_i(m^*) = x_i^*$ for all $i \in N$. Note that $p(m_i, m_{-i}^*) = p_i(m^*)$ and $Y_i(m_i, m_{-i}^*) = Y_i(m^*)$ for all $m_i \in M_i$ by the construction of the mechanisms, and $\nu_1(m_1, m_{-1}^*) = \nu_1(m^*) = 0$ for all $m_1 \in M_1$ by Lemma 7. Then for all $m_i \in M_i$,
$$p(m^*) \cdot X_i\big(m_i, m_{-i}^*\big) \leqq r_i\big(w_i, p\big(m_i, m_{-i}^*\big), \pi\big(p\big(m_i, m_{-i}^*\big), Y_i\big(m_i, m_{-i}^*\big)\big)\big)$$
$$= r_i\big(w_i, p(m^*), \pi\big(p(m^*), Y_i(m^*)\big)\big). \tag{20}$$
Hence, $X_i(m_i, m_{-i}^*)$ satisfies the budget constraint for all $m_i \in M_i$. Then, $X_i(m^*)\ R_i\ X_i(m_i, m_{-i}^*)$ for all $m_i \in M_i$, or it contradicts the fact that $(X(m^*), Y(m^*))$ is a loss-free pricing equilibrium allocation. Thus, $(X(m^*), Y(m^*))$ must be a Nash equilibrium allocation , i.e., $LFP(e) \subseteq N_{M,h}(e)$ and $ACP(e) \subseteq N_{M,h}(e)$.
2. Similarly, we have $x^* \in \mathbb{R}^L_{++}$, $\sum_{i=1}^n w_i + \sum_{j=1}^J y_j^* > 0$, and $p^* \in \Delta^{L-1}_{++}$. Since $\mathcal{Y}_j$ has a twice continuously differentiable hypersurface, $p^* = \nabla f_j(y_j^*)/\|\nabla f_j(y_j^*)\|_1 = \perp_{\mathcal{Y}_j}(y_j^*) = N_{\mathcal{Y}_j}(y_j^*)$ for $j = 1, 2, \ldots, J$, where $\|\cdot\|_1$ is the l_1-norm. Consequently, there exists a $\rho^* \geqq 0$ such that $p^* \cdot y_j^* \geqq p^* \cdot y_j - \rho^*\|y_j - y_j^*\|^2$ for all $y_j \in \mathcal{Y}_j$. We need to show that there is a message m^* such that (x^*, y^*) is a Nash equilibrium allocation. For each $i \in N$, define m_i^* by $p_i^* = p^*$ (for $i = 1, 2$), $\rho_1^* = \rho_2^* = \rho^*$, $y_1^* = q_1^* = s_1^* = y^*$, $y_2^* = y^*$, $z_i^* = x_i^*$, $\gamma_i^* = 1$, and $\eta_i^* = 1$. Then, it can be easily verified that $Y(m^*) = y_1^* = y_2^* = s_1^*$, $p_i(m^*) = p^*$, and $X_i(m^*) = x_i^*$ for all $i \in N$. Then, $p_i(m_i, m_{-i}^*) = p_i(m^*)$ and $Y_i(m_i, m_{-i}^*) = Y_i(m^*)$ for all $m_i \in M_i$, $p(m^*) \cdot Y_j(m^*) \geqq p(m^*) \cdot y_j - \rho_2^*\|y_j - Y_j(m^*)\|^2$ for all $y_j \in \mathcal{Y}_j$, and thus $\nu_1(m_1, m_{-1}^*) = \nu_1(m^*) = 0$, $\tau_1(m_1, m_{-1}^*) = \tau_1(m^*) = 0$ for all $m_1 \in M_1$, $p(m^*) = \nabla f_j(Y_j(m^*))/\|\nabla f_j(Y_j(m^*))\|_1 = \perp_{\mathcal{Y}_j}(Y_j(m^*)) = N_{\mathcal{Y}_j}(Y_j(m^*))$ for $j = 1, 2, \ldots, J$. Hence
$$p(m^*) \cdot X_i\big(m_i, m_{-i}^*\big) \leqq r_i\big(w_i, p(m^*), \pi\big(p(m^*), Y(m^*)\big)\big) \tag{21}$$
for all $m_i \in M_i$ Thus, $X_i(m_i, m_{-i}^*)$ satisfies the budget constraint for all $m_i \in M_i$. Thus, we must have $X_i(m^*)\ R_i\ X_i(m_i, m_{-i}^*)$ for all $m_i \in M_i$, or it contradicts the fact that $(X(m^*), Y(m^*))$ is a marginal cost pricing equilibrium allocation. Therefore, $(X(m^*), Y(m^*))$ is a Nash equilibrium allocation, i.e., $MCP(e) \subseteq N_{M,h}(e)$.
3. For each $i \in N$, define m_i^* by $p_i^* = p^*$ (for $i = 1, 2$), $y_1^* = q_1^* = t_1^* = y^*$, $y_2^* = y^*$, $z_i^* = x_i^*$, $\gamma_i^* = 1$, and $\eta_1^* = 1$. Then, it can be easily verified that $Y(m^*) = y_1^*$, $p_i(m^*) = p^*$, and $X_i(m^*) = x_i^*$ for all $i \in N$. Also, note that $p(m_i, m_{-i}^*) = p_i(m^*)$ and $Y_i(m_i, m_{-i}^*) = Y_i(m^*)$ for all $m_i \in M_i$ by the construction of the mechanism, $\nu_1(m_1, m_{-1}^*) = \nu_1(m^*) = 0$ and $\mu_1(m_1, m_{-1}^*) = \mu_1(m^*) = 0$ for all $m_1 \in M_1$ by Lemma 9. The remaining proof is the same as in (1), and thus $(X(m^*), Y(m^*))$ must be a Nash equilibrium allocation, i.e., $VTP(e) \subseteq N_{M,h}(e)$.

Combining Propositions 1 and 2, we have the following theorem that summarizes our main result. □

Theorem 1. *For the class of pricing economic environments E specified in Section 2, suppose the following assumptions are satisfied:*

(1) *$w_i > 0$ for all $i \in N$.*
(2) *For all $i \in N$, preference orderings, R_i, are continuous and convex on $\mathbb{R}^L_+$, strictly increasing on $\mathbb{R}^L_{++}$, and satisfy the Interiority Condition of Preferences.*

(3) *For all j, production sets $\mathcal{Y}_j$ are non-empty, closed, $0 \in \mathcal{Y}_j$, and $\{\mathcal{Y}_j - \mathbb{R}^L_+\} \subseteq \mathcal{Y}_j$. For the marginal cost pricing rule, it is assumed $\mathcal{Y}_j$ has a twice continuously differentiable hypersurface for all j.*[10]

(4) *For each $i \in N$, consumer i's wealth function $r_i : \mathbb{R}^L_+ \times \mathbb{R}^L_+ \times \mathbb{R}^J \to \mathbb{R}$ is increasing in w_i, continuous, $\sum_{i=1}^n r_i(w_i, p, \pi(p, y)) = \sum_{i=1}^n p \cdot w_i + \sum_{j=1}^J p \cdot y_j$, $r_i(w_i, tp, t\pi) = tr_i(w_i, p, \pi)$ for all $t > 0$, and $\sum_{i=1}^n p \cdot w_i + \sum_{j=1}^J p \cdot y_j > 0$ implies that $r_i(w_i, p, \pi) > 0$.*

Then, the mechanisms, which are continuous, feasible, and have message spaces of finite dimension, fully Nash-implement loss-free, average cost, marginal cost, and voluntary trading/quantity-taking pricing equilibrium allocations on E, respectively.

Proof. By Proposition 1, the set of Nash equilibrium allocations is a subset of the corresponding pricing equilibrium allocations, which means the mechanisms implement loss-free, average cost, marginal cost, and voluntary trading/quantity-taking pricing equilibrium allocations in Nash equilibrium on E, respectively. By Proposition 2, the set of loss-free, average cost, or voluntary trading/quantity-taking pricing equilibrium allocations is a subset of Nash equilibrium allocations of the corresponding mechanism. When $\mathcal{Y}_j$ has a twice continuously differentiable hypersurface, also by Proposition 2, $MCP(e) \subseteq N_{M,h}(e)$, and thus by combining Propositions 1 and 2, the mechanism fully Nash-implements loss-free, average cost, MCP, or voluntary trading/quantity-taking pricing equilibrium allocations. □

5. Concluding remarks

This paper considers the incentive aspects of the loss-free, average cost, marginal cost, voluntary trading, and quantity-taking pricing principles for production economies with increasing returns to scale or more general types of non-convexities. We present specific mechanisms that use finite-dimensional message spaces and fully implement loss-free, average cost, marginal cost, voluntary trading pricing equilibrium allocations, and competitive equilibrium allocations with quantity-taking producers when preferences and productions sets are all unknown to the designer.

The mechanisms constructed are well-behaved and have a number of desired properties:

(1) They use finite-dimensional message spaces. This desired property is the main motivation for the paper since the existing results are mainly negative in the sense that they all use infinite-dimensional message spaces for non-convex production economies. We are able to do so since the mechanisms mainly require consumers and producers to announce their consumption and production plans, but not require them to announce their preferences and production technologies.

(2) All the mechanisms, except the one that implements voluntary trading pricing allocations, are stable in the sense that they are continuous. A slight change of strategies does not result in a drastic change of the outcome.

(3) The mechanisms are credible in the sense that they are feasible. Every participant receives a consumption bundle in her consumption set. Every production plan is in the production set, and aggregate consumption does not exceed aggregate supply, even at non-equilibria. A mechanism would not be credible if it were infeasible.

(4) They are market type mechanisms. The price and quantity are components of the message spaces.

(5) The mechanisms are more realistic and relatively more informationally efficient. Our mechanisms not only use smaller message spaces, but are also more realistic since only managers and owners of firms are required to announce production plans.

(6) The mechanisms work not only for three or more agents, but also for two-agent economies. Thus, they are unified mechanisms that are irrespective of the number of agents.

Appendix A

Lemma 1. *Suppose $x_i(m)\ P_i\ x_i$. Then agent i can choose a very large γ_i such that $X_i(m)\ P_i\ x_i$.*

[10] Without this smooth assumption, the mechanism for the marginal cost pricing principle implements, although not fully implements, marginal cost pricing equilibrium allocations.

Proof. If agent i declares a large enough γ_i, then $\bar{\gamma}(m)$ becomes very small (since $\bar{\gamma}(m)\gamma_i \leqq 1$) and thus almost nullifies the effect of other agents in $\bar{\gamma}(m)\sum_{i=1}^{n}\gamma_i x_i(m) \leqq \sum_{i=1}^{n} w_i + \sum_{j=1}^{J} Y_j(m)$. Thus, $X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m)$ can arbitrarily approach as close to $x_i(m)$ as agent i wishes. From $x_i(m)\ P_i\ x_i$ and continuity of preferences, we have $X_i(m)\ P_i\ x_i$ if agent i chooses a very large γ_i. □

This lemma shows that consumer i can secure herself a consumption bundle determined the mechanism arbitrarily close to $x_i(m)$ as agent i wishes, and consequently the consumer can reach any consumption bundle in her budget set. We need use this basic lemma to prove all the lemmas below.

Lemma 2. *If $m^* \in V_{M,h}(e)$, then $X(m^*) \in \mathbb{R}^{nL}_{++}$, $x_i(m^*) \in \mathbb{R}^{nL}_{++}$ and $x'_{ii}(m^*) \in \mathbb{R}^{nL}_{++}$ for all $i \in N$.*

Proof. We argue by contradiction. Suppose $X(m^*) \in \partial\mathbb{R}^{nL}_{+}$. Then there is some $i \in N$ such that $X_i(m^*) \in \partial\mathbb{R}^{L}_{+}$. Since $\sum_{i=1}^{n} w_i + \sum_{j=1}^{J} Y_j(m^*) > 0$ by construction, $r_i(w_i, p_i(m^*), \pi(p_i(m^*), Y_i(m^*))) > 0$ by assumption. Then, there is some $x_i \in \mathbb{R}^{L}_{++}$ such that $p_i(m^*)\cdot x_i \leqq r_i(w_i, p_i(m^*), \pi(p_i(m^*), Y_i(m^*)))$, $x_i \leqq \sum_{i=1}^{n} w_i + \sum_{j=1}^{J} Y_j(m^*)$, and $x_i\ P_i\ X_i(m^*)$ by interiority of preferences. Now suppose that agent i chooses $z_i = x_i$, $\gamma_i > \gamma_i^*$, and keeps the other components of the message unchanged. Then $x_i \in B_i(m_i, m^*_{-i})$, and thus $x_i(m_i, m^*_{-i}) = x'_i(m_i, m^*_{-i}) = x_i > 0$. Hence, we have $x_i(m_i, m^*_{-i})\ P_i\ X_i(m^*)$ by interiority of preferences. Therefore, by Lemma 1, $X_i(m_i, m^*_{-i})\ P_i\ X_i(m^*)$ if agent i chooses a very large γ_i. This contradicts $m^* \in V_{M,h}(e)$ and thus we must have $X_i(m^*) \in \mathbb{R}^{L}_{++}$ for all $i \in N$. Since $X_i(m^*)$ is proportional to $x_i(m^*)$ and $x'_{ii}(m^*)$, $x_i(m^*) \in \mathbb{R}^{nL}_{++}$ and $x'_{ii}(m^*) \in \mathbb{R}^{nL}_{++}$ for all $i \in N$. □

This lemma shows that every allocation resulting from Nash equilibrium must be an interior consumption bundle, which is a necessary requirement for a Nash allocation to maximize consumers' preferences subject to their budget constrains.

The following lemma shows that the owner and manager of firms will reach an agreement about the pricing and production at equilibrium.

Lemma 3. *Suppose m^* is a Nash equilibrium. Then, we have*:

(1) *For the mechanisms for the loss-free and average cost pricing principles, $p_1^* = p_2^*$ and $y_1^* = q_1^* = y_2^*$. Consequently, $p(m^*) \equiv p_1(m^*) = \cdots = p_n(m^*) = p_1^* = p_2^*$, $Y(m^*) = Y_1(m^*) = \cdots = Y_n(m^*) = y_1^* = q_1^* = y_2^*$, and thus $\nu_1(m^*) = 0$ and $x_i(m^*) = x'_i(m^*)$.*
(2) *For the mechanism for the marginal cost pricing principle, $p_1^* = p_2^*$, $\rho_1^* = \rho_2^*$, and $y_1^* = y_2^* = q_1^* = s_1^*$. Consequently, $\nu_1(m^*) = 0$, $\tau_1(m^*) = 0$, $p(m^*) \equiv p_i(m^*) = p_1^* = p_2^*$, $Y(m^*) = Y_i(m^*) = y_1^* = q_1^* = s_1^* = y_2^*$, and $x_i(m^*) = x'_i(m^*)$ for all $i \in N$.*
(3) *For the mechanism for the voluntary trading/quantity-taking pricing principle, $p_1^* = p_2^*$ and $y_1^* = y_2^* = q_1^* = t_1^*$. Consequently, $\nu_1(m^*) = 0$, $\mu_1(m^*) = 0$, $p(m^*) \equiv p_i(m^*) = p_1^* = p_2^*$, $Y(m^*) = Y_i(m^*) = y_1^* = q_1^* = y_2^*$, and $x_i(m^*) = x'_i(m^*)$ for all $i \in N$.*

Proof. The proofs are basically the same for all the mechanisms. So we only give the proof of (1). For the mechanisms for the loss-free and average cost pricing principles, we prove the lemma by way of contradiction. Suppose that $p_1^* \neq p_2^*$ or $y_1^* \neq y_2^*$. Since $x'_i(m^*) > 0$ for all agent i by Lemma 2, agent 2 can choose $p_2 = p_1^*$ or $y_2 = y_1^*$ so that his consumption becomes larger and he would be better off by monotonicity of preferences. Hence, m^* is not a Nash equilibrium strategy if $p_1^* \neq p_2^*$ or $y_1^* \neq y_2^*$. Also, suppose $q_1^* \neq y_2^*$. Then agent 1 can choose a smaller $\eta_1 < \eta_1^*$ in (0, 1] so that her consumption becomes larger and she would be better off by monotonicity of preferences. Hence, no choice of η_1 could constitute part of the Nash equilibrium strategy when $q_1^* \neq y_2^*$. Thus, we must have $q_1^* = y_2^*$ at the Nash equilibrium. Thus, we must have $p_1^* = p_2^*$ and $y_1^* = q_1^* = y_2^*$ at the Nash equilibrium. Consequently, $p(m^*) \equiv p_1(m^*) = \cdots = p_n(m^*) = p_1^* = p_2^*$, $Y(m^*) = Y_1(m^*) = \cdots = Y_n(m^*) = y_1^* = q_1^* = y_2^*$, and thus $\nu_1(m^*) = 0$ and $x_i(m^*) = x'_i(m^*)$. □

Lemma 4. *Suppose $X(m^*) \in \mathbb{R}^{nL}_{++}$ for $m^* \in M$ and there is an $x_i \in \mathbb{R}^{L}_{++}$ for some $i \in N$ such that $p(m^*)\cdot x_i \leqq r_i(w_i, p(m^*), \pi(p(m^*), Y(m^*)))$, and $x_i\ P_i\ X_i(m^*)$. Then there is some $m_i \in M_i$ such that $X_i(m_i, m^*_{-i})\ P_i\ X_i(m^*)$.*

Proof. Since $X(m^*) > 0$, $X_i(m^*) < \sum_{j\in N} w_j + \sum_{j=1}^J Y_j(m^*)$. Let $x_{\lambda i} = \lambda x_i + (1-\lambda)X_i(m^*)$. Then, by convexity of preferences, we have $x_{\lambda i}\ P_i\ X_i(m^*)$ for any $0 < \lambda < 1$. Also $x_{\lambda i} \in \mathbb{R}^L_{++}$, $p(m^*)\cdot x_{\lambda i} \leqq r_i(w_i, p(m^*), \pi(p(m^*), Y(m^*)))$, and $x_{\lambda i} < \sum_{k=1}^n w_k + \sum_{j=1}^J Y_j(m^*)$ when λ is sufficiently close to 0. Now suppose agent i chooses $z_i = x_{\lambda i}$, $\gamma_i > \gamma_i^*$, and keeps the other components of the message unchanged, then $x_{\lambda i} \in B_i(m_i, m^*_{-i})$. Thus we have $x_i(m_i, m^*_{-i}) = x_i'(m_i, m^*_{-i}) = x_{\lambda i}$. From $x_{\lambda i}\ P_i\ X_i(m^*)$, we have $x_i(m_i, m^*_{-i})\ P_i\ X_i(m^*)$. Therefore, by Lemma 1, agent i can choose a very large γ_i such that $X_i(m_i, m^*_{-i})\ P_i\ X_i(m^*)$. □

This lemma shows consumer i could always improve her preference by changing her message if she did not maximize her preference at equilibrium. Furthermore, the budget constraint and the feasibility condition must hold with equality at Nash equilibrium as shown in the following lemma.

Lemma 5. *If $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, then $p(m^*)\cdot X_i(m^*) = r_i(w_i, p(m^*), \pi(p(m^*), Y(m^*)))$. Consequently, the feasibility condition must hold with equality, i.e., $\hat{X}(m^*) = \hat{w} + \hat{Y}(m^*)$.*

Proof. Suppose, by contradiction, that $p(m^*)\cdot X_i(m^*) < r_i(w_i, p(m^*), \pi(p(m^*), Y(m^*)))$. Then, there is an $x_i > X_i(m^*)$ such that $p(m^*)\cdot x_i \leqq r_i(w_i, p(m^*), \pi(p(m^*), Y(m^*)))$ and $x_i\ P_i\ X_i(m^*)$ by monotonicity of preferences. Since $X(m^*) \in \mathbb{R}^{nL}_{++}$ by Lemma 2, there is some $m_i \in M_i$ such that $X_i(m_i, m^*_{-i})\ P_i\ X_i(m^*)$ by Lemma 4. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, and thus $p(m^*)\cdot X_i(m^*) = r_i(w_i, p(m^*), \pi(p(m^*), Y(m^*)))$. Also, since $\sum_{i=1}^n r_i(w_i, p(m^*), \pi(p(m^*), Y(m^*))) = \sum_{i=1}^n p(m^*)\cdot w_i + \sum_{j=1}^J p(m^*)\cdot Y_j(m^*)$, $\sum_{i=1}^n p(m^*)\cdot X_i(m^*) = \sum_{i=1}^n p(m^*)\cdot w_i + \sum_{j=1}^J p(m^*)\cdot Y_j(m^*)$. Consequently, the feasibility condition must hold with equality. Otherwise, there exists an $i \in N$ such that $p(m^*)\cdot X_i(m^*) < r_i(w_i, p(m^*), \pi(p(m^*), Y(m^*)))$. □

As a direct corollary of Lemma 5, we have the following result that shows the coincidence of the final consumption and affordable consumption chosen from the constraint set.

Lemma 6. *If $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, then $\bar{\gamma}(m^*)\gamma_i^* = 1$ for all $i \in N$ and thus $X(m^*) = x(m^*) = x'(m^*)$.*

Proof. Suppose $\bar{\gamma}(m^*)\gamma_i^* < 1$ for some $i \in N$. Then $X_i(m^*) = \bar{\gamma}(m^*)\gamma_i^* x_i(m^*) < x_i(m^*)$, and therefore $p(m^*)\cdot X_i(m^*) < r_i(w_i, p(m^*), \pi(p(m^*), Y(m^*)))$. But this is impossible by Lemma 5. Thus, $\bar{\gamma}(m^*)\gamma_i^* = 1$ for all $i \in N$ and thus $X(m^*) = x(m^*) = x'(m^*)$. □

Lemma 7. *If m^* is a Nash equilibrium, then $Y(m^*) \in \partial\mathcal{Y}$, and thus $\nu_1(m_i, m^*_{-i}) = \nu_1(m^*) = 0$ for all $m_1 \in M_1$.*

Proof. First note that, under the assumptions that $\mathcal{Y}_j$ is closed, contains 0, and $\{\mathcal{Y}_j - \mathbb{R}^L_+\} \subseteq \mathcal{Y}_j$ for $j = 1, \dots, J$, $\partial\mathcal{Y}_j$, the boundary of the production set $\mathcal{Y}_j$, is exactly the set of weakly efficient production plans. Suppose, by way of contradiction, that $Y_j(m^*) \notin \partial\mathcal{Y}_j$ for some firm j. Then, there is a production plan $y_j \in \mathcal{Y}_j$ such that $y_j > Y_j(m^*)$.

Now suppose agent 1 chooses $q_{1j} = y_j$, $\gamma_1 > \gamma_1^*$, $\eta_1 < \eta_1^*$, and keeps the other components of the message unchanged. We then have $\nu_1(m_1, m^*_{-1}) = \prod_{l=1}^L (y_j^l - Y_j^l(m^*)) > 0$, and thus $x_1(m_1, m^*_{-1}) = [\nu_1(m_1, m^*_{-1}) + \frac{1}{1+\eta_1\|q_{1j}-Y(m^*)\|}]x'(m^*)$. Note that, when $\eta_1 \to 0$, $[\nu_1(m_1, m^*_{-1}) + \frac{1}{1+\eta_1\|q_{1j}-Y(m^*)\|}]x'(m^*) \to [\nu_1(m_1, m^*_{-1}) + 1]x'(m^*) > x(m^*) = X(m^*)$. Thus, when η_1 is sufficiently close to zero, from $[\nu_1(m_1, m^*_{-1}) + 1]x'(m^*) > x(m^*) = X(m^*)$, we have $x_1(m_i, m^*_{-1})\ P_1\ X_1(m^*)$ by continuity of preferences. Then, by Lemma 1, agent 1 can choose a very large γ_1 such that $X_1(m_1, m^*_{-1})\ P_1\ X_1(m^*)$, which contradicts the fact m^* is a Nash equilibrium. Hence $Y_j(m^*)$ must be a weakly efficient production plan, and thus $Y(m^*) \in \partial\mathcal{Y}$. Finally, since there is no production plan $y_j \in \mathcal{Y}_j$ such that $y_j > Y_j(m^*)$ for all j, we must have $\nu_1(m_1, m^*_{-1}) = \nu_1(m^*) = 0$ for all $m_1 \in M_1$. □

This result implies we must have efficient production at Nash equilibrium. The following two lemmas show that firms also must follow the marginal cost pricing rule and the voluntary trading pricing rule respectively at Nash equilibrium.

Lemma 8. *If m^* is a Nash equilibrium of the mechanism for the marginal cost pricing equilibrium principle, then $p(m^*)\cdot Y_j(m^*) \geqq p(m^*)\cdot y_j - \rho_2^*\|y_j - Y_j(m^*)\|^2$ for all $y_j \in \mathcal{Y}_j$, and thus $p(m^*) \in \perp_{\mathcal{Y}_j}(Y_j(m^*)) \subseteq N_{\mathcal{Y}_j}(Y_j(m^*))$*

for $j = 1, \dots, J$. Consequently, $p(m^*)$ satisfies the marginal cost pricing rule and $\tau_1(m_1, m^*_{-1}) = \tau_1(m^*) = 0$ for all $m_1 \in M_1$.

Proof. Suppose, by way of contradiction, $Y_j(m^*)$ does not maximize the quadratic profit function $p(m^*) \cdot y_j - \rho_2^* \|y_j - Y_j(m^*)\|^2$ in $\mathcal{Y}_j$ for some firm j. Then, there exists a production plan $y_j \in \mathcal{Y}_j$ such that $p(m^*) \cdot Y_j(m^*) < p(m^*) \cdot y_j - \rho \|y_j - Y_j(m^*)\|^2$.

Now suppose agent 1 chooses $s_{1j} = y_j$, $\gamma_1 > \gamma_1^*$, $\eta_1 < \eta_1^*$, and keeps the other components of the message unchanged. We have $\tau_1(m_1, m^*_{-1}) = p(m^*) \cdot y_j - \rho_2^* \|y_j - Y_j(m^*)\|^2 - p(m^*) \cdot Y_j(m^*) > 0$, and thus $x_1(m_1, m^*_{-1}) = [\tau_1(m_1, m^*_{-1}) + \frac{1}{1+\eta_1 \|s_{1j} - Y(m^*)\|}] x'(m^*)$. Note that, when $\eta_1 \to 0$, $[\tau_1(m_1, m^*_{-1}) + \frac{1}{1+\eta_1 \|s_{1j} - Y(m^*)\|}] x'(m^*) \to [\tau_1(m_1, m^*_{-1}) + 1] x'(m^*) > x(m^*) = X(m^*)$. Thus, when η_1 is sufficiently close to zero, from $[\tau_1(m_1, m^*_{-1}) + 1] x'(m^*) > x(m^*) = X(m^*)$, we have $x_1(m_i, m^*_{-1})\ P_1\ X_1(m^*)$ by continuity of preferences. Then, by Lemma 1, agent 1 can choose a very large γ_1 such that $X_1(m_1, m^*_{-1})\ P_1\ X_1(m^*)$, which contradicts the fact m^* is a Nash equilibrium. Hence, we must have $p(m^*) \cdot Y_j(m^*) \geqq p(m^*) \cdot y_j - \rho_2^* \|y_j - Y_j(m^*)\|^2$ for all $y_j \in \mathcal{Y}_j$, and thus $p(m^*) \in \perp_{\mathcal{Y}_j} (Y_j(m^*)) \subseteq N_{\mathcal{Y}_j}(Y_j(m^*))$ for $j = 1, \dots, J$, which means $p(m^*)$ satisfies the marginal cost pricing rule. Finally, since $Y_j(m^*)$ maximizes the quadratic profit function $p(m^*) \cdot y_j - \rho_2^* \|y_j - Y_j(m^*)\|^2$ over the production set $\mathcal{Y}_j$ for all j, we must have $\tau_1(m_1, m^*_{-1}) = \tau_1(m^*) = 0$ for all $m_1 \in M_1$ by the definition of $\tau_1(\cdot)$. □

Lemma 9. *If m^* is a Nash equilibrium of the mechanism for the voluntary trading pricing equilibrium principle, then $p(m^*)$ satisfies the voluntary trading pricing rule, i.e., for $j = 1, \dots, J$, $p(m^*) \cdot Y_j(m^*) \geqq p(m^*) \cdot y_j$ for all $y_j \in \mathcal{Y}_j$ such that $y_j \leqq Y_j^+(m^*)$, and thus $\mu_1(m_1, m^*_{-1}) = \mu_1(m^*) = 0$ for all $m_1 \in M_1$.*

Proof. Suppose, to the contrary, that for some firm j, there exists a production plan $y_j \in \mathcal{Y}_j$ such that $p(m^*) \cdot Y_j(m^*) < p(m^*) \cdot y_j$ and $y_j \leqq Y_j^+(m^*)$.

Now suppose agent 1 chooses $t_{1j} = y_j$, $\gamma_1 > \gamma_1^*$, $\eta_1 < \eta_1^*$, and keeps the other components of the message unchanged. We have $\mu_1(m_1, m^*_{-1}) = p(m^*) \cdot y_j - p(m^*) \cdot Y_j(m^*) > 0$, and thus $x_1(m_1, m^*_{-1}) = [\mu_1(m_1, m^*_{-1}) + \frac{1}{1+\eta_1 \|t_{1j} - Y(m^*)\|}] x'(m^*)$. Note that, when $\eta_1 \to 0$, $[\mu_1(m_1, m^*_{-1}) + \frac{1}{1+\eta_1 \|t_{1j} - Y(m^*)\|}] x'(m^*) \to [\tau_1(m_1, m^*_{-1}) + 1] x'(m^*) > x(m^*) = X(m^*)$. Thus, when η_1 is sufficiently close to zero, from $[\mu_1(m_1, m^*_{-1}) + 1] x'(m^*) > x(m^*) = X(m^*)$, we have $x_1(m_i, m^*_{-1})\ P_1\ X_1(m^*)$ by continuity of preferences. Then, by Lemma 1, agent 1 can choose a very large γ_1 such that $X_1(m_1, m^*_{-1})\ P_1\ X_1(m^*)$, which contradicts the fact m^* is a Nash equilibrium. Hence $p(m^*)$ must satisfy the voluntary trading pricing rule. Finally, since there is no production plan $y_j \in \mathcal{Y}_j$ such that $p(m^*) \cdot y_j > p(m^*) \cdot Y_j(m^*)$ and $y_j \leqq Y_j^+(m^*)$, we must have $\mu_1(m_i, m^*_{-1}) = \mu_1(m^*) = 0$ for all $m_1 \in M_1$. □

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