Games and Economic Behavior 105 (2017) 1–8

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Guaranteed size ratio of ordinally efficient and envy-free mechanisms in the assignment problem [☆]

Chao Huang [a], Guoqiang Tian [b,c,*,1]

[a] Institute for Social and Economic Research, Nanjing Audit University, Nanjing, 211815, China
[b] Department of Economics, Texas A&M University, College Station, TX 77843, USA
[c] School of Economics, Shanghai University of Finance and Economics, Shanghai, 200433, China

ARTICLE INFO

Article history:
Received 18 January 2016
Available online 18 July 2017

JEL classification:
D63
D70

Keywords:
Size of assignment
Ordinal efficiency
Envy-freeness

ABSTRACT

In the assignment problem where agents can stay unassigned, the size of the assignment is an important consideration for designers. Bogomolnaia and Moulin (2015) show that there is a tension between size and fairness: the guaranteed size ratio of any envy-free mechanism is at most r_m, which converges decreasingly to $1 - \frac{1}{e} \approx 63.2\%$ as the maximum size increases. They then ask whether r_m is also the guaranteed size ratio for any ordinally efficient and envy-free mechanism. We study this issue and show that the lower bound of the guaranteed size ratio of ordinally efficient and envy-free mechanisms converges to $\frac{1}{2}$ as the maximum size increases, which means that almost half of the maximum size is wasted at the lower bound. Moreover, the exact lower bound is $\frac{m+1}{2m}$ when the maximum size m is odd.

1. Introduction

We study the assignment problem of assigning a set of heterogeneous indivisible objects to a set of agents, where each agent can receive at most one object. Typical examples include student placement in public schools, office assignment, and public housing allocation. For instance, a public housing allocation problem concerns resource allocation where a set of public houses are assigned to a set of agents without monetary transfers. In most of these markets, participants reveal ordinal preferences rather than cardinal utilities. Since deterministic assignments suffer from the lack of fairness, randomization becomes a common tool to recover fairness. A *random assignment mechanism* returns lotteries over deterministic assignments. *Random Priority* (RP) mechanism is very popular in real-life applications, which randomly orders the agents and assigns the first agent her top choice, the next agent her top choice among the remaining objects, and so on.

Properties of efficiency, fairness and incentives of the mechanism are the main concerns for designers. In their seminal work, Bogomolnaia and Moulin (2001) discover incompatibility among these properties. RP is *strategy-proof*, i.e., immune to preference manipulation, while it may induce ex-ante efficiency loss and unfairness among agents. In the assignment problems, not every pair of random assignments is comparable with ordinal preferences. Two random assignments of an

☆ We thank Herve Moulin, an advisory editor and two anonymous referees for very helpful comments and suggestions that significantly improved the exposition of the paper. Huang is grateful to Ning Sun, Yongchao Zhang and especially Qianfeng Tang for their guidance in matching and assignment theory. All errors are our own.

* Corresponding author.
E-mail addresses: tommyhuang16@163.com (C. Huang), gtian@tamu.edu (G. Tian).

1 Financial support from the National Natural Science Foundation of China (NSFC-71371117) and the Key Laboratory of Mathematical Economics (SUFE) at Ministry of Education of China is gratefully acknowledged.

http://dx.doi.org/10.1016/j.geb.2017.07.001
0899-8256/

agent are comparable only if under one assignment, for every object a, the probability of receiving an object that is at least as good as a is at least as large as under the other assignment. We compare the assignments with respect to this first order stochastic dominance (see detailed definition in Section 2). A mechanism *dominates* another if for every preference profile, it assigns to every agent a random assignment that she weakly prefers to the random assignment of the other mechanism with respect to the first order stochastic dominance. Bogomolnaia and Moulin (2001) show that RP can be dominated by other mechanisms, and propose a contender to RP: the *Probabilistic Serial* (PS) mechanism. Although PS is not strategy-proof, it has superior efficiency and fairness properties. In particular, PS is efficient with respect to the first order stochastic dominance, which is called *ordinally efficient*, and further it is *envy-free* in the sense that no agent envies any other agent. RP and PS have been widely studied in recent literature.

In real-life applications, outside options are quite common. For instance, college students have the option not to live in the dormitories and may choose to live outside the campus. In public school choice, students may prefer to go to job markets, stay at home or go to private schools rather than go to some public schools.

When agents can always opt out, the number of objects actually assigned is also an important consideration for designers. The expected number of objects assigned in an assignment is called the *size* of the assignment. When a college assigns dormitories to students, the size of assignment reflects the utilization of resources. In school choice problem, the enrollment rate of high schools is also an important consideration for a social planner. In practice, the size of assignment is not only a measure of utilization of resources, but also an important social concern.

Given the preference profiles of agents, there is a potential *maximum size* of the problem, i.e., the maximum objects that can be assigned. The *size ratio* of an assignment is the ratio of its size to the maximum size of the problem. The worst size ratio of a random assignment mechanism is defined as the *guaranteed size ratio* of the mechanism. This paper investigates how much of the maximum size can be wasted by an ordinally efficient and envy-free mechanism, or equivalently, what is the lower bound for the guaranteed size ratio of ordinally efficient and envy-free mechanisms. This is motivated by the open question proposed by Bogomolnaia and Moulin (2015), in which they ask whether r_m, the guaranteed size ratio of PS, is the lower bound for any ordinally efficient and envy-free mechanism where r_m is an index that decreases and converges to $1 - \frac{1}{e} \approx 0.632$ as the maximum size m increases.

We show that some ordinally efficient and envy-free mechanism can assign, in fact, less than r_m of the maximum size. Our result can be interpreted from the two examples we present in Section 4. The constructions of preference profiles are modified from the *canonical diagonal problem* studied by Bogomolnaia and Moulin (2015). In the problems with odd maximum size m, we can construct a class of ordinally efficient and envy-free assignments with the size of $\frac{m+1}{2}$. Thus $\frac{m-1}{2}$ objects have been wasted, and the number of wasted objects approaches half of the maximum size as m increases.

In the assignment problem, there is a well known result: any ordinally efficient assignment is of size at least $\frac{m+1}{2}$ (resp. $\frac{m}{2}$) when m is odd (resp. even), the reason for which will be explained in Section 4. Since the size of our constructed assignment reaches $\frac{m+1}{2}$ when m is odd, we know that $\frac{m+1}{2}$ is also the lower bound for any ordinally efficient and envy-free assignment with an odd maximum size m.

In the problems with even maximum size m, our construction results in an ordinally efficient and envy-free assignment with the size of $\frac{m}{2} + \frac{m}{m+2}$. This does not reach the least size for ordinally efficient assignments. While we are not certain as to whether this is the exact lower bound for any ordinally efficient and envy-free assignment, the size ratio $\frac{1}{2} + \frac{1}{m+2}$ of the constructed assignment, nevertheless, converges decreasingly to $\frac{1}{2}$ as the maximum size m increases, which is also the least size ratio for any ordinally efficient assignment in limit.

Our results in both cases imply that there exist ordinally efficient and envy-free mechanisms that waste almost half of the maximum size as the maximum size becomes larger. Therefore, the lower bound of guaranteed size ratio of any ordinally efficient and envy-free mechanism converges to $\frac{1}{2}$ as the maximum size increases.

1.1. Related literature

Maximizing the size of assignment has been the main object in the literature of online bilateral matching algorithm (see, e.g., Procaccia and Tennenholtz, 2009). The size of assignment recently appears in the literature on its connection with efficiency, fairness and incentives.

Bogomolnaia and Moulin (2015) show the tension between size and fairness. The tension can be easily understood from the elementary example presented in their work. There are two objects a, b and two agents, say, Ana and Bob who both prefer a to b. Only a is acceptable to Ana (which means she prefers her outside option to b) while the two objects are both acceptable to Bob. To maximize the size we can only assign a to Ana and b to Bob, and get the maximum size 2. However, this is unfair to Bob who envies Ana's assignment. The outcome of PS is fair but assigns only the expected size of 1.5 in this problem. In order to depict this tension, Bogomolnaia and Moulin (2015) first show that PS reaches the worst case in the canonical diagonal problems, and derive the guaranteed size ratio of PS (denoted by r_m), which converges decreasingly to $1 - \frac{1}{e} \approx 63.2\%$ as the maximum size m increases. They then show that the guaranteed size ratio of any envy-free mechanism is at most r_m. In other words, in the worst case, an envy-free mechanism assigns at most r_m of the maximum feasible size.

Erdil (2014) shows that the ex-ante efficiency of some strategy-proof mechanism (like RP) can be improved by assigning more objects, while the strategy-proofness is preserved. Such strategy-proof improvement cannot be induced by a mere reshuffling of objects (in probability shares) among agents, but must involve assigning more objects. Also he shows that RP is dominated within the class of ex-post efficient strategy-proof mechanisms which satisfy equal treatment of equals.

There are several studies concerning the guaranteed size ratio of RP. Bhalgat et al. (2011) and Krysta et al. (2014) provide the lower bound $1-(1-\frac{1}{m+1})^m-\frac{1}{m}$, a sequence converging increasingly to $1-\frac{1}{e}$, for the guaranteed size ratio of RP. In a radically different way, the results in Cres and Moulin (2001) and Bogomolnaia and Moulin (2015) also imply that the guaranteed size ratio of RP is always bounded above by r_m. These results are consistent with the asymptotic equivalence results in Che and Kojima (2010), in which they show that RP and PS become equivalent when the market becomes large.

Our work also relates to the characterization results of random assignment mechanisms. PS is ordinally efficient and envy-free, but it is not the only mechanism that satisfies these two properties. The characterization of PS requires an extra invariance axiom (see Bogomolnaia and Heo, 2012, and Hashimoto et al., 2014).[2] Ordinally efficient and envy-free mechanisms which do not produce the assignments of PS can also be seen from the examples we present in Section 4.

The paper proceeds as follows. Section 2 specifies the framework. Section 3 introduces the canonical diagonal problem which is closely related to our result. Section 4 provides two illustrative examples. Section 5 presents the main result. Section 6 concludes.

2. Model

We follow the notations of Bogomolnaia and Moulin (2015). Let N denote a finite set of agents and A a finite set of objects, with respective cardinalities n and q. Each agent $i \in N$ has a strict, complete and transitive preference relation R_i over $\widetilde{A} = A \cup \{\widetilde{\emptyset}\}$ so that there are no two objects indifferent to the agent, where $\widetilde{\emptyset}$ is the outside option. For simplicity, we write $R_i = (a_1, a_2, \cdots, a_k)$ where a_1 is the best object for i and a_k her least preferred acceptable object. We write $a \in R_i$ if a is an acceptable object for i, and $R_i = \emptyset$ if no object is acceptable to i.

Denote by $\mathcal{R}(A)$ the set of all possible preference profiles. A profile of preference $R \in \mathcal{R}(A)$ defines a compatibility bipartite graph $E \subseteq N \times A : ia \in E(R) \Leftrightarrow a \in R_i$, describing which objects are acceptable to which agents. An assignment problem is a triple $\Delta = (N, A, R)$, and its compatibility graph is written as $E(\Delta)$. An assignment is an $|N| \times |A|$ matrix $P = (p_{ia})_{i \in N, a \in A}$ where $\sum_{i \in N} p_{ia} \leq 1$ for all a and $s(i) = \sum_{a \in A} p_{ia} \leq 1$ for all i. An assignment is *feasible* at R if $ia \in E(\Delta)$ whenever $p_{ia} > 0$. It is *deterministic* if $p_{ia} \in \{0, 1\}$ for all i and a. Denote by $\mathcal{P}(E(\Delta))$, or simply $\mathcal{P}(E)$, the set of feasible assignments at Δ, and $\mathcal{P}^d(E)$ the subset of deterministic feasible assignments.

The total number of objects allocated by P will be called the *size* of an assignment P, and is formally defined as $s(P) = \sum_{a \in A} \sum_{i \in N} p_{ia}$. A random assignment is implemented by deterministic assignments of (almost) equal size: any $P \in \mathcal{P}(E)$ is a convex combination of deterministic assignments of size $\lfloor s(P) \rfloor$ or $\lceil s(P) \rceil$ (lower and upper integral part).[3] In particular, the program

$$s^*(E) = \max_{P \in \mathcal{P}(E)} s(P)$$

has at least one deterministic solution, and every solution is a convex combination of such deterministic assignments. We call $s^*(E(\Delta))$ the *size of the problem* Δ, i.e., the maximum number of objects/agents it is feasible to assign. The set of assignment problems of size m is denoted by $\mathcal{A}^m$. Given an assignment in the problem of size m, the *size ratio* of the assignment is the ratio of the size of the assignment to the maximum size m.

An assignment mechanism F is a function which associates to each preference profile an assignment matrix. The *guaranteed m-size ratio* of F is defined as the worst ratio of the actual expected size to the maximum feasible size:

$$\sigma_m(F) = \min_{\Delta \in \mathcal{A}^m} \frac{1}{m} s(F(\Delta)).$$

We compare an agent's assignments by means of first order stochastic dominance. Let $R_i = (a_1, \cdots, a_k)$ be agent i's preference and P_i, Q_i be two assignments for agent i. We say that P_i *weakly stochastically dominates* Q_i at R_i, denoted as $P_i \succeq_i^{sd} Q_i$, if

$$\sum_1^t p_{ia_t} \geq \sum_1^t q_{ia_t}, \quad \text{for all} \quad t, \quad 1 \leq t \leq k.$$

P_i is said to *stochastically dominate* Q_i, if in addition to the above, the inequality is strict for at least one t. An assignment P *weakly dominates* another assignment Q if for each agent i, $P_i \succeq_i^{sd} Q_i$. If we also have P_j stochastically dominates Q_j for some j, we say that P *dominates* Q. If an assignment is not dominated by any other assignment, then it is called *ordinally efficient*.

If P is a convex combination of Pareto efficient deterministic assignments, then it is called *ex-post efficient*. Every ordinally efficient assignment is ex-post efficient, but the converse does not hold. We say that P is *envy-free* at R if for each pair $i, j \in N$, $P_i \succeq_i^{sd} P_j$. Envy-freeness is a property of fairness. We say that an assignment P *treats equals equally* (or is *symmetric*) at R if for each $i, j \in N$, $R_i = R_j$ implies $P_i = P_j$. Equal treatment of equals is a notion of symmetry, which is weaker than

[2] See Bogomolnaia (2015) for a new characterization of PS, which applies to non-strict preferences and non-integer quantities of objects.

[3] This is a refinement of Birkhoff's Theorem. It follows from the results in Budish et al. (2013).

envy-freeness. A mechanism F is *strategy-proof*, if for all problems, all $i \in N$, and all $R_i' \in \mathcal{R}(A)$, we have $P_i \succeq_i^{sd} P_i'$, where $F(N, A, R) = P$ and $F(N, A, (R_i', R_{-i})) = P'$.

Now we introduce the *Probabilistic Serial* mechanism (PS) proposed by Bogomolnaia and Moulin (2001). Regard each object as a divisible object of probability shares. Each agent "eats" the best available object with speed one at every moment of time $t \in [0, 1]$. Agents stop eating by time 1, or before time 1 when they find that no available object is acceptable (object a is available at time t if less than one share of a has been eaten away by time t). The resulting profile of shares of objects eaten by agents corresponds to the outcome of PS, which we call the *probabilistic serial assignment*.

3. Canonical diagonal problem

Our construction of ordinally efficient and envy-free assignments is closely related with the following special profile of preferences, which is called the *canonical diagonal problem*:

5	4	3	2	1
a_5	a_5	a_5	a_5	a_5
$\widetilde{\emptyset}$	a_4	a_4	a_4	a_4
	$\widetilde{\emptyset}$	a_3	a_3	a_3
		$\widetilde{\emptyset}$	a_2	a_2
			$\widetilde{\emptyset}$	a_1.

In the canonical diagonal problem of size m (denoted by Δ_m^*), there are m agents $N = \{1, \cdots, m\}$ and m objects $A = \{a_1, \cdots, a_m\}$, and agent $i's$ preferences are $R_i = (a_m, a_{m-1}, \cdots, a_i)$. Problem Δ_m^* is in $\mathcal{A}_m$ because we can assign object a_i to agent i for all i. The canonical diagonal problem has been studied in Karp et al. (1990); Cres and Moulin (2001), and Bogomolnaia and Moulin (2002; 2015).

Bogomolnaia and Moulin (2002) show that PS is the only mechanism that satisfies ordinal efficiency and envy-freeness in the canonical diagonal problems. In particular, consider the class $\mathcal{D}^m$ of problems where for a common ordering $\{a_1, \cdots, a_m\}$ of the objects, all individual preferences take the form $R^k = (a_m, a_{m-1}, \cdots, a_k)$. D^m contains Δ_m^*, as well as problems with different numbers of preferences R^k for each k. Theorem 4.1 in Bogomolnaia and Moulin (2002) states that PS is the only mechanism that satisfies ordinal efficiency and envy-freeness on $\mathcal{D}^m$.

Bogomolnaia and Moulin (2015) show that the assignment of PS in the canonical diagonal problem achieves the worst possible size ratio. We reproduce the calculation of this ratio from their work, since it is an important index and provides a nice intuition. When applying Probabilistic Serial algorithm in the problem Δ_m^*, all the agents eat a_m simultaneously at the beginning, and they each get $\frac{1}{m}$ of a_m; then object a_{m-1} is eaten by agents $1, \cdots, m-1$, who each get a share $\frac{1}{m-1}$; the process goes on until the critical object a_{k_m} such that

$$\frac{1}{k_m+1} + \frac{1}{k_m+2} + \cdots + \frac{1}{m} \leq 1 < \frac{1}{k_m} + \frac{1}{k_m+1} + \cdots + \frac{1}{m}.$$

Thus a_{k_m+1} has been eaten up, but object a_{k_m} hasn't. Objects $a_{k_m-1}, \cdots, a_1$ have not been eaten at all. Define for any integer $1 \leq k < m$

$$S(m,k) = \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{m},$$

so that k_m is defined by the inequalities $S(m, k_m) \leq 1 < S(m, k_m - 1)$. Then, the size of the assignment of $PS(\Delta_m^*)$ is as follows:

$$\begin{aligned} s(PS(\Delta_m^*)) &= \sum_{1 \leq i, j \leq m} p_{ia_j} \\ &= m - k_m + k_m(1 - S(m, k_m)) \\ &= m - k_m S(m, k_m). \end{aligned}$$

Thus the size ratio of PS in the canonical diagonal problem, denoted by r_m, is calculated as follows:

$$r_m \equiv \frac{1}{m} s(PS(\Delta_m^*)) = 1 - \frac{k_m}{m} S(m, k_m).$$

Bogomolnaia and Moulin (2015) prove that PS achieves the worst possible size ratio in the canonical diagonal problem. Hence the guaranteed m-size ratio of PS is r_m. The sequence r_m is decreasing and converges to $1 - \frac{1}{e} \approx 0.632$ at the speed $O(\frac{1}{n})$. For instance, $r_2 = 0.75, r_3 \approx 0.722, r_4 \approx 0.708, r_{10} \approx 0.662, r_{20} \approx 0.648$.

Bogomolnaia and Moulin (2015) next show that the guaranteed m-size ratio of any envy-free mechanism is at most r_m and further ask whether the guaranteed m-size ratio of any ordinally efficient and envy-free mechanism is r_m. Since the results of Bogomolnaia and Moulin (2002) imply that in the problem of Δ_m^*, PS is the only mechanism that satisfies ordinal efficiency and envy-freeness, the question is equivalent to asking whether the problems Δ_m^* are also the worst case configuration for any ordinally efficient and envy-free mechanisms.

4. Illustrative examples

Our answer to this question is negative. We first present the following example. The assignment in the example is ordinally efficient and envy-free, while the size of the assignment is less than mr_m, i.e., the maximum size multiplies r_m.

Example 1. We consider the following preference profile which resembles the canonical diagonal problem Δ_3^*. The only difference is that a_3 is not acceptable to agent 2. The number behind the object is the share of this object assigned to the agent. The maximum size is 3, since we can assign a_i to agent i for all i.

3	2	1
$a_3, \frac{1}{2}$		$a_3, \frac{1}{2}$
$\widetilde{\emptyset}$	$a_2, \frac{1}{2}$	$a_2, \frac{1}{2}$
	$\widetilde{\emptyset}$	$a_1, 0$

It is straightforward to check that this assignment is envy-free. For the ordinal efficiency, suppose there exists an assignment Q such that for $i = 1, 2, 3$, $Q_i \succeq_i^{sd} P_i$, where P is the assignment in the example. Since $q_{3a_3} \geq p_{3a_3} = \frac{1}{2}$, $q_{1a_3} \geq p_{1a_3} = \frac{1}{2}$, we have $q_{3a_3} = q_{1a_3} = \frac{1}{2}$. Then $q_{1a_3} + q_{1a_2} \geq p_{1a_3} + p_{1a_2} = 1$ implies $q_{1a_2} = \frac{1}{2}$. Finally, $q_{2a_2} \geq p_{2a_2} = \frac{1}{2}$ implies $q_{2a_2} = \frac{1}{2}$. Hence, $Q = P$ and P is ordinally efficient. The size of assignment P is 2. We then compare it with $3r_3$, which is the size PS achieves in the canonical diagonal problem Δ_3^*. PS runs in Δ_3^* as follows: each of agents 3, 2 and 1 eats $\frac{1}{3}$ of object a_3, then each of agents 2 and 1 eats $\frac{1}{2}$ of object a_2, and finally agent 1 eats $\frac{1}{6}$ of object a_1. Thus $3r_3 = 2\frac{1}{6}$, which is larger than the size of assignment P in the example. Therefore, any ordinally efficient and envy-free mechanism producing the assignment P in this problem has a guaranteed m-size ratio less than r_m for $m = 3$.

In any problem of size m, any Pareto efficient deterministic assignment is of size at least $\frac{m}{2}$ (resp. $\frac{m+1}{2}$), when m is even (resp. odd). This is because for any agent i and object a matched in the assignment of maximum size, at least one of them is matched in a Pareto efficient deterministic assignment; otherwise assigning a to i would be a Pareto improvement. Since every ex-post efficient assignment is a convex combination of Pareto efficient deterministic assignments, any ex-post efficient assignment is also of size at least $\frac{m}{2}$ (resp. $\frac{m+1}{2}$), when the maximum size m is even (resp. odd). This is also true for ordinally efficient assignment because every ordinally efficient assignment is ex-post efficient.

The statement that "every ordinally efficient assignment is ex-post efficient" has been proven in the Lemma 2 of Bogomolnaia and Moulin (2001) in the model without outside option. Their argument is also valid in the model with outside option. Suppose that an assignment Q is not ex-post efficient, then every decomposition of Q as a convex combination of deterministic assignments contains at least one Pareto inefficient deterministic assignment. Making Pareto improvement on this deterministic assignment, and replacing the original one in the decomposition, we obtain a random assignment that dominates Q, since the stochastic dominance is preserved by convex combinations.

Therefore, in any problem of size m, any ordinally efficient assignment is of size at least $\frac{m}{2}$ (resp. $\frac{m+1}{2}$), when m is even (resp. odd). Examples that reach the lower bound can be constructed in the canonical diagonal problems. For instance, consider the problem of Δ_m^* where m is even, we assign a_m to agent 1 (with probability 1), a_{m-1} to agent 2,$\cdots$, $a_{\frac{m}{2}+1}$ to agent $\frac{m}{2}$. Other objects are not assigned at all. Such an assignment is ordinally efficient while wasting half of the maximum objects.

Note that the assignment in Example 1 reaches the least size of any ordinally efficient assignment. Generally, in similar problems of size m where m is odd, we can construct a class of ordinally efficient and envy-free assignments that reach the least size of any ordinally efficient assignment (see the construction in the proof of Proposition 1). Thus the lower bound of the size of any ordinally efficient and envy-free assignment is also $\frac{m+1}{2}$.

Another finding in Example 1 is that the probabilistic serial assignment in this problem maximizes the size within the class of ordinally efficient and envy-free assignments. In any ordinally efficient and envy-free assignment, a_3 must have been fully assigned; otherwise assigning the remaining share of a_3 to agent 3 is an improvement. For the same reason, a_2 is fully assigned. By envy-freeness, agents 1 and 3 both get $\frac{1}{2}$ of a_3. Suppose agent 2 gets x of a_2 and agent 1 gets $1 - x$ of a_2. Again, by envy-freeness, $x \leq \frac{1}{2} + 1 - x$ such that agent 1 does not envy agent 2. Thus $x \leq \frac{3}{4}$ and $1 - x \geq \frac{1}{4}$, which implies agent 1 is assigned at most $\frac{1}{4}$ of a_1. Therefore, the maximum size within the class of ordinally efficient and envy-free assignments is $2\frac{1}{4}$ in the problem of Example 1. This is achieved by the probabilistic serial assignment:

3	2	1
$a_3, \frac{1}{2}$		$a_3, \frac{1}{2}$
$\widetilde{\emptyset}$	$a_2, \frac{3}{4}$	$a_2, \frac{1}{4}$
	$\widetilde{\emptyset}$	$a_1, \frac{1}{4}$

In the problem of size m when m is even, such constructions result in the size of $\frac{m}{2} + \frac{m}{m+2}$, which does not reach the lower bound of the size of any ordinally efficient assignment. This can be seen from the following example.

Example 2. In the canonical diagonal problem Δ_4^*, assume a_4 is unacceptable to agent 3, while the preferences of the other agents remain unchanged. The maximum size of the new problem is 4 since we can assign a_i to agent i for all i.

4	3	2	1
$a_4, \frac{1}{3}$		$a_4, \frac{1}{3}$	$a_4, \frac{1}{3}$
$\widetilde{\emptyset}$	$a_3, \frac{1}{3}$	$a_3, \frac{1}{3}$	$a_3, \frac{1}{3}$
	$\widetilde{\emptyset}$	$a_2, \frac{1}{3}$	$a_2, \frac{1}{3}$
		$\widetilde{\emptyset}$	$a_1, 0$

It follows the same way in Example 1 to check that the assignment above is both ordinally efficient and envy-free. The size of the assignment in this example is $2\frac{2}{3}$, which is $\frac{2}{3}$ of the maximum size. The ratio is less than $r_4 \approx 0.708$. However, unlike the assignment in Example 1, the size ratio in this example does not reach $\frac{1}{2}$, which is the lower bound of the ratio for ordinally efficient assignments in this problem.

Moreover, it is not difficult to find out again that the probabilistic serial assignment maximizes the size within the class of ordinally efficient and envy-free assignments in this example. In any ordinally efficient and envy-free assignment, a_4 must have been fully assigned; otherwise assigning the remaining share of a_4 to agent 4 is an improvement. For the same reason, a_3 is fully assigned. By envy-freeness, each of agents 1, 2 and 4 gets $\frac{1}{3}$ of a_4. Suppose agent 3 gets x of a_3, then agents 1 and 2 both get $\frac{1-x}{2}$ of a_3 due to envy-freeness. Again, by envy-freeness, $\frac{1-x}{2} \le x \le \frac{1}{3} + \frac{1-x}{2}$ such that agents 2 and 3 do not envy each other. Thus we have $\frac{1}{3} \le x \le \frac{5}{9}$.

4	3	2	1
$a_4, \frac{1}{3}$		$a_4, \frac{1}{3}$	$a_4, \frac{1}{3}$
$\widetilde{\emptyset}$	$a_3, \frac{5}{9}$	$a_3, \frac{2}{9}$	$a_3, \frac{2}{9}$
	$\widetilde{\emptyset}$	$a_2, \frac{4}{9}$	$a_2, \frac{4}{9}$
		$\widetilde{\emptyset}$	$a_1, 0$

Then it comes to see that a_2 is not fully assigned, and the size ratio would be in $[\frac{2}{3}, \frac{13}{18}]$. The upper bound is exactly achieved by probabilistic serial assignment as above.

5. Result

Similar constructions of the assignments in Examples 1 and 2 can be applied for any maximum sizes. Now we state our result as follows.

Proposition 1. *For economic environments under consideration, the following statements are true:*

i) *When the maximum size m is odd, the size of any ordinally efficient and envy-free assignment is at least $\frac{m+1}{2}$. This lower bound is reached by some ordinally efficient and envy-free assignment under special preference profile.*
ii) *When the maximum size m is even, there exists ordinally efficient and envy-free assignment with the size of $\frac{m}{2} + \frac{m}{m+2}$ under special preference profile.*

Proof. i) Modify the preference profile in Δ_m^* where m is odd and $m \ge 3$ as follows. Let $a_{m-1}, a_{m-2}, \ldots, a_{\frac{m+1}{2}}$ be the only acceptable object for agent $m-1, m-2, \ldots, \frac{m+1}{2}$ respectively. The preferences of agents $1, 2, \ldots, \frac{m-1}{2}$ and m remain unchanged. In the resulting problem, we can still assign a_i to agent i for all i, and thus the maximum size is m. Denote this problem by Δ_m'. Consider the following assignment P' in Δ_m': each agent in $\{m, m-1, \ldots, \frac{m+1}{2}\}$ gets $\frac{2}{m+1}$ of the only acceptable object. Each agent in $\{\frac{m-1}{2}, \ldots, 2, 1\}$ gets $\frac{2}{m+1}$ of every object in $\{a_m, a_{m-1}, \ldots, a_{\frac{m+1}{2}}\}$. It is straightforward to check that P' is envy-free.

m	$m-1$	$\cdots$	$\frac{m+1}{2}$	$\frac{m-1}{2}$	$\cdots$	2	1
$a_m, \frac{2}{m+1}$				$a_m, \frac{2}{m+1}$	$\cdots$	$a_m, \frac{2}{m+1}$	$a_m, \frac{2}{m+1}$
$\widetilde{\emptyset}$	$a_{m-1}, \frac{2}{m+1}$			$a_{m-1}, \frac{2}{m+1}$	$\cdots$	$a_{m-1}, \frac{2}{m+1}$	$a_{m-1}, \frac{2}{m+1}$
	$\widetilde{\emptyset}$			$\vdots$		$\vdots$	$\vdots$
			$a_{\frac{m+1}{2}}, \frac{2}{m+1}$	$a_{\frac{m+1}{2}}, \frac{2}{m+1}$	$\cdots$	$a_{\frac{m+1}{2}}, \frac{2}{m+1}$	$a_{\frac{m+1}{2}}, \frac{2}{m+1}$
			$\widetilde{\emptyset}$	$a_{\frac{m-1}{2}}, 0$	$\cdots$	$a_{\frac{m-1}{2}}, 0$	$a_{\frac{m-1}{2}}, 0$
				$\widetilde{\emptyset}$		$\vdots$	$\vdots$
					$\widetilde{\emptyset}$	$a_2, 0$	$a_2, 0$
						$\widetilde{\emptyset}$	$a_1, 0.$

We now check the ordinal efficiency. Suppose there exists an assignment Q' such that for $i = 1, 2, \dots, m$, $Q'_i \succeq^{sd}_i P'_i$. For object a_m, we have $q'_{ia_m} \geq p'_{ia_m} = \frac{2}{m+1}$ for $i = 1, 2, \dots, \frac{m-1}{2}$, and m. It must be that $q'_{ia_m} = \frac{2}{m+1}$ for $i = 1, 2, \dots, \frac{m-1}{2}$, and m. For object a_{m-1}, we then have $q'_{ia_{m-1}} \geq p'_{ia_{m-1}} = \frac{2}{m+1}$ for $i = 1, 2, \dots, \frac{m-1}{2}$, and $m-1$. It must be $q'_{ia_{m-1}} = \frac{2}{m+1}$ for $i = 1, 2, \dots, \frac{m-1}{2}$, and $m-1$. From this point it is straightforward to prove $Q' = P'$ by an induction. Thus P' is ordinally efficient.

The size of P' is $\frac{m+1}{2}$, which reaches the least size for ordinally efficient assignments. When $m = 1$, the problem reduces to Δ_1^* and the statement is trivially true. As a result, when the maximum size m is odd, the lower bound of the size of any ordinally efficient and envy-free assignment is $\frac{m+1}{2}$.

ii) Modify the preference profile in Δ_m^* where m is even and $m \geq 4$ as follows. Let $a_{m-1}, a_{m-2}, \dots, a_{\frac{m}{2}+1}$ be the only acceptable object for agent $m-1, m-2, \dots, \frac{m}{2}+1$ respectively. The preferences of agents $1, 2, \dots, \frac{m}{2}$ and m remain unchanged. In the resulting problem, we can still assign a_i to agent i for all i, and thus the maximum size is m. Denote this problem by Δ''_m. Consider the following assignment P'' in Δ''_m: each agent in $\{m, m-1, \dots, \frac{m}{2}+1\}$ gets $\frac{2}{m+2}$ of the only acceptable object. Each agent in $\{\frac{m}{2}, \dots, 2, 1\}$ gets $\frac{2}{m+2}$ of every object in $\{a_m, a_{m-1}, \dots, a_{\frac{m}{2}}\}$. The size of P'' is $m(\frac{1}{2} + \frac{1}{m+2})$, and it follows the same way in i) to check that P'' is both ordinally efficient and envy-free.

$$
\begin{array}{cccccccc}
m & m-1 & \cdots & \frac{m}{2}+1 & \frac{m}{2} & \cdots & 2 & 1 \\
a_m, \frac{2}{m+2} & & & & a_m, \frac{2}{m+2} & \cdots & a_m, \frac{2}{m+2} & a_m, \frac{2}{m+2} \\
\widetilde{\emptyset} & a_{m-1}, \frac{2}{m+2} & & & a_{m-1}, \frac{2}{m+2} & \cdots & a_{m-1}, \frac{2}{m+2} & a_{m-1}, \frac{2}{m+2} \\
 & \widetilde{\emptyset} & & & \vdots & & \vdots & \vdots \\
 & & & a_{\frac{m}{2}+1}, \frac{2}{m+2} & a_{\frac{m}{2}+1}, \frac{2}{m+2} & \cdots & a_{\frac{m}{2}+1}, \frac{2}{m+2} & a_{\frac{m}{2}+1}, \frac{2}{m+2} \\
 & & & \widetilde{\emptyset} & a_{\frac{m}{2}}, \frac{2}{m+2} & \cdots & a_{\frac{m}{2}}, \frac{2}{m+2} & a_{\frac{m}{2}}, \frac{2}{m+2} \\
 & & & & \widetilde{\emptyset} & \cdots & a_{\frac{m}{2}-1}, 0 & a_{\frac{m}{2}-1}, 0 \\
 & & & & & & \vdots & \vdots \\
 & & & & & \widetilde{\emptyset} & a_2, 0 & a_2, 0 \\
 & & & & & & \widetilde{\emptyset} & a_1, 0.
\end{array}
$$

When $m = 2$, the problem reduces to the canonical diagonal problem Δ_2^* and obviously the outcome of PS reaches $m(\frac{1}{2} + \frac{1}{m+2}) = \frac{3}{2}$. $\square$

There is a main feature of the constructed assignments: each agent shares the same probability of obtaining each acceptable object that has been assigned. Hence the constructed assignment can be viewed as an equal division of the acceptable objects among agents.

In the problem with an odd maximum size m, our result shows that the lower bound of the size of any ordinally efficient and envy-free assignment is $\frac{m+1}{2}$. In the problem with an even maximum size m, our construction results in an ordinally efficient and envy-free assignment with the size of $\frac{m}{2} + \frac{m}{m+2}$. Interestingly, unlike the problems with odd sizes, this does not reach the least size for ordinally efficient assignments, which is $\frac{m}{2}$. However, the size ratio of the constructed assignment is $\frac{1}{2} + \frac{1}{m+2}$, which converges decreasingly to $\frac{1}{2}$ as the maximum size m increases. This is also the least size ratio for any ordinally efficient assignment in the limit. Hence, we obtain the limit bound $\frac{1}{2}$ when the maximum size is even.

When m is large, both cases of Proposition 1 imply that there exist ordinally efficient and envy-free mechanisms that waste almost half of the maximum size. Summarizing the above discussion, we have the following theorem:

Theorem 1. *The lower bound of guaranteed size ratio of any ordinally efficient and envy-free mechanism approaches $\frac{1}{2}$ as the maximum size increases.*

6. Concluding remarks

In many applications the designer wants to improve the size within the class of efficient and fair assignments. For instance, suppose the designer wants to assign school seats to students and faces the problem in Example 1 or 2. He would like to make an ordinally efficient and envy-free assignment, and improve the expected enrollment rate as much as possible. Then the designer would prefer the probabilistic serial assignments to the assignments we present in the examples, since the former assigns more objects than the latter.

How to maximize the size within the class of ordinally efficient and envy-free assignments is an open question. We show that, in both problems of Examples 1 and 2, the probabilistic serial assignments maximize the size among all ordinally efficient and envy-free assignments. Bogomolnaia and Moulin (2002) show that PS is the only mechanism that satisfies ordinal efficiency and envy-freeness in a class of problems that contains the canonical diagonal problems. Thus PS also maximizes the size in such problems since it is the only candidate. These facts make us to conjecture that the Probabilistic

Serial mechanism is the answer to this question. If the conjecture is not true, an efficient and fair mechanism that improves the size on PS will find its use in real-life applications.

Erdil (2014) shows that there exists symmetric, ex-post efficient and strategy-proof mechanism that assigns more objects than RP. The notion of "assign more objects" is stronger in his statement: not only assign a larger size of objects, but assign a larger set of objects that contains the original set. As a counterpart, it is interesting to know whether there exists ordinally efficient and envy-free mechanism that assigns more objects than PS. However, the notion of "assign more objects" in our question simply means assigning a larger size of objects.

Finally, our construction of ordinally efficient and envy-free assignments presents an open question. The constructed assignment does not reach the least size of ordinally efficient assignments when the maximum size is even. We do not know whether it reaches the least size of ordinally efficient and envy-free assignments. It is an interesting question to find the exact lower bound for the size ratio of any ordinally efficient and envy-free assignment with even maximum size.

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