**M412: Theory of Partial Differential Equations**
**Final TEST, December 8th, 2006**
**Notes, books, and calculators are not authorized.**

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

**Question 1**

Solve the following PDE (Hint: use the method of characteristics):

$$\partial_t w + 3\partial_x w = 0, \quad x > 0,\ t > 0$$
$$w(x,0) = f(x), \quad x > 0, \quad \text{and} \quad w(0,t) = h(t), \quad t > 0.$$

---

Define the characteristics by

$$\frac{dX}{dt} = 3, \quad \text{with} \quad \begin{cases} X(0) = X_0, & \text{if } X_0 > 0, \\ X(\tau) = 0, & \text{if } \tau > 0. \end{cases}$$

The general solution is $X(t) = 3(t - \tau) + X_0$. Then $X_0 = X - 3t$ if $X - 3t > 0$ and $\tau = t - X/3$ if $X - 3t \le 0$.

Now we set $\phi(t) = w(X(t), t)$ and we insert this ansatz in the equation. This gives $d\phi/dt = 0$, i.e., $\phi(t)$ is a constant. In other words

$$\phi(t) = \begin{cases} \phi(0) = f(X_0) = f(x - 3t) & \text{if } x - 3t > 0, \\ \phi(\tau) = h(\tau) = h(t - x/3) & \text{if } x - 3t \le 0. \end{cases}$$

---

## Question 2

Consider the following wave equation

$$\partial_{tt} w - c^2 \partial_{xx} w = 0, \quad x > 0,\ t > 0$$
$$w(x,0) = f(x), \quad x > 0, \qquad \partial_t w(x,0) = 0, \quad x > 0, \quad \text{and} \quad w(0,t) = 0, \quad t > 0.$$

(a) Solve the equation (Hint: recall that the solution can always be put in the form $F(x-ct) + G(x+ct)$)

If $x - ct > 0$, we can apply D'Alembert's formula $u(x,t) = F(x-ct) + G(x+ct)$

$$F(z) = \frac{1}{2} f(z) + \frac{1}{2c} \int_z^0 g(\tau) d\tau, \quad G(z) = \frac{1}{2} f(z) + \frac{1}{2c} \int_0^z g(\tau) d\tau,$$

where $g(x) = \partial_t w(x,0) = 0$. In other words

$$w(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)), \quad \text{if } x \geq ct.$$

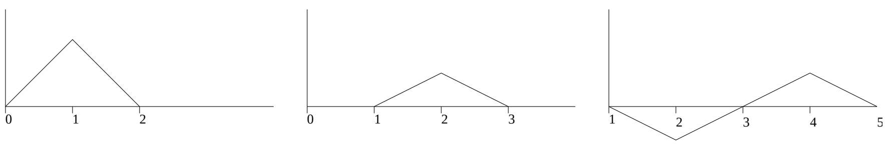
If $x - ct < 0$ we apply the boundary condition at $x = 0$

$$w(0,t) = 0 = F(-ct) + G(ct), \quad \forall t \geq 0.$$

This means $f(-z) = -G(z) = -\frac{1}{2} f(z)$ for all $z \geq 0$. In other words we have obtained

$$w(x,t) = \frac{1}{2}(-f(-x+ct) + f(x+ct)), \quad \text{if } x \leq ct.$$

(b) We now set $c = 1$, $f(x) = x$, if $x \in [0,1]$, $f(x) = 2 - x$, if $x \in [1,2]$, and $f(x) = 0$ otherwise. Draw the graph of the solution at $t = 0$, $t = 1$, and $t = 2$ (draw three different graphs).



**Question 3**

Consider the following conservation equation

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{6} & \text{if } x < 0, \\ \frac{1}{3} & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2 - 3\rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

---

The characteristics are defined by

$$\frac{dX(t)}{dt} = q'(\rho) = 2(1 - 3\rho(x(t), t)), \quad X(0) = X_0.$$

Set $\phi(t) = \rho(X(t), t)$, then we obtain that $\phi$ is constant, i.e., $\rho$ is constant along the characteristics: $\rho(X(t), t) = \rho(X_0, 0) = \rho_0(X_0)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t) = 2(1 - 3\rho_0(X_0))t + X_0$. We then have two cases depending whether $X_0$ is positive or negative.

1. $X_0 < 0$, then $\rho_0(X_0) = \frac{1}{6}$ and $X(t) = t + X_0$. This means

$$\rho(x,t) = \frac{1}{6} \quad \text{if} \quad x < t.$$

2. $X_0 > 0$, then $\rho_0(X_0) = \frac{1}{3}$ and $X(t) = X_0$. This means

$$\rho(x,t) = \frac{1}{3} \quad \text{if} \quad x > 0.$$

We see that the characteristics cross in the region $\{t > x > 0\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock:

$$s = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{\frac{1}{6}\frac{3}{2} - \frac{1}{3}}{\frac{1}{6} - \frac{1}{3}} = \frac{1}{12}6 = \frac{1}{2}.$$

In conclusion

$$\rho = \frac{1}{6}, \quad x < \frac{t}{2},$$
$$\rho = \frac{1}{3}, \quad x > \frac{t}{2}.$$

---

**Question 4**

Consider the following equation

$$\partial_t u - k\partial_{xx}u + \gamma u = 0, \quad x \in (-\infty, +\infty),\ t > 0, \quad \text{with} \quad u(x,0) = f(x), \quad x \in (-\infty, +\infty).$$

Use the Fourier transform method together with the convolution theorem[1] to compute the solution.

---

Taking the FT of the PDE gives

$$\partial_t \hat{u} + (\gamma + \omega^2)\hat{u} = 0.$$

The solution is then

$$\hat{u}(\omega, t) = \hat{f}e^{-(\gamma+\omega^2)t} = e^{-\gamma t}\hat{f}e^{-\omega^2 t}.$$

But we have $\sqrt{\frac{\pi}{t}}\mathcal{F}(e^{-x^2/4t}) = e^{-\alpha\omega^2}$, i.e.,

$$\hat{u}(\omega, t) = \sqrt{\tfrac{\pi}{t}}e^{-\gamma t}\hat{f}\mathcal{F}(e^{-x^2/4t})$$

The convolution theorem then implies

$$u(x,t) = \sqrt{\tfrac{\pi}{t}}e^{-\gamma t}\frac{1}{2\pi}f * e^{-x^2/4t}.$$

Using the definition of convolution, this gives

$$u(x,t) = \tfrac{1}{\sqrt{4\pi t}}e^{-\gamma t}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4t}dy.$$

---

[1] $f = \frac{1}{2\pi}h * g \Leftrightarrow \hat{f} = \hat{h}\hat{g}$, and $\sqrt{\frac{\pi}{\alpha}}\mathcal{F}(e^{-x^2/4\alpha}) = e^{-\alpha\omega^2}$

**Question 5**

(a) Compute all the solutions set of the equation

$$u'' + u = 0, x \in (0, 2\pi) \quad \text{with} \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi).$$

---

clearly $u(x) = a\cos(x) + b\sin(x)$ where $a$ and $b$ are arbitrary numbers. The solution set is a two-dimensional vector space spanned by $\cos(x)$ and $\sin(x)$.

---

(b) Do the following equation have solution(s)? (Hint: think of the Fredholm alternative)

$$u'' + u = \sin(2x), x \in (0, 2\pi) \quad \text{with} \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi).$$

---

We are in the second case of the Fredholm alternative, i.e., the null space of the operator is not $\{0\}$. We have to verify that $\sin(2x)$ is orthogonal to $\cos(x)$ and $\sin(x)$, which is clearly true. In conclusion the equation has solutions.

---

(c) Do the following equation have solution(s)? (Hint: think of the Fredholm alternative)

$$u'' + u = \sin(x), x \in (0, 2\pi) \quad \text{with} \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi).$$

---

We are in the second case of the Fredholm alternative, i.e., the null space of the operator is not $\{0\}$. We have to verify that $\sin(x)$ is orthogonal to $\cos(x)$ and $\sin(x)$, which is clearly wrong. In conclusion the equation has no solution.

---

### Question 6

Let $\Omega$ be a three-dimensional domain and consider the PDE

$$\nabla^2 u = f(x), \quad x \in \Omega, \quad \text{with} \quad u(x) = h(x) \quad \text{on the boundary of } \Omega, \text{ say } \Gamma.$$

Let $G(x, x_0)$ be the Green's function of this problem (the exact expression of $G$ does not matter; just assume that $G$ is known). Give a representation[2] of $u(x)$ in terms of $G$, $f$ and $h$.

---

By definition

$$\nabla_x^2 G(x, x_0) = \delta(x - x_0), \quad x \in \Omega, \quad \text{with} \quad G(x, x_0) = 0 \quad x \in \Gamma.$$

Then using the integration by parts formula, we obtain

$$\int_\Omega u(x)\nabla_x^2(G(x,x_0))dx = \int_\Omega \nabla_x^2(u(x))G(x,x_0)dx + \int_\Gamma u(x)\partial_n(G(x,x_0))dx - \int_\Gamma \partial_n(u(x))G(x,x_0)dx.$$

which can also be rewritten

$$u(x_0) = \int_\Omega f(x)G(x,x_0)dx + \int_\Gamma h(x)\partial_n(G(x,x_0))dx.$$

---

[2] Hint: use $\int_\Omega \psi\nabla^2(\phi) = \int_\Omega \nabla^2(\psi)\phi + \int_\Gamma \psi\partial_n(\phi) - \int_\Gamma \partial_n(\psi)\phi$