M602: Methods and Applications of Partial Differential Equations
Final TEST, December, 2008
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

$$\mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{2}$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}, \qquad \mathcal{F}(\frac{2\alpha}{x^2+\alpha^2})(\omega) = e^{-\alpha|\omega|}, \tag{3}$$

$$\mathcal{F}(f(x-\beta))(\omega) = e^{i\beta\omega}\mathcal{F}(f)(\omega), \tag{4}$$

Question 1

Consider the PDE

$$
\begin{aligned}
&u_{tt} - u_{xx} = 0, && 0 \le x \le 2, \quad 0 < t,\\
&u(0,t) = 0, \quad u(2,t) = 0 && 0 < t,\\
&u_t(x,0) = 0, \quad u(x,0) = f(x) := \begin{cases} x & 0 \le x \le 1, \\ 2-x & 1 \le x \le 2. \end{cases} && 0 < x < +2.
\end{aligned}
$$

(a) Give u(x,t) for all $x \in [0, +2]$, $t > 0$. (Hint use an extension technique).

We notice first that the wave speed is 1. We define f_o to be the odd extension of f over $(-2, +2)$, the we define f_{op} to be the periodic extension of f_o over $(-\infty, +\infty)$ with period 4. From class we know that the solution to the above problem is given by the D'Alembert formula

$$u(x,t) = \frac{1}{2}(f_{op}(x-t) + f_{op}(x+t)).$$

(b) Using (a), compute $u(x, \frac{1}{2})$, for all $x \in [0, +2]$.

We have to compute $f_{op}(x - \frac{1}{2})$ and $f_{op}(x + \frac{1}{2})$.

Case 1: $0 \le x \le \frac{1}{2}$. Then $-\frac{1}{2} \le x - \frac{1}{2} \le 0$ and by definition of f_{op}, $f_{op}(x - \frac{1}{2}) = -f(-x + \frac{1}{2}) = x - \frac{1}{2}$. We also have $\frac{1}{2} \le x + \frac{1}{2} \le 1$, which means $f_{op}(x + \frac{1}{2}) = f(x + \frac{1}{2}) = x + \frac{1}{2}$. Finally $u(x, \frac{1}{2}) = \frac{1}{2}(x - \frac{1}{2} + x + \frac{1}{2}) = x$ for all $x \in [0, \frac{1}{2}]$.

Case 2: $\frac{1}{2} \le x \le \frac{3}{2}$. Then $0 \le x - \frac{1}{2} \le 1$ and $f_{op}(x - \frac{1}{2}) = f(x - \frac{1}{2}) = x - \frac{1}{2}$. We also have $1 \le x + \frac{1}{2} \le 2$, which means $f_{op}(x + \frac{1}{2}) = f(x + \frac{1}{2}) = 2 - (x + \frac{1}{2}) = -x + \frac{3}{2}$. Finally $u(x, \frac{1}{2}) = \frac{1}{2}(x - \frac{1}{2} - x + \frac{3}{2}) = \frac{1}{2}$ for all $x \in [\frac{1}{2}, \frac{3}{2}]$.

Case 3: $\frac{3}{2} \le x \le 2$. Then $1 \le x - \frac{1}{2} \le \frac{3}{2}$ and $f_{op}(x - \frac{1}{2}) = f(x - \frac{1}{2}) = 2 - (x - \frac{1}{2}) = \frac{5}{2} - x$. We also have $2 \le x + \frac{1}{2} \le \frac{5}{2}$, which means by periodicity that $f_{op}(x + \frac{1}{2}) = f_{op}(x + \frac{1}{2} - 4) = f_{op}(x - \frac{7}{2})$. Now we observe that $-2 \le x - \frac{7}{2} \le -\frac{3}{2}$, which means $f_{op}(x + \frac{1}{2}) = f_{op}(x - \frac{7}{2}) = -f(\frac{7}{2} - x) = -(2 - (\frac{7}{2} - x)) = -(-\frac{3}{2} + x) = \frac{3}{2} - x$. In conclusion $u(x, \frac{1}{2}) = \frac{1}{2}(\frac{5}{2} - x + \frac{3}{2} - x) = 2 - x$ for all $x \in [\frac{3}{2}, 2]$.

Conclusion: We now put everything togther

$$u(x, \frac{1}{2}) = \begin{cases} x, & x \in [0, \frac{1}{2}], \\ \frac{1}{2}, & x \in [\frac{1}{2}, \frac{3}{2}], \\ 2 - x, & x \in [\frac{3}{2}, 2]. \end{cases}.$$

Question 2

Let $\Omega = \{(t,x) \in \mathbb{R}^2 : \ t > 0, \ x \geq -\sqrt{t}\}$. Let Γ be defined by the following parameterization $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s), s \in \mathbb{R}\}$, with $x_\Gamma(s) = s$ and $t_\Gamma(s) = s^2$ if $s \leq 0$, $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s \geq 0$. Solve the following PDE (give the implicit and explicit representations):

$$u_t + 2u_x + 3u = 0, \quad \text{in } \Omega, \qquad \text{and} \quad u(x_\Gamma(s), t_\Gamma(s)) := e^{-t_\Gamma(s) - x_\Gamma(s)}, \quad \forall s \in (-\infty, +\infty).$$

We define the characteristics by

$$\frac{dX(t,s)}{dt} = 2, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

This gives $X(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s))$. Upon setting $\phi(t,s) = u(X(t,s),t)$, we observe that $\partial_t \phi(t,s) + 3\phi(t,s) = 0$, which means

$$\phi(t,s) = ce^{-3t}.$$

The initial condition implies $\phi(t_\Gamma(s), s) = u(x_\Gamma(s), t_\Gamma(s)) = e^{-t_\Gamma(s) - x_\Gamma(s)} = ce^{-3t_\Gamma(s)}$; as a result $c = e^{2t_\Gamma(s) - x_\Gamma(s)}$ and

$$\phi(t,s) = e^{2t_\Gamma(s) - x_\Gamma(s) - 3t}.$$

The implicit representation of the solution is

$$u(X(t,s),t) = e^{2t_\Gamma(s) - x_\Gamma(s) - 3t}, \qquad X(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s)).$$

Now we give the explicit representation.
We observe the following:

$$2t_\Gamma(s) - x_\Gamma(s) = 2t - X(t,s),$$

which gives

$$u(X(t,s),t) = e^{2t - X(t,s) - 3t} = e^{-X(t,s) - t}.$$

In conclusion, the explicit representation of the solution to the problem is the following:

$$u(x,t) = e^{-x-t}.$$

Question 3

Solve the integral equation: $f(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f(y)}{(x-y)^2+1} dy = \frac{1}{x^2+4} + \frac{1}{x^2+1}$, for all $x \in (-\infty, +\infty)$.

The equation can be re-written

$$f(x) + \frac{1}{2\pi} f * \frac{1}{x^2+1} = \frac{1}{x^2+4} + \frac{1}{x^2+1}.$$

We take the Fourier transform of the equation and we apply the Convolution Theorem (see (2))

$$\mathcal{F}(f) + \frac{1}{2\pi} 2\pi \mathcal{F}(\frac{1}{x^2+1})\mathcal{F}(f) = \mathcal{F}(\frac{1}{x^2+4}) + \mathcal{F}(\frac{1}{x^2+1}).$$

Using (3), we obtain

$$\mathcal{F}(f) + \frac{1}{2} e^{-|\omega|}\mathcal{F}(f) = \frac{1}{4} e^{-2|\omega|} + \frac{1}{2} e^{-|\omega|},$$

which gives

$$\mathcal{F}(f)(1 + \frac{1}{2} e^{-|\omega|}) = \frac{1}{2} e^{-|\omega|}(\frac{1}{2} e^{-|\omega|} + 1).$$

We then deduce

$$\mathcal{F}(f) = \frac{1}{2} e^{-|\omega|}.$$

Taking the inverse Fourier transform, we finally obtain $f(x) = \frac{1}{x^2+1}$.

Question 4

Consider the following conservation equation

$$\partial_t u + u\partial_x u = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad u(x,0) = u_0(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 \leq x \leq 1, \\ 2-x & \text{if } 1 \leq x \leq 2 \\ 0 & \text{if } 2 \leq x \end{cases}$$

(i) Solve this problem using the method of characteristics for $0 \leq t < 1$.

The characteristics are defined by

$$\frac{dX(t,x_0)}{dt} = u(X(t,x_0),t), \qquad X(0,x_0) = x_0.$$

From class we know that $u(X(t,x_0),t)$ does not depend on time, that is to say

$$X(t,x_0) = u(X(0,x_0),0)t + x_0 = u(x_0,0)t + x_0 = u_0(x_0)t + x_0.$$

Case 1: If $x_0 \leq 0$, we have $u_0(x_0) = 0$ and $X(t,x_0) = x_0$; as a result, $x_0 = X(t,x_0)$, and

$$u(x,t) = 0, \qquad \text{if } x \leq 0.$$

Case 2: If $0 \leq x_0 \leq 1$, we have $u_0(x_0) = x_0$ and $X(t,x_0) = tx_0 + x_0$; as a result $x_0 = X/(1+t)$, and

$$u(x,t) = x/(1+t), \qquad \text{if } 0 \leq x \leq 1+t.$$

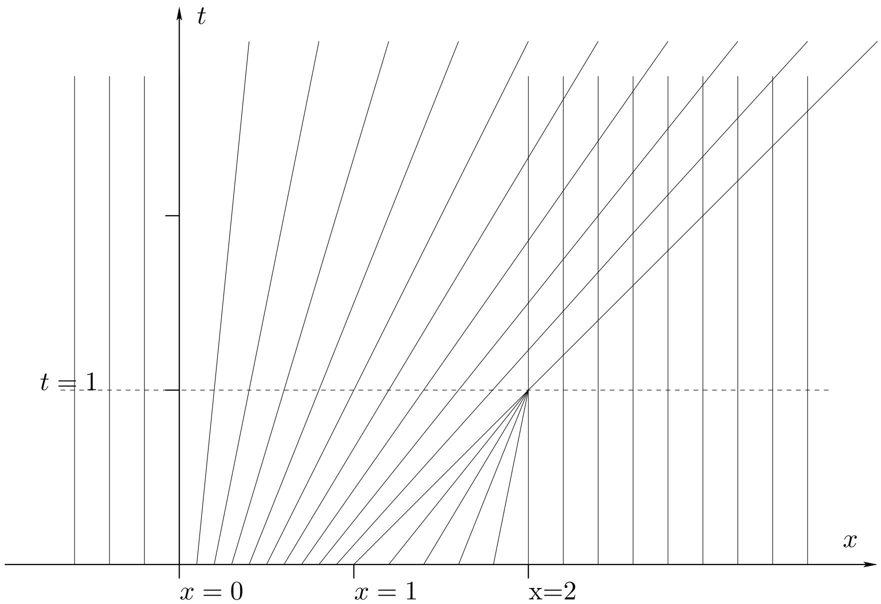
case 3: If $1 \leq x_0 \leq 2$, we have $u_0(x_0) = 2 - x_0$ and $X(t,x_0) = t(2-x_0) + x_0$; as a result $x_0 = (X(t,x_0) - 2t)/(1-t)$, which implies

$$u(x,t) = 2 - (x-2t)/(1-t) = (2-x)/(1-t), \qquad \text{if } 1+t \leq x \leq 2.$$

Case 4: If $2 \leq x_0$, we have $u_0(x_0) = 0$ and $X(t,x_0) = x_0$; as a result $x_0 = X(t,x_0)$, which implies

$$u(x,t) = 0 \qquad \text{if } 2 \leq x.$$

(ii) Draw the characteristics for all $t > 0$ and all $x \in \mathbb{R}$.



(iii) There is a shock forming at $t = 1$ and $x = 2$. Let $x_s(t)$ be the location of the shock as a function of t. Compute $x_s(t)$ using the fact that the solution for $t > 1$ is given by

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ u^-(t)\frac{x}{x_s(t)} & \text{if } 0 \le x < x_s(t), \text{ where } u^-(t) \text{ is the value of } u \text{ at the left of the shock} \\ 0 & \text{if } x_s(t) \le x, \end{cases}$$

Let $u^-(t)$ be the value of u at the left of the shock. Conservation of mass implies

$$\frac{1}{2}u^-(t)x_s(t) = \int_{-\infty}^{+\infty} u_0(x)dx = 1.$$

The Rankin-Hugoniot formula gives

$$\dot{x}_s(t) = \frac{\frac{1}{2}(u^-(t))^2}{u^-(t)} = \frac{1}{2}u^-(t) = \frac{1}{x_s(t)}.$$

This implies

$$x_s(t)\dot{x}_s(t) = \frac{1}{2}\frac{d}{dt}(x_s(t)^2) = 1, \quad \text{with} \quad x_s(1) = 2.$$

The Fundamental Theorem of Calculus implies

$$x_s(t)^2 - 2^2 = 2(t-1),$$

which in turn implies $x_s(t) = \sqrt{2t+2}$, for all $t \geq 1$.

Question 5

Use the Fourier transform method to compute the solution of $u_{tt} - a^2 u_{xx} = 0$, where $x \in \mathbb{R}$ and $t \in (0, +\infty)$, with $u(0, x) = f(x) := \sin^2(x)$ and $u_t(0, x) = 0$ for all $x \in \mathbb{R}$.

Take the Fourier transform in the x direction:

$$\mathcal{F}(u)_{tt} + \omega^2 a^2 \mathcal{F}(u) = 0.$$

This is an ODE. The solution is

$$\mathcal{F}(u)(t, \omega) = c_1(\omega)\cos(\omega a t) + c_2(\omega)\sin(\omega a t).$$

The initial boundary conditions give

$$\mathcal{F}(u)(0, \xi) = \mathcal{F}(f)(\omega) = c_1(\omega)$$

and $c_2(\omega) = 0$. Hence

$$\mathcal{F}(t, \omega) = \mathcal{F}(f)(\omega)\cos(\omega a t) = \frac{1}{2}\mathcal{F}(f)(\omega)(e^{ia\omega t} + e^{-ia\omega t}).$$

Using the shift lemma (i.e., formula (4)) we obtain

$$u(t, x) = \frac{1}{2}(f(x - at) + f(x + at)) = \frac{1}{2}(\sin^2(x + at) + \sin^2(x - at)).$$

Note that this is the D'Alembert formula.

Question 6

Consider the equation $u''(x) = f(x)$ for $x \in (0, 1)$ with $u(0) = 1$ and $u'(1) = 1$. Let $G(x, x_0)$ be the associated Green's function.
(a) Give an expression of $u(x)$ in terms of G, f and the boundary data.

The Green's function is defined by

$$G''(x, x_0) = \delta(x - x_0), \quad G(0, x_0) = 0, \quad G'(1, x_0) = 0.$$

We multiply the equation by u and we integrate (in the distribution sense),

$$\int_0^1 G''(x, x_0)u(x)dx = u(x_0).$$

We integrates by parts twice and we obtain,

$$\begin{aligned} u(x_0) &= -\int_0^1 G'(x, x_0)u'(x)dx + G'(1, x_0)u(1) - G'(0, x_0)u(0) \\ &= \int_0^1 G(x, x_0)u''(x)dx - G(1, x_0)u'(1) + G(0, x_0)u'(0) + G'(1, x_0)u(1) - G'(0, x_0)u(0). \end{aligned}$$

Then, using the boundary conditions for G and u, we obtain

$$u(x_0) = \int_0^1 G(x, x_0)f(x)dx - G'(0, x_0) - G(1, x_0), \quad \forall x_0 \in (0, 1).$$

(b) Compute $G(x, x_0)$.

For $x < x_0$ we have

$$G(x, x_0) = ax + b.$$

The boundary condition $G(0, x_0) = 0$ implies $b = 0$. For $x_0 < x$ we have

$$G(x, x_0) = cx + d.$$

The boundary condition $G'(1, x_0) = 0$ implies $c = 0$. Moreover we have

$$1 = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} G''(x, x_0)dx = G'(x_0^+, x_0) - G'(x_0^-, x_0) = -a,$$

meaning $a = -1$. The continuity of G at x_0 implies

$$ax_0 = d,$$

implying $d = -x_0$. As a result,

$$G(x, x_0) = \begin{cases} -x, & \text{if } 0 \le x \le x_0, \\ -x_0, & \text{if } x_0 \le x \le 1. \end{cases}$$