M602: Methods and Applications of Partial Differential Equations. Final TEST, December, 2010. Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

$$\mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \qquad \mathcal{F}(e^{-\alpha x^2}) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}} \tag{2}$$

$$\cos(a) - \cos(b) = -2\sin(\tfrac{1}{2}(a+b))\sin(\tfrac{1}{2}(a-b)) \tag{3}$$

The implicit representation of the solution to the equation $\partial_t v + \partial_x q(v) = 0$, $v(x,0) = v_0(x)$, is

$$X(s,t) = q'(v_0(s))t + s; \quad v(X(s,t),t) = v_0(s). \tag{4}$$

Question 1: Solve $u_{tt} - 4u_{xx} = 0$, $x \in (0,1)$ and $t \geq 0$, with $u(0,t) = u(1,t) = 0$, $u(x,0) = 0$, $\partial_t u(x,0) = g(x) := 2\pi\sin(\pi x)$. (Hint: use an extension technique).

We notice first that the wave speed is 2. We define g_o to be the odd extension of g over $(-1,+1)$. Clearly $g_o(x) = 2\pi\sin(\pi x)$ since $\sin(\pi x)$ is odd. We define g_{op} to be the periodic extension of g_o over $(\infty,+\infty)$ with period 2. Clearly, $g_{op}(x) = 2\pi\sin(\pi x)$ since 2 is a period for $\sin(\pi x)$. From class we know that the solution to the above problem is given by the restriction of the D'Alembert formula to the interval $[0,1]$:

$$\begin{aligned}
u(x,t) &= \frac{1}{4}\int_{x-2t}^{x+2t} g_{op}(\xi)\mathsf{d}\xi = \frac{1}{4}\int_{x-2t}^{x+2t} 2\pi\sin(\pi x)\mathsf{d}\xi, \\
&= -\frac{1}{2}\left(\cos(\pi(x+2t)) - \cos(\pi(x-2t))\right) \\
&= \sin(\frac{\pi}{2}(2x))\sin(\frac{\pi}{2}(4t)) \\
&= \sin(\pi x)\sin(2\pi t), \qquad \forall x \in [0,1], \forall t \geq 0.
\end{aligned}$$

Question 2: Let $\Omega = \{(x,t) \in \mathbb{R}^2;\ x + 2t \geq 0\}$. Solve the following PDE in explicit form with the method of characteristics:

$$\partial_t u(x,t) + 3\partial_x u(x,t) = u(x,t), \quad \text{in } \Omega, \quad \text{and} \quad u(x,t) = 1 + \sin(x), \text{ if } x + 2t = 0.$$

(i) First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with $x_\Gamma(s) = -2s$ and $t_\Gamma(s) = s$. This choice implies

$$u(x_\Gamma(s), t_\Gamma(s)) := u_\Gamma(s) := 1 + \sin(-2s).$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = 3, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

The solution is $X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s)$.

(iii) Set $\Phi(t,s) := u(X(t,s),t)$ and compute $\partial_t \Phi(t,s)$. This gives

$$\begin{aligned}\partial_t \Phi(t,s) &= \partial_t u(X(t,s),t) + \partial_x u(X(t,s),t)\partial_t X(t,s) \\ &= \partial_t u(X(t,s),t) + 3\partial_x u(X(t,s),t) = u(X(t,s),t) = \Phi(t,s).\end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s), s)e^{t - t_\Gamma(s)}$.

(iv) The implicit representation of the solution is

$$X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s) \quad u(X(t,s)) = u_\Gamma(s)e^{t - t_\Gamma(s)}.$$

(v) The explicit representation is obtained by using the definitions of $-t_\Gamma(s)$, $x_\Gamma(s)$ and $u_\Gamma(s)$.

$$X(s,t) = 3(t - s) - 2s = 3t - 5s,$$

which gives

$$s = \frac{1}{5}(3t - X).$$

The solution is

$$\begin{aligned}u(x,t) &= (1 + \sin(\tfrac{2}{5}(x - 3t)))e^{t - \frac{1}{5}(3t - x)} \\ &= (1 + \sin(\tfrac{2(x-3t)}{5}))e^{\frac{x+2t}{5}}.\end{aligned}$$

Question 3: Solve the following integral equation (Hint: $x^2 - 3xa + 2a^2 = (x-a)(x-2a)$):

$$\int_{-\infty}^{+\infty} f(y)f(x-y)\mathrm{d}y - 3\sqrt{2}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\pi}} f(x-y)\mathrm{d}y = -4\pi e^{-\frac{x^2}{4\pi}}. \qquad \forall x \in \mathbb{R}.$$

This equation can be re-written using the convolution operator:

$$f * f - 3\sqrt{2}e^{-\frac{x^2}{2\pi}} * f = -4\pi e^{-\frac{x^2}{4\pi}}.$$

We take the Fourier transform and use (2) to obtain

$$\begin{aligned} 2\pi\mathcal{F}(f)^2 - 2\pi 3\sqrt{2}\mathcal{F}(f)\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{2\pi}}} &= -4\pi\frac{1}{\sqrt{4\pi\frac{1}{4\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{4\pi}}} \\ \mathcal{F}(f)^2 - 3\mathcal{F}(f)e^{-\omega^2\frac{\pi}{2}} + 2e^{-\omega^2\pi} &= 0 \\ (\mathcal{F}(f) - e^{-\omega^2\frac{\pi}{2}})(\mathcal{F}(f) - 2e^{-\omega^2\frac{\pi}{2}}) &= 0. \end{aligned}$$

This implies

$$\text{either} \quad \mathcal{F}(f) = e^{-\omega^2\frac{\pi}{2}}, \quad \text{or} \quad \mathcal{F}(f) = 2e^{-\omega^2\frac{\pi}{2}}.$$

Taking the inverse Fourier transform, we obtain

$$\text{either} \quad f(x) = \sqrt{2}e^{-\frac{x^2}{2\pi}}, \quad \text{or} \quad f(x) = 2\sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

Question 4: Consider the quasilinear Klein-Gordon equation: $\partial_{tt}\phi(x,t)-c^2\partial_{xx}\phi(x,t)+m^2\phi(x,t)+\beta^2\phi^3(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $\phi(x,0) = f(x)$, $\partial_t\phi(x,0) = g(x)$ and $\phi(\pm\infty,t) = 0$, $\partial_t\phi(\pm\infty,t) = 0$, $\partial_x\phi(\pm\infty,t) = 0$. Find an energy $E(t)$ which is invariant with respect to time (Hint: test with $\partial_t\phi(x,t)$ and use $\phi^p\phi' = (\frac{1}{p+1}\phi^{p+1})'$.)

Testing with $\partial_t\phi(x,t)$ and integrating over $\mathbb{R}$ and using the property $\partial_t\phi(\pm\infty,t) = 0$, $\partial_x\phi(\pm\infty,t) = 0$, we obtain

$$\begin{aligned}
0 &= \int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}(\partial_t\phi)^2)\mathsf{d}x - c^2\int_{-\infty}^{+\infty} \partial_{xx}\phi\partial_t\phi\mathsf{d}x + m^2\int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}\phi^2)\mathsf{d}x + \beta^2\int_{-\infty}^{+\infty} \partial_t(\frac{1}{4}\phi^4)\mathsf{d}x \\
&= d_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + c^2\int_{-\infty}^{+\infty} \partial_x\phi\partial_t\partial_x\phi\mathsf{d}x + d_t\int_{-\infty}^{+\infty} (\frac{m^2}{2}\phi^2 + \frac{\beta^2}{4}\phi^4)\mathsf{d}x \\
&= d_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + d_t\int_{-\infty}^{+\infty} \frac{c^2}{2}(\partial_x\phi)^2\mathsf{d}x + d_t\int_{-\infty}^{+\infty} (\frac{m^2}{2}\phi^2 + \frac{\beta^2}{4}\phi^4)\mathsf{d}x \\
&= d_t\int_{-\infty}^{+\infty} \left(\frac{1}{2}(\partial_t\phi)^2 + \frac{c^2}{2}(\partial_x\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\beta^2}{4}\phi^4\right)\mathsf{d}x.
\end{aligned}$$

Introduce

$$E(t) = \int_{-\infty}^{+\infty} \left(\frac{1}{2}(\partial_t\phi)^2 + \frac{c^2}{2}(\partial_x\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\beta^2}{4}\phi^4\right))\mathsf{d}x.$$

Then

$$d_tE(t) = 0.$$

The fundamental Theorem of calculus gives

$$E(t) = E(0).$$

In conclusion the quantity $E(t)$ is invariant with respect to time, as requested.

Question 5: Solve the conservation equation $\partial_t \rho + \partial_x q(\rho) = 0$, $x \in (\infty, +\infty)$, $t > 0$ with flux $q(\rho) = \rho^2 + \rho$, and with the initial condition $\rho(x, 0) = -1$, if $x < 0$, $\rho(x, 0) = 1$, if $x > 0$. Do we have a shock or an expansion wave here?

The solution is given by the implicit representation

$$\rho(X(s,t),t) = \rho_0(s), \quad X(s,t) = s + (2\rho_0(s) + 1)t.$$

Case 1: $s < 0$. Then $\rho_0(s) = -1$ and $X(s,t) = s + (-2+1)t$. This means $s = X + t$. The solution is

$$\rho(x,t) = -1, \quad \text{if } x < t.$$

Case 2: $s < 0$. Then $\rho_0(s) = 1$ and $X(s,t) = s + (2+1)t$. This means $s = X - 3t$. The solution is

$$\rho(x,t) = 1, \quad \text{if } 3t < x.$$

We have a expansion wave. We need to consider the case $\rho_0 \in [-, 1]$ at $s = 0$.
Case 3: $s = 0$ and $\rho_0 \in [-1, 1]$. Then $X(s,t) = s + (2\rho_0 + 1)t = (2\rho_0 + 1)t$. This means $\rho_0 = (X/t - 1)2$. In conclusion

$$\rho(x,t) = \frac{1}{2}\left(\frac{x}{t} - 1\right), \quad \text{if } -t < x < 3t.$$

Question 6: Solve the conservation equation $\partial_t \rho + \partial_x q(\rho) = 0$, $x \in (\infty, +\infty)$, $t > 0$ with flux $q(\rho) = \rho^4 + 2\rho$, and with the initial condition $\rho(x, 0) = 1$, if $x < 0$, $\rho(x, 0) = -1$, if $x > 0$. Do we have a shock or an expansion wave here?

The solution is given by the implicit representation

$$\rho(X(s,t),t) = \rho_0(s), \quad X(s,t) = s + (4\rho_0(s)^3 + 2)t.$$

We then have two cases depending whether s is positive or negative.
Case 1: $s < 0$, then $\rho_0(s) = 1$ and $X(s,t) = (4+2)t + s = 6t + s$. This means

$$\rho(x,t) = 1 \quad \text{if} \quad x < 6t.$$

Case 2: $s > 0$, then $\rho_0(s) = -1$ and $X(s,t) = (-4+2)t + s = -2t + s$. This means

$$\rho(x,t) = -1 \quad \text{if} \quad x > -2t.$$

We see that the characteristics cross in the region $\{6t > x > -2t\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock with $\rho^- = 1$ and $\rho^+ = -1$:

$$\frac{dx_s(t)}{dt} = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{-1-3}{-1-1} = 2, \qquad x_s(0) = 0.$$

In conclusion the location of the shock is $x_s(t) = 2t$ and the solution is as follows:

$$\begin{aligned} \rho = 1, &\quad x < x_s(t) = 2t, \\ \rho = -1, &\quad x > x_s(t) = 2t. \end{aligned}$$

Question 7: Consider the equation $u'(x) + u = f(x)$ for $x \in (0, 1)$ with $u(0) = a$. Let $G(x, x_0)$ be the associated Green's function. (Pay attention to the number of derivatives).
(a) Give the equation and boundary condition defining G and give an integral representation of $u(x_0)$ in terms of G, f and the boundary data a. (Do not compute G.)

The Green's function is defined by

$$-G'(x, x_0) + G(x, x_0) = \delta(x - x_0), \quad G(1, x_0) = 0.$$

We multiply the equation by u and we integrate over $(0, 1)$ (in the distribution sense),

$$\int_0^1 -G'(x, x_0)u(x)\mathsf{d}x + \int_0^1 G(x, x_0)u(x)\mathsf{d}x = u(x_0).$$

We integrates by parts and we obtain,

$$u(x_0) = \int_0^1 G(x, x_0)(u'(x) + u(x))\mathsf{d}x - G(1, x_0)u(1) + G(0, x_0)u(0)$$

Then, using the fact that $u' + u = f$ and using the boundary conditions for G and u, we obtain

$$u(x_0) = \int_0^1 G(x, x_0)f(x)\mathsf{d}x + 2G(0, x_0). \quad \forall x_0 \in (0, 1).$$

(b) Compute $G(x, x_0)$.

For $x < x_0$ and $x_0 > x$ we have

$$-G'(x, x_0) + G(x, x_0) = 0.$$

The solution is

$$G(x, x_0) = \begin{cases} \alpha e^x & \text{for } x < x_0 \\ \beta e^x & \text{for } x > x_0. \end{cases}$$

The boundary condition $G(1, x_0) = 0$ implies $\beta = 0$.

For every $\epsilon > 0$ we have

$$\begin{aligned} 1 &= \int_{x_0-\epsilon}^{x_0+\epsilon} (-G'(x, x_0) + G(x, x_0))\mathrm{d}x \\ &= G(x_0 - \epsilon, x_0) - G(x_0 + \epsilon, x_0) + \int_{x_0-\epsilon}^{x_0+\epsilon} G(x, x_0)\mathrm{d}x \end{aligned}$$

The term $R_\epsilon = \int_{x_0-\epsilon}^{x_0+\epsilon} G(x, x_0)\mathrm{d}x$ can be bounded as follows:

$$|R_\epsilon| \leq 2\epsilon \max_{x\in[0,1]} |G(x, x_0)| = 2\epsilon\alpha e^{x_0}.$$

Clearly R_ϵ goes to 0 with ϵ. As a result we obtain the jump condition

$$1 = G(x_0^-, x_0) - G(x_0^+, x_0) = \alpha e^{x_0}.$$

This implies

$$\alpha = e^{-x_0}.$$

Finally

$$G(x, x_0) = \begin{cases} e^{x-x_0} & \text{for } x < x_0 \\ 0 & \text{for } x > x_0. \end{cases}$$