M602: Methods and Applications of Partial Differential Equations. Final TEST, Dec, 2012. Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

$$\mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{2}$$

$$\mathcal{F}(e^{-\alpha x^2}) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}} \tag{3}$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}, \tag{4}$$

$$\mathcal{F}(H(x)e^{-ax})(\omega) = \frac{1}{2\pi}\frac{1}{a-i\omega}, \qquad a \geq 0, \text{ where } H \text{ is the Heaviside function} \tag{5}$$

The implicit representation of the solution to the equation $\partial_t v + \partial_x q(v) = 0$, $v(x,0) = v_0(x)$, is

$$X(s,t) = q'(v_0(s))t + s; \quad v(X(s,t),t) = v_0(s). \tag{6}$$

Question 1: Consider the following PDE's:

$$\begin{aligned}
\partial_y u(x,y) + 3\partial_{xx}u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L)\\
u(x,0) = 1, \quad u(0,y) &= 3, \quad \partial_x u(L,y) = 2 \qquad (7)\\
\partial_y u(x,y) - 3\partial_{xx}u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L)\\
u(x,0) = 1, \quad u(0,y) &= 3, \quad \partial_x u(0,y) = 2, \quad \partial_x u(L,y) = 2 \qquad (8)\\
\partial_y u(x,y) - 3\partial_{xx}u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L)\\
u(x,0) = 1, \quad u(0,y) &= 3, \quad \partial_x u(L,y) = 2 \qquad (9)\\
\partial_{yy} u(x,y) - 3\partial_{xx}u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L)\\
u(x,0) = 1, \quad u(0,y) &= 3, \quad \partial_x u(L,y) = 2 \qquad (10)\\
\partial_{yy} u(x,y) - 3\partial_{xx}u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L)\\
u(x,0) = 1, \quad \partial_y u(x,0) &= 1, \quad u(0,y) = 2, \quad \partial_x u(L,y) = 3 \qquad (11)\\
\partial_{yy} u(x,y) + 3\partial_{xx}u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L)\\
u(x,0) = 1, \quad \partial_y u(x,0) &= 1, \quad u(0,y) = 2, \quad \partial_x u(L,y) = 3 \qquad (12)\\
-\partial_{yy} u(x,y) - \partial_{xx}u(x,y) &= f(x,y), \quad y\in(0,H), \quad x\in(0,L)\\
u(x,0) = 1, \quad u(x,H) &= 1, \quad u(0,y) = 2, \quad \partial_x u(L,y) = 3 \qquad (13)\\
-\partial_{yy} u(x,y) - \partial_{xx}u(x,y) &= f(x,y), \quad y\in(0,H), \quad x\in(0,L)\\
u(x,0) = 1, \quad u(x,H) &= 1, \quad \partial_x u(x,H) = 2, \quad \partial_x u(L,y) = 3 \qquad (14)\\
-\partial_{yy} u(x,y) - \partial_{xx}u(x,y) &= f(x,y), \quad y\in(0,H), \quad x\in(0,L)\\
u(x,0) = 1, \quad u(L,y) &= 2, \quad \partial_x u(0,y) = 3 \qquad (15)\\
\partial_y u(x,y) + 3\partial_x u(x,y) &= f(x,y), \quad y\in(0,H), \quad x\in(0,L)\\
u(x,0) = 1, \quad u(x,H) &= 1, \quad u(L,y) = 2, \quad \partial_x u(0,y) = 3 \qquad (16)\\
\partial_y u(x,y) + 3\partial_x u(x,y) &= f(x,y), \quad y>0 \quad x\in(0,L)\\
u(x,0) = 1, \quad u(0,y) &= 2, \quad \partial_x u(L,y) = 3 \qquad (17)\\
\partial_y u(x,y) + 3\partial_x u(x,y) &= f(x,y), \quad y>0 \quad x\in(0,L)\\
u(x,0) = 1, \quad u(0,y) &= 2 \qquad (18)
\end{aligned}$$

Which one is the

- Heat equation? ____________ 9
- Laplace equation? ____________ 13
- Transport equation? ____________ 18
- Wave equation? ____________ 11

Question 2: Solve by the Fourier transform technique the following equation: $\partial_{xx}\phi(x) - 2\partial_x\phi(x) + \phi(x) = H(x)e^{-x}$, $x \in (-\infty, +\infty)$, where $H(x)$ is the Heaviside function. (Hint: use the factorization $i\omega^3 + \omega^2 + i\omega + 1 = (1+\omega^2)(1+i\omega)$ and recall that $\mathcal{F}(f(x))(-\omega) = \mathcal{F}(f(-x))(\omega)$).

Applying the Fourier transform with respect to x gives

$$(-\omega^2 + 2i\omega + 1)\mathcal{F}(\phi)(\omega) = \mathcal{F}(H(x)e^{-x})(\omega) = \frac{1}{2\pi}\frac{1}{1 - i\omega}.$$

where we used (??). Then, using the hint gives

$$\begin{aligned}\mathcal{F}(\phi)(\omega) &= \frac{1}{2\pi}\frac{1}{(1 - i\omega)(-\omega^2 + 2i\omega + 1)} = \frac{1}{2\pi}\frac{1}{i\omega^3 + \omega^2 + i\omega + 1} \\ &= \frac{1}{2\pi}\frac{1}{1+\omega^2}\frac{1}{1+i\omega}.\end{aligned}$$

We now use again (??) and (??) to obtain

$$\mathcal{F}(\phi)(\omega) = \pi\frac{1}{\pi}\frac{1}{1+\omega^2}\frac{1}{2\pi}\frac{1}{1 - i(-\omega)} = \pi\mathcal{F}(e^{-|x|})(\omega)\mathcal{F}(H(x)e^{-x})(-\omega).$$

Now we use $\mathcal{F}(H(x)e^{-x})(-\omega) = \mathcal{F}(H(-x)e^{x})(\omega)$ and we finally have

$$\mathcal{F}(\phi)(\omega) = \pi\mathcal{F}(e^{-|x|})(\omega)\mathcal{F}(H(-x)e^{x})(\omega).$$

The Convolution Theorem (??) gives

$$\mathcal{F}(\phi)(\omega) = \pi\frac{1}{2\pi}\mathcal{F}(e^{-|x|} * (H(-x)e^{x}))(\omega).$$

We obtain ϕ by using the inverse Fourier transform

$$\phi(x) = \frac{1}{2}e^{-|x|} * (H(-x)e^{x}),$$

i.e.,

$$\phi(x) = \int_{-\infty}^{+\infty}\frac{1}{2}e^{-|x-y|}H(-y)e^{y}\mathrm{d}y$$

and recalling that H is the Heaviside function we finally have

$$\phi(x) = \frac{1}{2}\int_{-\infty}^{0}e^{y-|x-y|}\mathrm{d}y = \begin{cases}\frac{1}{4}e^{-x} & \text{if } x \geq 0\\ (\frac{1}{4} - x)e^{x} & \text{if } x \leq 0.\end{cases}$$

Question 3: Let $\Omega = (0, L)$ and let (λ, u) be an eigenpair of the Laplace equation over Ω with zero Dirichlet condition. Assume that $\lambda \in \mathbb{C}$ and that the function $u(x)$ is complex-valued.

(i) Write the PDE solved by u.

u and λ are such

$$-\partial_{xx}u(x) = \lambda u(x), \quad u(0) = 0, \quad u(L) = 0.$$

(ii) Let $\bar{u}$ be the complex conjugate of u. Write the PDE solved by $\bar{u}$ (Hint: take the conjugate of (i)).

Taking the complex conjugate of (i) we obtain

$$-\partial_{xx}\bar{u}(x) = \overline{-\partial_{xx}u(x)} = \overline{\lambda u(x)} = \bar{\lambda}\bar{u}(x), \quad \bar{u}(0) = 0, \quad \bar{u}(L) = 0,$$

which gives

$$-\partial_{xx}\bar{u}(x) = \bar{\lambda}\bar{u}(x), \quad \bar{u}(0) = 0, \quad \bar{u}(L) = 0.$$

Note that we used the fact that $\overline{\partial_x u} = \partial_x \bar{u}$.

(iii) Prove that $\lambda \in \mathbb{R}$ (Hint: Use an energy argument with $\bar{u}$ in (i) and an energy argument with u (ii) and conclude that $\lambda = \bar{\lambda}$. Recall $u \neq 0$ and $|z|^2 = z\bar{z}$ for all $z \in \mathbb{C}$).

Multiply (i) by $\bar{u}$ and integrate over Ω

$$\begin{aligned}
\int_\Omega -\partial_{xx}u(x)\bar{u}(x)\mathsf{d}x &= \int_\Omega \lambda u(x)\bar{u}(x)\mathsf{d}x \\
\int_\Omega \partial_x u(x)\partial_x\bar{u}(x)\mathsf{d}x - [\partial_x u(x)\bar{u}(x)]_0^L &= \lambda \int_\Omega |u(x)|^2\mathsf{d}x \\
\int_\Omega \partial_x u(x)\overline{\partial_x u(x)}\mathsf{d}x &= \lambda \int_\Omega |u(x)|^2\mathsf{d}x \\
\int_\Omega |\partial_x u(x)|^2\mathsf{d}x &= \lambda \int_\Omega |u(x)|^2\mathsf{d}x.
\end{aligned}$$

Multiply (ii) by u and integrate over Ω

$$\begin{aligned}
\int_\Omega -\partial_{xx}\bar{u}(x)u(x)\mathsf{d}x &= \int_\Omega \bar{\lambda}\bar{u}(x)u(x)\mathsf{d}x \\
\int_\Omega \partial_x\bar{u}(x)\partial_x u(x)\mathsf{d}x - [\partial_x\bar{u}(x)u(x)]_0^L &= \bar{\lambda} \int_\Omega |u(x)|^2\mathsf{d}x \\
\int_\Omega \overline{\partial_x u(x)}\partial_x u(x)\mathsf{d}x &= \bar{\lambda} \int_\Omega |u(x)|^2\mathsf{d}x \\
\int_\Omega |\partial_x u(x)|^2\mathsf{d}x &= \bar{\lambda} \int_\Omega |u(x)|^2\mathsf{d}x.
\end{aligned}$$

In conclusion

$$\lambda \int_\Omega |u(x)|^2\mathsf{d}x = \bar{\lambda} \int_\Omega |u(x)|^2\mathsf{d}x,$$

which means

$$(\lambda - \bar{\lambda}) \int_\Omega |u(x)|^2\mathsf{d}x.$$

This in turn implies that $\lambda = \bar{\lambda}$ since $\int_\Omega |u(x)|^2\mathsf{d}x$ is not zero (recall $u \neq 0$). In conclusion λ is real. Note in passing that this also prove that $\lambda \geq 0$.

Question 4: Give an explicit solution to the equation $\partial_t u + \partial_x(u^4) = 0$, where $x \in (-\infty, +\infty)$, $t > 0$, with initial data $u_0(x) = 0$ if $x < 0$, $u_0(x) = x^{\frac{1}{3}}$ if $0 < x < 1$, and $u_0(x) = 0$ if $1 < x$.

The implicit representation of the solution is

$$u(X(s,t),t) = u_0(s), \quad X(s,t) = s + 4u_0(s)^3 t.$$

Case 1: $s < 0$, then $u_0(s) = 0$ and $X(s,t) = s$. This means

$$u(x,t) = 0 \quad \text{if } x < 0.$$

Case 2: $0 < s < 1$, then $u_0(s) = s^{\frac{1}{3}}$ and $X(s,t) = s + 4st$. This means $s = X/(1+4t)$

$$u(x,t) = \left(\frac{x}{1+4t}\right)^{\frac{1}{3}} \quad \text{if } 0 < x < 1+4t.$$

Case 3: $1 < s$, then $u_0(s) = 0$ and $X(s,t) = s$. This means

$$u(x,t) = 0 \quad \text{if } 1 < x.$$

There is a shock starting at $x = 1$ (this is visible when one draws the characteristics).

Solution 1: The speed of the shock is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = \frac{u_+^4 - u_-^4}{u_+ - u_-}, \qquad \text{and } x_s(0) = 1,$$

where $u_+(t) = 0$ and $u_-(t) = \left(\frac{x_s(t)}{1+4t}\right)^{\frac{1}{3}}$. This gives

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = u_-(t)^3 = \frac{x_s(t)}{1+4t},$$

which we re-write as follows:

$$\frac{\mathrm{d}\log(x_s(t))}{\mathrm{d}t} = \frac{1}{1+4t} = \frac{1}{4}\frac{\mathrm{d}\log(1+4t)}{\mathrm{d}t}.$$

Applying the fundamental of calculus between 0 and t gives

$$\log(x_s(t)) - \log(1) = \frac{1}{4}(\log(1+4t) - \log(1)).$$

This give

$$x_s(t) = (1+4t)^{\frac{1}{4}}.$$

Solution 2: Another (equivalent) way of solving this problem, that does not require to solve the Rankin-Hugoniot relation, consists of writing that the value of u_- is such that the total mass is conserved:

$$\int_0^{x_s(t)} u(x,t)\mathrm{d}x = \int_0^{x_s(0)} u_0(x)\mathrm{d}x = \int_0^1 x^{\frac{1}{3}}\mathrm{d}x = \frac{3}{4}$$

i.e., using the fact that $u(x,t) = (x/(1+4t))^{\frac{1}{3}}$ for all $0 \le x \le x_s(t)$, we have

$$\frac{3}{4} = (1+4t)^{-\frac{1}{3}} \int_0^{x_s(t)} x^{\frac{1}{3}}\mathrm{d}x = (1+4t)^{-\frac{1}{3}}\frac{3}{4}x_s(t)^{\frac{4}{3}}.$$

This again gives

$$x_s(t) = (1+4t)^{\frac{1}{4}}.$$

Conclusion: The solution is finally expressed as follows:

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \left(\frac{x}{1+4t}\right)^{\frac{1}{3}} & \text{if } 0 < x < (1+4t)^{\frac{1}{4}} \\ 0 & \text{if } (1+4t)^{\frac{1}{4}} < x \end{cases}$$

Question 5: Consider the equation $-\partial_x(x\partial_x u(x)) = f(x)$ for all $x \in (1,2)$ with $u(1) = a$ and $u(2) = b$. Let $G(x, x_0)$ be the associated Green's function.

(i) Give the equation and boundary conditions satisfied by G and give the integral representation of $u(x_0)$ for all $x_0 \in (1,2)$ in terms of G, f, and the boundary data. (Do not compute G in this question).

We have a second-order PDE and the operator is clearly self-adjoint. The Green's function solves the equation

$$-\partial_x(x\partial_x G(x,x_0)) = \delta(x-x_0), \quad G(1,x_0) = 0, \quad G(2,x_0) = 0.$$

We multiply the equation by u and integrate over the domain $(1,2)$ (in the distribution sense).

$$\langle \delta(x-x_0), u\rangle = u(x_0) = -\int_1^2 \partial_x(x\partial_x G(x,x_0))u(x)\mathrm{d}x.$$

We integrate by parts and we obtain,

$$\begin{aligned} u(x_0) &= \int_1^2 x\partial_x G(x,x_0)\partial_x u(x)\mathrm{d}x - [x\partial_x G(x,x_0)u(x)]_1^2 \\ &= -\int_1^2 G(x,x_0)\partial_x(x\partial_x u(x))\mathrm{d}x - 2\partial_x G(2,x_0)u(2) + \partial_x G(1,x_0)u(1). \end{aligned}$$

Now, using the boundary conditions and the fact that $-\partial_x(x\partial_x u(x)) = f(x)$, we finally have

$$u(x_0) = \int_1^2 G(x,x_0)f(x)\mathrm{d}x - 2\partial_x G(2,x_0)b + \partial_x G(1,x_0)a.$$

(ii) Compute $G(x, x_0)$ for all $x, x_0 \in (1,2)$.

For all $x \neq x_0$ we have

$$-\partial_x(x\partial_x G(x,x_0)) = 0.$$

The solution is

$$G(x,x_0) = \begin{cases} a\log(x) + b & \text{if } 1 < x < x_0 \\ c\log(x) + d & \text{if } x_0 < x < 2 \end{cases}$$

The boundary conditions give $b = 0$ and $d = -c\log(2)$; as a result,

$$G(x,x_0) = \begin{cases} a\log(x) & \text{if } 1 < x < x_0 \\ c\log(x/2) & \text{if } x_0 < x < 2 \end{cases}$$

G must be continuous at x_0,

$$a\log(x_0) = c\log(x_0) - c\log(2)$$

and must satisfy the gap condition

$$-\int_{x_0-\epsilon}^{x_0+\epsilon} \partial_x(x\partial_x G(x,x_0))\mathrm{d}x = 1, \qquad \forall\epsilon > 0.$$

This gives

$$\begin{aligned} -x_0\left(\partial_x G(x_0^+,x_0) - G(x_0^-,x_0)\right) &= 1 \\ -x_0\left(\frac{c}{x_0} - \frac{a}{x_0}\right) &= 1 \end{aligned}$$

This gives

$$a - c = 1.$$

In conclusion $\log(x_0) = -c\log 2$ and

$$c = -\log(x_0)/\log(2), \quad a = 1 - \log(x_0)/\log(2) = \log(2/x_0)/\log(2).$$

This means

$$G(x,x_0) = \begin{cases} \frac{\log(2/x_0)}{\log(2)}\log(x) & \text{if } 1 < x < x_0 \\ \frac{\log(x_0)}{\log(2)}\log(2/x) & \text{if } x_0 < x < 2 \end{cases}$$

Question 6: Solve $u_{tt} - 4u_{xx} = 0$, $x \in (0,1)$ and $t \geq 0$, with $u(0,t) = 0$, $\partial_x u(1,t) = 0$, $u(x,0) = 0$, $\partial_t u(x,0) = g(x) := 2\pi \sin(\frac{\pi}{2}x)$. (Hint: Pay attention to the boundary conditions. Use three extensions.)

To be able to apply the d'Alembert formula, we need to extend the above problem to the $(-\infty, +\infty)$. the Dirichlet condition a $x = 0$ requires an odd extension and the Neumann condition requires an even extension.

We define g_o to be the odd extension of g over $(-1,+1)$ to account for the Dirichlet boundary condition at $x = 0$.

$$g_o(x) = \begin{cases} g(x) & \text{for all } x \in (0,1), \\ -g(-x) & \text{for all } x \in (-1,0) \end{cases}$$

Clearly $g_o(x) = 2\pi \sin(\frac{\pi}{2}x)$ since $\sin(\frac{\pi}{2}x)$ is odd. More precisely,

$$g_o(x) = \begin{cases} g(x) = 2\pi \sin(\frac{\pi}{2}x) & \text{for all } x \in (0,1), \\ -g(-x) = -2\pi \sin(\frac{\pi}{2}(-x)) = 2\pi \sin(\frac{\pi}{2}x) & \text{for all } x \in (-1,0). \end{cases}$$

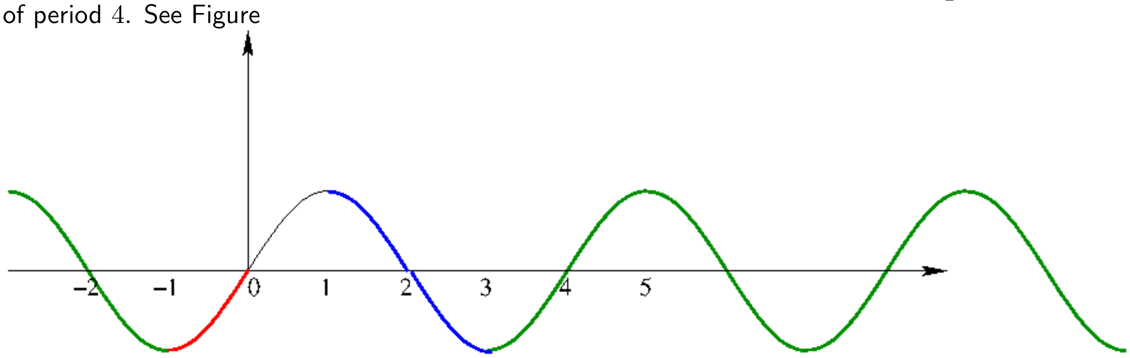
Now we need to consider the even extension of g_o about the point $x = 1$ to account for the Neumann boundary condition at $x = 1$. Let us denote $g_{oe}(x)$ this extension. The function $g_{oe}(x)$ is such that

$$g_{oe}(x) = \begin{cases} g_{oe}(x) = g_o(x) & \text{for all } x \in (-1,1), \\ g_{oe}(x) = g_o(2-x) & \text{for all } x \in (1,3) \end{cases}$$

Now we observe that $\sin(\frac{\pi}{2}(2-x)) = \sin(\pi - \frac{\pi}{2}x) = \sin(\frac{\pi}{2}x)$, which means that $g_{oe}(x) = 2\pi \sin(\frac{\pi}{2}x)$. More precisely,

$$g_{oe}(x) = \begin{cases} g_{oe}(x) = g_o(x) = 2\pi \sin(\frac{\pi}{2}x) & \text{for all } x \in (-1,1), \\ g_{oe}(x) = g_o(2-x) = 2\pi \sin(\frac{\pi}{2}(2-x)) = 2\pi \sin(\frac{\pi}{2}x) & \text{for all } x \in (1,3). \end{cases}$$

Now we consider the periodic extension of g_{oe} of period 4, say g_{oep}. Clearly $g_{oep} = 2\pi \sin(\frac{\pi}{2}x)$, since $42\pi \sin(\frac{\pi}{2}x)$ is periodic of period 4. See Figure



We notice finally that the wave speed is 2. From class we know that the solution to the above problem is given by the restriction of the D'Alembert formula to the interval $[0,1]$:

$$\begin{aligned} u(x,t) &= \frac{1}{4} \int_{x-2t}^{x+2t} g_{oep}(\xi) \mathrm{d}\xi = \frac{1}{4} \int_{x-2t}^{x+2t} 2\pi \sin(\frac{\pi}{2}\xi) \mathrm{d}\xi, \\ &= -\Big(\cos(\frac{\pi}{2}(x+2t)) - \cos(\frac{\pi}{2}(x-2t)) \Big) \\ &= 2\sin(\frac{\pi}{2}(x)) \sin(\frac{\pi}{2}(2t)) \\ &= 2\sin(\frac{\pi}{2}x) \sin(\pi t), \qquad \forall x \in [0,1], \forall t \geq 0. \end{aligned}$$

Question 7: Let $\Omega = \{(x,t) \in \mathbb{R}^2 \mid t > 0, x \geq \frac{1}{t}\}$. Solve the following PDE in explicit form with the method of characteristics: (Solution: $u(x,t) = (2+\cos(s))e^{\frac{1}{s}-t}$ with $s = \frac{1}{2}[(x-2t) + \sqrt{(x-2t)^2+8}]$)

$$\partial_t u(x,t) + 2\partial_x u(x,t) = -u(x,t), \quad \text{in } \Omega, \quad \text{and} \quad u(x,t) = 2 + \cos(x), \text{ if } x = 1/t.$$

(i) First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with $x_\Gamma(s) = s$ and $t_\Gamma(s) = \frac{1}{s}$. This choice implies

$$u(x_\Gamma(s), t_\Gamma(s)) := u_\Gamma(s) := 2 + \cos(s).$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = 2, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

The solution is $X(t,s) = 2(t - t_\Gamma(s)) + x_\Gamma(s)$.

(iii) Set $\Phi(t,s) := u(X(t,s),t)$ and compute $\partial_t \Phi(t,s)$. This gives

$$\begin{aligned}\partial_t \Phi(t,s) &= \partial_t u(X(t,s),t) + \partial_x u(X(t,s),t)\partial_t X(t,s) \\ &= \partial_t u(X(t,s),t) + 2\partial_x u(X(t,s),t) = u(X(t,s),t) = -\Phi(t,s).\end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s), s)e^{-t+t_\Gamma(s)}$.

(iv) The implicit representation of the solution is

$$X(t,s) = 2(t - t_\Gamma(s)) + x_\Gamma(s), \qquad u(X(t,s)) = u_\Gamma(s)e^{-t+t_\Gamma(s)}.$$

(v) The explicit representation is obtained by using the definitions of $-t_\Gamma(s)$, $x_\Gamma(s)$ and $u_\Gamma(s)$.

$$X(s,t) = 2(t - \frac{1}{s}) + s = 2t - \frac{2}{s} + s$$

which gives the equation

$$s^2 - s(X - 2t) - 2 = 0$$

The solutions are $s_\pm = \frac{1}{2}\left((X-2t) \pm \sqrt{(X-2t)^2+8}\right)$. The only legitimate solution is the positive one:

$$s = \frac{1}{2}\left((X-2t) + \sqrt{(X-2t)^2+8}\right)$$

The solution is

$$\begin{aligned}&u(x,t) = (2+\cos(s))e^{\frac{1}{s}-t} \\ &\text{with } s = \frac{1}{2}((x-2t) + \sqrt{(x-2t)^2+8})\end{aligned}$$

Question 8: Consider the quasilinear Klein-Gordon equation: $\partial_{tt}\phi(x,t) - c^2\partial_{xx}\phi(x,t) + m^2\phi(x,t) + \beta^2\phi^3(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $\phi(x,0) = f(x)$, $\partial_t\phi(x,0) = g(x)$ and $\phi(\pm\infty,t) = 0$, $\partial_t\phi(\pm\infty,t) = 0$, $\partial_x\phi(\pm\infty,t) = 0$. Find an energy $E(t)$ which is invariant with respect to time (Hint: test with $\partial_t\phi(x,t)$ and use $\phi^p\phi' = (\frac{1}{p+1}\phi^{p+1})'$.)

Testing with $\partial_t\phi(x,t)$ and integrating over $\mathbb{R}$ and using the property $\partial_t\phi(\pm\infty,t) = 0$, $\partial_x\phi(\pm\infty,t) = 0$, we obtain

$$
\begin{aligned}
0 &= \int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}(\partial_t\phi)^2)\mathsf{d}x - c^2\int_{-\infty}^{+\infty} \partial_{xx}\phi\partial_t\phi\mathsf{d}x + m^2\int_{-\infty}^{+\infty} \partial_t(\frac{1}{2}\phi^2)\mathsf{d}x + \beta^2\int_{-\infty}^{+\infty} \partial_t(\frac{1}{4}\phi^4)\mathsf{d}x \\
&= d_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + c^2\int_{-\infty}^{+\infty} \partial_x\phi\partial_t\partial_x\phi\mathsf{d}x + d_t\int_{-\infty}^{+\infty} (\frac{m^2}{2}\phi^2 + \frac{\beta^2}{4}\phi^4)\mathsf{d}x \\
&= d_t\int_{-\infty}^{+\infty} \frac{1}{2}(\partial_t\phi)^2\mathsf{d}x + d_t\int_{-\infty}^{+\infty} \frac{c^2}{2}(\partial_x\phi)^2\mathsf{d}x + d_t\int_{-\infty}^{+\infty} (\frac{m^2}{2}\phi^2 + \frac{\beta^2}{4}\phi^4)\mathsf{d}x \\
&= d_t\int_{-\infty}^{+\infty} \left(\frac{1}{2}(\partial_t\phi)^2 + \frac{c^2}{2}(\partial_x\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\beta^2}{4}\phi^4\right)\mathsf{d}x.
\end{aligned}
$$

Introduce

$$
E(t) = \int_{-\infty}^{+\infty} \left(\frac{1}{2}(\partial_t\phi)^2 + \frac{c^2}{2}(\partial_x\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\beta^2}{4}\phi^4\right))\mathsf{d}x.
$$

Then

$$
d_tE(t) = 0.
$$

The fundamental Theorem of calculus gives

$$
E(t) = E(0).
$$

In conclusion the quantity $E(t)$ is invariant with respect to time, as requested.