M602: Methods and Applications of Partial Differential Equations. Final TEST, Dec17, 2014. Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}\mathrm{d}x, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}\mathrm{d}\omega, \qquad \mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{1}$$

The implicit representation of the solution to the equation $\partial_t v + \partial_x q(v) = 0$, $v(x,0) = v_0(x)$, is

$$X(s,t) = q'(v_0(s))t + s; \quad v(X(s,t),t) = v_0(s). \tag{2}$$

Question 1: (a) Prove that $\partial_\omega \mathcal{F}(f)(\omega) = i\mathcal{F}(xf(x))(\omega)$ for all $f \in L^1(\mathbb{R})$.

Solution: Let $f \in L^1(\mathbb{R})$, then

$$\partial_\omega \mathcal{F}(f)(\omega) = \partial_\omega \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\mathrm{d}x\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\partial_\omega e^{i\omega x}\mathrm{d}x$$
$$= i\frac{1}{2\pi}\int_{-\infty}^{\infty} xf(x)e^{i\omega x}\mathrm{d}x,$$

which prove that $\partial_\omega \mathcal{F}(f)(\omega) = i\mathcal{F}(xf(x))(\omega)$.

(b) Let $\alpha \in \mathbb{R}$ with $\alpha > 0$. Prove that $\mathcal{F}(\partial_x e^{-\alpha x^2})(\omega) = 2\alpha i \partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega)$. (Hint: use (a).)

Solution: We use (a) to deduce that

$$\mathcal{F}(\partial_x e^{-\alpha x^2})(\omega) = \mathcal{F}(-2\alpha x e^{-\alpha x^2})(\omega) = -2\alpha \mathcal{F}(xe^{-\alpha x^2})(\omega)$$
$$= 2\alpha i \partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega).$$

(c) Show that $\partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega) = -\frac{\omega}{2\alpha}\mathcal{F}(e^{-\alpha x^2})(\omega)$.

Solution: We use the property $\mathcal{F}(\partial_x f(x))(\omega) = -i\omega\mathcal{F}(f(x))(\omega)$ and (b)

$$-i\omega\mathcal{F}(e^{-\alpha x^2})(\omega)) = \mathcal{F}(\partial_x e^{-\alpha x^2})(\omega) = 2\alpha i \partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega),$$

which implies the desired result.

(d) Given that $\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$, compute $\mathcal{F}(e^{-\alpha x^2})(\omega)$. (Hint: Note that (c) is an ODE.)

Solution: The solution to the ODE $\partial_\omega g(\omega) = -\frac{\omega}{2\alpha}g(\omega)$ is $g(\omega) = g(0)e^{-\frac{\omega^2}{4\alpha}}$. We apply this formula to $g(\omega) = \mathcal{F}(e^{-\alpha x^2})(\omega)$,

$$\mathcal{F}(e^{-\alpha x^2})(\omega) = \mathcal{F}(e^{-\alpha x^2})(0)e^{-\frac{\omega^2}{4\alpha}}.$$

We now need to compute $\mathcal{F}(e^{-\alpha x^2})(0)$,

$$\mathcal{F}(e^{-\alpha x^2})(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha x^2}\mathrm{d}x = \frac{1}{2\pi}\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-\alpha x^2}\sqrt{\alpha}\mathrm{d}x = \frac{1}{2\pi}\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{x^2}\mathrm{d}x = \frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{\alpha}} = \frac{1}{4\pi\alpha}.$$

Finally

$$\mathcal{F}(e^{-\alpha x^2})(\omega) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}.$$

Question 2: Consider the Schrödinger equation $i\partial_t u + (1-i\epsilon)\partial_{xx}u = 0$, $u(x,0) = u_0(x)$, $x \in \mathbb{R}$, $t > 0$, where $\epsilon > 0$ and $i^2 = -1$. Note that u is complex-valued. (a) Solve the equation by using the Fourier technique assuming that $u_0 \in L^1(\mathbb{R})$ and decreases fast enough at infinity. (Hint: $\mathcal{F}(\sqrt{\frac{\pi}{\alpha}}e^{-\frac{x^2}{4\alpha}})(\omega) = e^{-\alpha\omega^2}$ for all $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$).)

Solution: We take the Fourier transform of the equation.

$$i\partial_t \mathcal{F}(u) + (i\omega)^2(1-i\epsilon)\mathcal{F}(u) = 0,$$

which gives the ODE

$$\partial_t \mathcal{F}(u) - \omega^2(-\epsilon - i)\mathcal{F}(u) = 0.$$

The solution is

$$\mathcal{F}(u)(\omega) = \mathcal{F}(u)(0)e^{-(\epsilon+i)t\omega^2} = \mathcal{F}(u_0)e^{-(\epsilon+i)t\omega^2}.$$

Observing that $\Re(\epsilon + i)t > \epsilon t > 0$, we can use the hint with $\alpha = (\epsilon + i)t$,

$$\mathcal{F}(u)(\omega) = \mathcal{F}(u_0)\mathcal{F}(\sqrt{\frac{\pi}{(\epsilon+i)t}}e^{-\frac{x^2}{4(\epsilon+i)t}})(\omega) = \frac{1}{2\pi}\mathcal{F}(u_0 * \sqrt{\frac{\pi}{(\epsilon+i)t}}e^{-\frac{x^2}{4(\epsilon+i)t}})\omega).$$

In conclusion

$$u(x,t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{(\epsilon+i)t}}\int_{-\infty}^{\infty} u_0(y)e^{-\frac{(x-y)^2}{4(\epsilon+i)t}}\mathrm{d}y = \sqrt{\frac{1}{4\pi(\epsilon+i)t}}\int_{-\infty}^{\infty} u_0(y)e^{-\frac{(x-y)^2}{4(\epsilon+i)t}}\mathrm{d}y.$$

(b) Let $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}|u(x,t)|^2\mathrm{d}x$ and $P(t) = \int_{-\infty}^{\infty}|\partial_x u(x,t)|^2\mathrm{d}x$. Prove that $\partial_t E(t) + \epsilon P(t) = 0$ assuming that u decreases fast enough at infinity. (Hint: (1) Apply the energy method to the Schrödinger equation with $\bar{u}$, (2) Apply the energy method to the complex conjugate of the Schrödinger equation with $-u$ (3) Sum the two results. Recall that $|v|^2 = v\bar{v}$.)

Solution: We follow the hint. (1) we test the equation with $\bar{u}$ and integrate ove $\mathbb{R}$,

$$\begin{aligned}
0 &= i\int_{\infty}^{\infty} \partial_t u\bar{u}\mathrm{d}x + (1-i\epsilon)\int_{\infty}^{\infty} \partial_{xx}u\bar{u}\mathrm{d}x \\
&= i\int_{\infty}^{\infty} \partial_t u\bar{u}\mathrm{d}x - (1-i\epsilon)\int_{\infty}^{\infty} \partial_x u\partial_x\bar{u}\mathrm{d}x, \qquad \text{we used } \bar{u}(\pm\infty,t) = 0 \\
&= i\int_{\infty}^{\infty} \partial_t u\bar{u}\mathrm{d}x + (i\epsilon - 1)\int_{\infty}^{\infty} \partial_x u\overline{\partial_x u}\mathrm{d}x, \\
&= i\int_{\infty}^{\infty} \partial_t u\bar{u}\mathrm{d}x + (i\epsilon - 1)\int_{\infty}^{\infty} |\partial_x u|^2\mathrm{d}x.
\end{aligned}$$

(2) Now we take the conjugate of the equation and use the energy method with $-u$,

$$\begin{aligned}
0 &= i\int_{\infty}^{\infty} \partial_t\bar{u}u\mathrm{d}x - (1+i\epsilon)\int_{\infty}^{\infty} \partial_{xx}\bar{u}u\mathrm{d}x \\
&= i\int_{\infty}^{\infty} \partial_t u\bar{u}\mathrm{d}x + (1+i\epsilon)\int_{\infty}^{\infty} \partial_x\bar{u}\partial_x u\mathrm{d}x, \qquad \text{we used } u(\pm\infty,t) = 0 \\
&= i\int_{\infty}^{\infty} \partial_t u\bar{u}\mathrm{d}x + (i\epsilon + 1)\int_{\infty}^{\infty} \partial_x u\overline{\partial_x u}\mathrm{d}x, \\
&= i\int_{\infty}^{\infty} \partial_t u\bar{u}\mathrm{d}x + (i\epsilon + 1)\int_{\infty}^{\infty} |\partial_x u|^2\mathrm{d}x.
\end{aligned}$$

(3) Now we sum the two results

$$\begin{aligned}
0 &= i\int_{\infty}^{\infty} (\partial_t u\bar{u} + \partial_t u\bar{u})\mathrm{d}x + 2i\epsilon\int_{\infty}^{\infty} |\partial_x u|^2\mathrm{d}x \\
&= i\int_{\infty}^{\infty} \partial_t(u\bar{u})\mathrm{d}x + 2i\epsilon\int_{\infty}^{\infty} |\partial_x u|^2\mathrm{d}x \\
&= i\int_{\infty}^{\infty} \partial_t|u|^2\mathrm{d}x + 2i\epsilon\int_{\infty}^{\infty} |\partial_x u|^2\mathrm{d}x.
\end{aligned}$$

We obtain the desired result after dividing by $2i$, $\partial_t E(t) + \epsilon P(t) = 0$.

Question 3: Compute $E(t)$ for $\epsilon = 0$,

Solution: We have $\partial_t E(t) = 0$, which implies that $E(t) = E(0)$,

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty}|u_0(x)|^2\mathrm{d}x.$$

Question 4: Let u solve $\partial_t u = \partial_{xx} u$, $u(x,0) = u_0(x)$, $x \in \mathbb{R}$, $t > 0$. Let us define $\mathcal{L}(u)(x,s) = \int_0^\infty e^{-st} u(x,t)\mathrm{d}t$ for all $s \in \mathbb{R}$, $s > 0$. (a) Compute $\partial_{xx}\mathcal{L}(u)(x,s)$ and show that $\partial_{xx}\mathcal{L}(u)(x,s) = s\mathcal{L}(u)(x,s) - u_0(x)$ for all $s > 0$.

Solution: We apply the definition

$$\begin{aligned}\partial_{xx}\mathcal{L}(u)(x,s) &= \partial_{xx}\int_0^\infty e^{-st}u(x,t)\mathsf{d}t = \int_0^\infty e^{-st}\partial_{xx}u(x,t)\mathsf{d}t = \int_0^\infty e^{-st}\partial_t u(x,t)\mathsf{d}t \\ &= -\int_0^\infty \partial_t(e^{-st})u(x,t)\mathsf{d}t + e^{-st}u(x,t)|_0^\infty = s\int_0^\infty e^{-st}u(x,t)\mathsf{d}t - u_0(x),\end{aligned}$$

which proves the desired result $\partial_{xx}\mathcal{L}(u)(x,s) = s\mathcal{L}(u)(x,s) - u_0(x)$.

(b) Let $s > 0$ and consider the equation $sv - \partial_{xx}v = u_0(x)$, $v(\pm\infty) = 0$, $x \in \mathbb{R}$. Compute Green's function, $G(x,x_0)$, $x, x_0 \in \mathbb{R}$.

Solution: Let $G(x,x_0)$ be Green's function. Since the operator is self-adjoint (shown in class many times), G satisfies

$$sG(x,x_0) - \partial_{xx}G(x,x_0) = \delta_{x_0}, \qquad G(\pm\infty, x_0) = 0.$$

Case 1: Assume $x < x_0$, then $G(x,x_0) = ae^{\sqrt{s}x} + be^{-\sqrt{s}x}$. The condition $G(-\infty,x_0) = 0$ implies that $b = 0$. Hence $G(x,x_0) = ae^{\sqrt{s}x}$ when $x < x_0$.
Case 2: Assume $x > x_0$, then $G(x,x_0) = ce^{\sqrt{s}x} + de^{-\sqrt{s}x}$. The condition $G(+\infty,x_0) = 0$ implies that $c = 0$. Hence $G(x,x_0) = de^{-\sqrt{s}x}$ when $x < x_0$.
Now we impose the continuity at x_0: $ae^{\sqrt{s}x_0} = de^{-\sqrt{s}x_0}$. We conclude with the jump condition,

$$\int_{x_0-\epsilon}^{x_0+\epsilon}(sG(x,x_0) - \partial_{xx}G(x,x_0))\mathsf{d}x = 1,$$

implying that $-\partial_x G(x_0^+,x_0) + \partial_x G(x_0^-,x_0) = 1$ when passing to the limit $\epsilon \to 0$. Hence $d\sqrt{s}e^{-\sqrt{s}x_0} + a\sqrt{s}e^{\sqrt{s}x_0} = 1$. Then using $ae^{\sqrt{s}x_0} = de^{-\sqrt{s}x_0}$, we infer that $d\sqrt{s}e^{-\sqrt{s}x_0} + d\sqrt{s}e^{-\sqrt{s}x_0} = 1$, i.e., $d = \frac{1}{2\sqrt{s}}e^{\sqrt{s}x_0}$ and $a = \frac{1}{2\sqrt{s}}e^{-\sqrt{s}x_0}$. In conclusion

$$G(x,x_0) = \begin{cases} \frac{1}{2\sqrt{s}}e^{\sqrt{s}(x-x_0)} & \text{if } x < x_0, \\ \frac{1}{2\sqrt{s}}e^{\sqrt{s}(x_0-x)} & \text{otherwise,}\end{cases}$$

which can also be re-written $G(x,x_0) = \frac{1}{2\sqrt{s}}e^{-\sqrt{s}|x-x_0|}$.

(c) Using (b), compute the solution to the equation $sv - \partial_{xx}v = u_0(x)$, $v(\pm\infty) = 0$, $x \in \mathbb{R}$.

Solution: We have

$$\begin{aligned} v(x_0) &= \int_{-\infty}^\infty (sG(x,x_0) - \partial_{xx}G(x,x_0))v(x)\mathsf{d}x, \qquad \text{with the usual abuse of notation} \\ &= \int_{-\infty}^\infty (sG(x,x_0)v(x) + \int_{-\infty}^\infty \partial_x G(x,x_0)\partial_x v(x)\mathsf{d}x = \int_{-\infty}^\infty (sG(x,x_0)v(x) - \int_{-\infty}^\infty G(x,x_0)\partial_{xx}v(x)\mathsf{d}x \\ &= \int_{-\infty}^\infty G(x,x_0)(sv(x) - \partial_{xx}v(x))\mathsf{d}x = \int_{-\infty}^\infty G(x,x_0)u_0(x)\mathsf{d}x.\end{aligned}$$

Hence

$$v(x_0) = \int_{-\infty}^\infty \frac{1}{2\sqrt{s}}e^{-\sqrt{s}|x-x_0|}u_0(x)\mathsf{d}x.$$

Question 5: Let us denote $\alpha = e^{\frac{\pi}{2}}$. Consider the operator

$$L : D(L) := \{v \in \mathcal{C}^2(1,\alpha) |\ v(1) = 0,\ v(\alpha) = 0\} \ni u \longmapsto 13u - 5x\partial_x u + x^2\partial_{xx}u \in \mathcal{C}^0(1,\alpha).$$

(a) Compute the formal adjoint of L, L^T, and its domain, $D(L^T)$.

Solution: Let w be a smooth function, say $w \in \mathcal{C}^2(1,\alpha)$. Then

$$\begin{aligned}
\int_1^\alpha Lv(x)w(x)\mathsf{d}x &= \int_1^\alpha (13v(x) - 5x\partial_x v(x) + x^2\partial_{xx}v(x))w(x)\mathsf{d}x \\
&= \int_1^\alpha (13v(x)w(x) + v(x)\partial_x(5xw(x)))\mathsf{d}x - 5xv(x)w(x)|_1^\alpha - \int_1^\alpha \partial_x v(x)\partial_x(x^2w(x))\mathsf{d}x + x^2\partial_x v(x)w(x)|_1^\alpha \\
&= \int_1^\alpha v(x)(13w(x) + \partial_x(5xw(x)))\mathsf{d}x + \int_1^\alpha v(x)\partial_{xx}(x^2w(x))\mathsf{d}x + x^2\partial_x v(x)w(x)|_1^\alpha - v(x)\partial_x(x^2w(x))|_1^\alpha \\
&= \int_1^\alpha v(x)(13w(x) + \partial_x(5xw(x))\partial_{xx}(x^2w(x)))\mathsf{d}x + x^2\partial_x v(x)w(x)|_1^\alpha.
\end{aligned}$$

We get rid of the boundary term by enforcing $w(1) = 0$ and $w(\alpha) = 0$. Hence the formal adjoint is defined by

$$L^T : D(L^T) = \{v \in \mathcal{C}^2(1,\alpha) |\ v(1) = 0,\ v(\alpha) = 0\} \ni w \longmapsto 13w(x) + \partial_x(5xw(x)) + \partial_{xx}(x^2w(x)) \in \mathcal{C}^0(1,\alpha).$$

(b) The general solution to $L(v) = 0$ is $v(x) = ax^3\cos(2\log(|x|)) + bx^3\sin(2\log(|x|))$, $a, b \in \mathbb{R}$ and the general solution to $L^T(w) = 0$ is $cx^{-4}\cos(2\log(x)) + dx^{-4}\sin(2\log(x))$. Compute the null spaces of L and L^T.

Solution: Let $v \in N(L)$, i.e., $L(v) = 0$ and $v \in D(L)$, then $v(x) = ax^3\cos(2\log(|x|)) + bx^3\sin(2\log(|x|))$. The boundary conditions gives

$$v(1) = 0, \quad v(\alpha) = a\alpha^3\cos(2\tfrac{\pi}{2}) + bx^3\sin(2\tfrac{\pi}{2}) = -a\alpha^3 = 0,$$

which in turn implies that $a = 0$, i.e., $N(L) = \text{span}\{x^3\sin(2\log(|x|))\}$.

Let $w \in N(L^T)$, i.e., $L^T(w) = 0$ and $w \in D(L^T)$, then $w(x) = cx^{-4}\cos(2\log(x)) + dx^{-4}\sin(2\log(x))$. The boundary conditions give

$$w(1) = 0, \quad w(\alpha) = c\alpha^{-4}\cos(2\tfrac{\pi}{2}) + dx^{-4}\sin(2\tfrac{\pi}{2}) = -c\alpha^{-4} = 0,$$

which implies $c = 0$, i.e., $N(L^T) = \text{span}\{x^{-4}\sin(2\log(x))\}$.

(c) Under which condition does the following problem have a solution: $Lu = f$, $u(1) = 0$, $u(\alpha) = 0$?

Solution: We are in the second case of the Fredholm alternative since $N(L) \neq \{0\}$. Hence this problem has a solution only if f is orthogonal to $N(L^T)$, that is to say $\int_1^\alpha f(x)x^{-4}\sin(2\log(|x|))\mathsf{d}x = 0$.

Question 6: Consider the conservation equation $\partial_t \rho + \partial_x(\rho^3) = 0$, $x \in \mathbb{R}$, $t > 0$, with initial data $\rho_0(x) = 0$ if $x < 0$, $\rho_0(x) = \sqrt{x}$ if $0 < x < 1$, and $\rho_0(x) = 0$ if $1 < x$. Draw the characteristics and give the explicit representation of the solution without computing the location of the shock.

Solution: The implicit representation of the solution to the equation $\partial_t \rho + \partial_x q(\rho) = 0$, $\rho(x, 0) = \rho_0(x)$, is

$$X(s,t) = q'(\rho_0(s))t + s; \quad \rho(X(s,t), t) = \rho_0(s). \tag{3}$$

The explicit representation is obtained by expressing s in terms of X and t and using $q(\rho) = \rho^3$.
Case 1: $s < 0$, we have $\rho_0(s) = 0$, $q'(\rho_0(s)) = 3\rho_0(s) = 0$, which implies $X = s$. Then

$$\rho(x,t) = 0 \text{ if } x < 0.$$

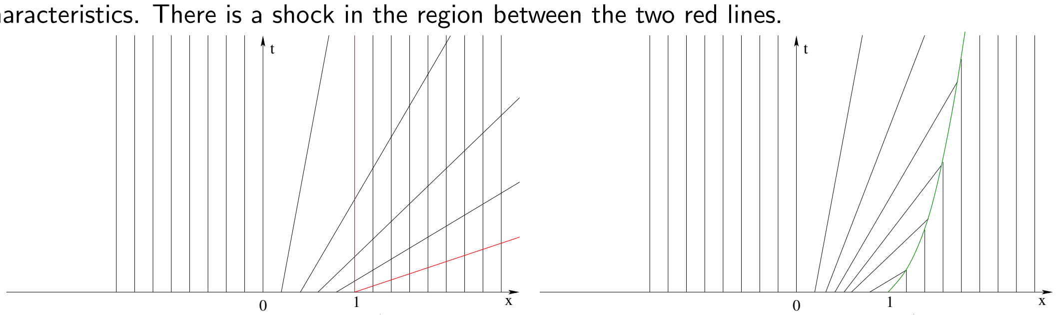
Case 2: $0 < s < 1$, we have $\rho_0(s) = \sqrt{s}$, $q'(\rho_0(s)) = 3\rho_0(s)^2 = 3s$, $X = s + 3ts$, which means $s = X/(1 + 3t)$. Then

$$\rho(x,t) = \sqrt{\frac{x}{1+3t}} \text{ if } 0 < x < 1 + 3t.$$

Case 3: $1 < 1$. we have $\rho_0(s) = 0$, $q'(\rho_0(s)) = 3\rho_0(s) = 0$, which implies $X = s$. Then

$$\rho(x,t) = 0 \text{ if } 1 < x.$$

Here are the characteristics. There is a shock in the region between the two red lines.



(b) Compute the position of the shock and give the expression for the solution for all $x \in \mathbb{R}$ and all $t > 0$.

Solution: We apply the Rankin Hugoniot formula. On the left of the shock the solution is $\rho^- = \sqrt{x_s(t)/(1+3t)}$, on the right it is $\rho^+ = 0$, then

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = \frac{(\rho^-)^3}{\rho^-} = (\rho^-)^2 = x_s(t)(1+3t).$$

Hence

$$\frac{1}{x_s(t)} \frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = \frac{1}{3} \frac{3}{(1+3t)},$$

which implies

$$\frac{\mathrm{d}\log(x_s(t))}{\mathrm{d}t} = \frac{1}{3} \frac{\mathrm{d}\log(1+3t)}{\mathrm{d}t}.$$

Applying the fundamental theorem of calculus, we obtain $\log(x_s(t)) - \log(1) = \frac{1}{3}(\log(1+3t) - \log(1))$. This implies that

$$x_s(t) = (1+3t)^{\frac{1}{3}}.$$

Finally the solution is as follows:

$$\rho(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \sqrt{\frac{x}{1+3t}} & \text{if } 0 < x < (1+3t)^{\frac{1}{3}} \\ 0 & \text{if } (1+3t)^{\frac{1}{3}} < x. \end{cases}$$

See figure above.

Question 7: Let $c > 0$ and $y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\mathrm{d}\xi$. Compute $\partial_t y(x,t)$, $\partial_{tt} y(x,t)$, $\partial_{ttt} y(x,t)$, $\partial_x y(x,t)$, $\partial_{xx} y(x,t)$, $\partial_{txx} y(x,t)$ and $\partial_{ttt} y - c^2\partial_{xxt} y$.

Solution: This exercise is meant to check whether you understand the notion of partial derivatives and the chain rule

$$\begin{aligned}
\partial_t y(x,t) &= \frac{1}{2}(h(x+ct) + h(x-ct)) \\
\partial_{tt} y(x,t) &= \frac{1}{2}(ch'(x+ct) - ch'(x-ct)) \\
\partial_{ttt} y(x,t) &= \frac{1}{2}(c^2 h'(x+ct) + c^2 h'(x-ct)) \\
\partial_x y(x,t) &= \frac{1}{2c}(h(x+ct) - h(x-ct)) \\
\partial_{xx} y(x,t) &= \frac{1}{2c}(h'(x+ct) - h'(x-ct)) \\
\partial_{txx} y(x,t) &= \frac{1}{2}(h'(x+ct) + h'(x-ct)).
\end{aligned}$$

In conclusion

$$\partial_{ttt} y - c^2\partial_{xxt} y = \frac{1}{2}(c^2 h'(x+ct) + c^2 h'(x-ct)) - c^2\frac{1}{2c}(ch'(x+ct) + ch'(x-ct)) = 0,$$

that is to say, $\partial_t y(x,t)$ solve the wave equation $\partial_{tt}(\partial_t y) - c^2\partial_{xx}(\partial_t(y)) = 0$.

Question 8: Consider the eigenvalue problem $-\frac{d}{dt}(t^{\frac{1}{2}}\frac{d}{dt}\phi(t)) = \lambda t^{-\frac{1}{2}}\phi(t)$, $t \in (0,1)$, supplemented with the boundary condition $\phi(0) = 0$, $\partial_t\phi(1) = 0$.
(a) What is the sign of λ? Is 0 an eigenvalue?

Solution: (i) Let ϕ be a non-zero smooth solution to the problem. Multiply the equation by ϕ and integrate over the domain. Use the Fundamental Theorem of calculus (i.e., integration by parts) to obtain

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t - [t^{\frac{1}{2}}\phi'(t)\phi(t)]_0^1 = \lambda\int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t.$$

Using the boundary conditions, we infer

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t = \lambda\int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t,$$

which means that λ is non-negative since ϕ is non-zero.

(ii) If $\lambda = 0$, then $\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2\mathrm{d}t = 0$, which implies that $\phi'(t) = 0$ for all $t \in (0,1]$. This implies that $\phi(t)$ is constant, and this constant is zero since $\phi(0) = 0$. Hence, ϕ is zero if $\lambda = 0$. Since we want a nonzero solution, this implies that λ cannot be zero.

(iii) In conclusion, it is necessary that λ be positive for a nonzero smooth solution to exist.