M602: Methods and Applications of Partial Differential Equations
Final TEST, May 9, Tuesday, 2006
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Here are some formulae that you may want to use:

$$\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g} \tag{1}$$

$$\widehat{\sqrt{\tfrac{\pi}{2}}e^{-|x|}} = (1+\xi^2)^{-1} \tag{2}$$

$$\widehat{e^{-\frac{1}{2}ax^2}} = \tfrac{1}{\sqrt{a}}e^{-\frac{1}{2}\xi^2/a} \tag{3}$$

$$\cos(\xi at) = \tfrac{1}{2}(e^{i\xi at} + e^{-i\xi at}) \tag{4}$$

Question 1

Compute the complex Fourier series of the function $f(x) = x$ defined on $[-\pi, \pi]$.

By definition $FS(f)(x) = \sum_{-\infty}^{+\infty} c_n e^{-in\pi x/L}$ where L is half the size of the interval on which f is defined. Here $L = \pi$. Hence, by integrating by parts once we obtain

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx} = -\frac{1}{2\pi}\frac{1}{-in}\int_{-\pi}^{\pi} e^{-inx} + \frac{1}{2\pi}\frac{1}{-in}(\pi e^{-in\pi} + \pi e^{in\pi})$$
$$= \frac{1}{2\pi}\frac{1}{-in}2\pi(-1)^n.$$

That is $c_n = \frac{(-1)^{n-1}}{in}$ and

$$FS(f)(x) = \sum_{-\infty}^{+\infty}\frac{(-1)^{n-1}}{in}e^{-inx}$$

Question 2

(a) Compute the Fourier transform of the function $f(x)$ defined by

$$f(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

By definition

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\xi x} = = \frac{1}{\sqrt{2\pi}} \frac{1}{-i\xi}(e^{-i\xi} - e^{i\xi})$$
$$= \frac{1}{\sqrt{2\pi}} \frac{2\sin(\xi)}{\xi}.$$

Hence

$$\hat{f}(\xi) = \sqrt{\frac{2}{\pi}} \frac{\sin(\xi)}{\xi}.$$

(b) Find the inverse Fourier transform of $\hat{g}(\xi) = \frac{\sin(\xi)}{\xi}$.

Using (a) we deduce that $g(x) = \sqrt{\frac{\pi}{2}} f(x)$ wherever $f(x)$ is of class C^1 and $g(x) = \sqrt{\frac{\pi}{2}} \frac{1}{2}(f(x^-) + f(x^+))$ where f is discontinuous. As a result:

$$g(x) = \begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } |x| < 1 \\ \frac{1}{2}\sqrt{\frac{\pi}{2}} & \text{at } |x| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Question 3

Use the Fourier transform technique to solve the following ODE $y''(x) - y(x) = f(x)$ for $x \in (-\infty, +\infty)$, where f is absolutely integrable.

By taking the Fourier transform of the ODE one obtains

$$-\xi^2 \hat{y} - \hat{y} = \hat{f}.$$

That is

$$\hat{y} = -\hat{f}\frac{1}{1+\xi^2}.$$

And by the convolution Theorem, that gives

$$y = -\frac{1}{\sqrt{2\pi}} f * (\sqrt{\frac{\pi}{2}} e^{-|x|}) = -\frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} e^{-|x-y|} f(y)$$

That is

$$y(x) = -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-y|} f(y).$$

Question 4

Solve the following integral equation $\int_{-\infty}^{+\infty} e^{-(x-y)^2} g(y)dy = e^{-\frac{1}{2}x^2}$ for all $x \in (-\infty, +\infty)$, i.e. find the function g that solves the above equation.

The left-hand side of the equation is a convolution; hence,

$$e^{-x^2} * g(x) = e^{-\frac{1}{2}x^2}.$$

By taking the Fourier transform, we obtain

$$\sqrt{2\pi}\frac{1}{\sqrt{2}}e^{-\frac{1}{4}\xi^2}\hat{g}(\xi) = e^{-\frac{1}{2}\xi^2}.$$

That yields

$$\hat{g}(\xi) = \frac{1}{\sqrt{\pi}}e^{-\frac{1}{4}\xi^2}.$$

By taking the inverse Fourier transform, we deduce

$$g(x) = \sqrt{\frac{2}{\pi}}e^{-x^2}.$$

Question 5

Use the Fourier transform method to compute the solution to the PDE: $u_{tt} - a^2 u_{xx} = 0$, for all $x \in (-\infty, +\infty)$ and $t \in (0, +\infty)$, with $u(0, x) = f(x) := \sin^2(x)$ and $u_t(0, x) = 0$ for all $x \in (-\infty, +\infty)$.

Take the Fourier transform in the x direction:

$$\hat{u}_{tt} + \xi^2 a^2 \hat{u} = 0.$$

This is an ODE. The solution is

$$\hat{u}(t, \xi) = c_1(\xi) \cos(\xi a t) + c_2(\xi) \sin(\xi a t).$$

The initial boundary conditions give

$$\hat{u}(0, \xi) = \hat{f}(\xi) = c_1(\xi)$$

and $c_2(\xi) = 0$. Hence

$$\hat{u}(t, \xi) = \hat{f}(\xi) \cos(\xi a t) = \frac{1}{2} \hat{f}(\xi)(e^{a\xi t} + e^{-a\xi t}).$$

Taking the inverse transform gives

$$\begin{aligned} u(t, x) &= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-\infty}^{+\infty} \hat{f}(\xi)(e^{a\xi t + \xi x} + e^{-a\xi t + \xi x}) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-\infty}^{+\infty} \hat{f}(\xi)(e^{\xi(at + x)} + e^{\xi(x - at)}) d\xi \\ &= \frac{1}{2}(f(x + at) + f(x - at)) = \frac{1}{2}(\sin^2(x + at) + \sin^2(x - at)). \end{aligned}$$

Note that this is the D'Alembert formula.

Question 6

(a) Find a function $U(x,y) = a + bx + cy + dxy$, such that $U(x,0) = x$, $U(1,y) = 1+y$, $U(x,1) = 3x-1$, and $U(0,y) = -y$.

$$U(x,y) = x - y + 2xy$$

solves the problem.

(b) Use (a) to solve the PDE $u_{xx} + u_{yy} = 0, \quad \forall (x,y) \in (0,1)\times(0,1)$, with the boundary conditions $u(x,0) = 3\sin(\pi x) + x$, $u(1,y) = 1+y$, $u(x,1) = 3x-1$,and $u(0,y) = \sin(2\pi y) - y$.

By setting $\phi = u - U$, we observe that $\phi_{xx} + \phi_{yy} = 0$ and at the boundary of the domain we have

$$\phi(x,0) = 3\sin(\pi x), \quad \phi(1,y) = 0, \quad \phi(x,1) = 0, \quad \text{and} \quad \phi(0,y) = \sin(2\pi y).$$

It is clear that

$$\phi(x,y) = 3\sin(\pi x)\frac{\sinh(\pi(1-y))}{\sinh(\pi)} + \frac{\sinh(2\pi(1-x))}{\sinh(2\pi)}\sin(2\pi y)$$

Then,

$$u(x,y) = \phi(x,y) + U(x,y)$$