M602: Methods and Applications of Partial Differential Equations
Final TEST, May 9, Wednesday, 2007
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Here are some formulae that you may want to use:

$$\widehat{f*g} = \sqrt{2\pi}\hat{f}\hat{g}; \qquad \widehat{xf}(\xi) = i\hat{f}'(\xi); \qquad \hat{\hat{h}}(x) = h(-x) \tag{1}$$

$$\sqrt{\tfrac{\pi}{2}}\widehat{e^{-|x|}} = (1+\xi^2)^{-1}; \qquad \sin(bx) = \tfrac{1}{2i}(e^{ibx} - e^{-ibx}) \tag{2}$$

$$\widehat{e^{-\frac{1}{2}x^2}} = e^{-\frac{1}{2}\xi^2}; \qquad Ai(x) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_0^{+\infty} \cos(\tfrac{1}{3}\xi^3 + x\xi)d\xi \tag{3}$$

Question 1

(i) Let Ω be an open connected set in $\mathbb{R}^2$. Let u be a real-valued nonconstant function continuous on $\overline{\Omega}$ and harmonic on Ω. Assume that there exists x_0 in Ω such $\nabla u(x_0) = 0$. Do we have a minimum, a maximum, or a saddle point at x_0? (explain)

The Maximum principle implies that u cannot be either minimum or maximum at x_0. This point is a saddle point.

(ii) Let $\Omega = (0,1)$ (note that $\overline{\Omega} = [0,1]$), and let $u : \overline{\Omega} \longrightarrow \mathbb{R}$ be such that $u(x) = 1$ for all $x \in \Omega$, $u(0) = 0$, and $u(1) = -1$. Is u harmonic on Ω? Find a point in $\overline{\Omega}$ where u reaches its maximum? Does this example contradict the Maximum Principle? (explain)

Yes u is harmonic on Ω since $u''(x) = 0$ for all x in Ω. Note however that u is not continuous on $\overline{\Omega}$; as a consequence, the hypotheses for the Maximum priciple are not satisfied. In other words, the above example does not contradict the Maximum principle .

Question 2

(i) Let f be an integrable function on $(-\infty, +\infty)$. Prove that for all $a, b \in \mathbb{R}$, and for all $\xi \in \mathbb{R}$, $[\widehat{e^{ibx} f(ax)}](\xi) = \frac{1}{a}\hat{f}(\frac{\xi - b}{a})$.

The definition of the Fourier transform together with the change of variable $ax \longmapsto x'$ implies

$$\begin{aligned}
[\widehat{e^{ibx} f(ax)}](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(ax) e^{ibx} e^{-i\xi x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(ax) e^{i(b-\xi)x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{a} f(x') e^{-i\frac{(\xi-b)}{a}x'} dx' \\
&= \tfrac{1}{a}\hat{f}(\tfrac{\xi-b}{a}).
\end{aligned}$$

(ii) Let c be a positive real number. Compute the Fourier transform of $f(x) = e^{-\frac{1}{2}cx^2} \sin(bx)$.

Using the fact that $\sin(bx) = -i\frac{1}{2}(e^{ibx} - e^{-ibx})$ and setting $a = \sqrt{c}$, we infer

$$\begin{aligned}
f(x) &= \tfrac{1}{2i} e^{-\frac{1}{2}(ax)^2} (e^{ibx} - e^{-ibx}) \\
&= \tfrac{1}{2i} e^{-\frac{1}{2}(ax)^2} e^{ibx} - \tfrac{1}{2i} e^{-\frac{1}{2}(ax)^2} e^{i(-b)x})
\end{aligned}$$

and using (i) and (**??**) we deduce

$$\hat{f}(\xi) = \tfrac{1}{2i}\tfrac{1}{\sqrt{c}}(e^{-\frac{1}{2c}(\xi-b)^2} - e^{-\frac{1}{2c}(\xi+b)^2}).$$

(iii) Let $g(x) = (x^2 + a^2)^{-1}$ and prove $\hat{g}(\xi) = \frac{1}{a}\sqrt{\frac{\pi}{2}}e^{-a|x|}$.

Let $f(x) = (1 + x^2)^{-1}$, then $g(x) = \frac{1}{a^2}f(xa^{-1})$. As a result, using (i) we infer

$$\hat{g}(\xi) = \frac{1}{a^2}\frac{1}{a^{-1}}\hat{f}(a\xi).$$

Moreover, using the property $\hat{\hat{h}}(x) = h(-x)$, we infer $\hat{f}(\xi) = \sqrt{\pi/2}e^{-\xi}$ and

$$\hat{g}(\xi) = \frac{1}{a}\sqrt{\frac{\pi}{2}}e^{-a|x|}.$$

Question 3

Let N be a positive integer and let $\mathbb{P}_N$ be the set of trigonometric polynomials of degree at most N; that is, $\mathbb{P}_N = \mathrm{span}\{1, \cos(x), \sin(x), \dots, \cos(Nx), \sin(Nx)\}$.

(i) Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n^5}\sin(7n)\cos(2nx).$$

Compute the best L^2-approximation of f in $\mathbb{P}_7$ over $(0, 2\pi)$.

The best L^2-approximation of f in $\mathbb{P}_7$ over $(0, 2\pi)$ is the truncated Fourier series $S_7(f)$. Clearly

$$S_7(f)(x) = \sum_{n=0}^{3} \frac{1}{n^5}\sin(7n)\cos(2nx)$$

(ii) Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{11} \frac{1}{n^3+1}\cos(35n^2)\sin(3nx).$$

Compute the best L^2-approximation of f in $\mathbb{P}_{35}$ over $(0, 2\pi)$.

The best L^2-approximation of f in $\mathbb{P}_{35}$ over $(0, 2\pi)$ is the truncated Fourier series $S_{35}(f)$. Observing that $f \in \mathbb{P}_{33} \subset \mathbb{P}_{35}$ it is clear that $S_{35}(f) = f$.

Question 4

Use the Fourier transform technique to solve the following boundary value problem $y''(x) - xy(x) = 0$ for $x \in (-\infty, +\infty)$, with the boundary conditions $y(\pm\infty) = 0$ and $y(0) = Ai(0)$.

By taking the Fourier transform of the equation one obtains

$$-\xi^2 \hat{y} - \widehat{xy} = 0.$$

Using (**??**), i.e. $\widehat{xy} = i\widehat{y}$,

$$\hat{y}' = i\xi^2 \hat{y}.$$

The solution of this ODE is

$$\hat{y} = ce^{i\xi^3/3}.$$

We now apply the inverse Fourier transform

$$\begin{aligned} y(x) &= c\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\xi^3/3} e^{ix\xi} d\xi \\ &= c\frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} \left(e^{i(\xi^3/3 + x\xi)} + e^{-i(\xi^3/3 + x\xi)} \right) d\xi \\ &= c\frac{2}{\sqrt{2\pi}} \int_{0}^{+\infty} \cos(\tfrac{1}{3}\xi^3 + x\xi) d\xi. \end{aligned}$$

The condition on $y(0)$ then implies

$$y(x) = \frac{1}{\pi} \int_{0}^{+\infty} \cos(\tfrac{1}{3}\xi^3 + x\xi) d\xi.$$

In other words $y(x)$ is equal to the Airy function $Ai(x)$.

Question 5

Solve the following integral equation $\int_{-\infty}^{+\infty} \frac{g(y)}{(x-y)^2+1} dy = \frac{1}{x^2+4}$ for all $x \in (-\infty, +\infty)$, i.e. find the function g that solves the above equation.

The left-hand side of the equation is a convolution; hence,

$$\frac{1}{z^2+1} * g = \frac{1}{x^2+4}.$$

By taking the Fourier transform, we obtain

$$\sqrt{2\pi}\sqrt{2\pi}\frac{1}{2}e^{-|\xi|}\hat{g}(\xi) = \sqrt{2\pi}\frac{1}{4}e^{-2|\xi|}.$$

That yields

$$\hat{g}(\xi) = \frac{1}{2\sqrt{2\pi}}e^{-|\xi|}.$$

By taking the inverse Fourier transform, we deduce

$$g(x) = \frac{1}{2\pi}\frac{1}{1+x^2}.$$

Question 6

Let u solve the following equation: $\partial_{tt}u + \partial_t u - a^2\partial_{xx}u = 0$ for all $x \in (-\infty, +\infty)$ and $t > 0$ with the boundary and initial conditions $u(\pm\infty, t) = 0$, $u(x, 0) = u_0(x)$, $\partial_t u(x, 0) = u_1(x)$.
(i) Assuming u is smooth, prove that the following holds for all $T > 0$:

$$\tfrac{1}{2}\|\partial_t u(\cdot, T)\|_{L^2}^2 + \tfrac{1}{2}a^2\|\partial_x u(\cdot, T)\|_{L^2}^2 + \int_0^T \|\partial_t u(\cdot, t)\|_{L^2}^2 dt = \tfrac{1}{2}\|u_1\|_{L^2}^2 + \tfrac{1}{2}a^2\|\partial_x u_0\|_{L^2}^2$$

We multiply by $\partial_t u$ and integrate over space to obtain

$$\begin{aligned} 0 &= \int_{-\infty}^{+\infty} \left(\partial_{tt}u\partial_t u + (\partial_t u)^2 - a^2\partial_{xx}u\partial_t u\right) dx \\ &= \int_{-\infty}^{+\infty} \frac{1}{2}\partial_t(\partial_t u)^2 + (\partial_t u)^2 + a^2\partial_x u\partial_t(\partial_x u) dx \\ &= \frac{1}{2}\frac{d}{dt}\|\partial_t u(\cdot, t)\|_{L^2}^2 + \|\partial_t u(\cdot, t)\|_{L^2}^2 + a^2\frac{1}{2}\frac{d}{dt}\|\partial_x u(\cdot, t)\|_{L^2}^2. \end{aligned}$$

Now we integrate over time between 0 and T,

$$0 = \frac{1}{2}\|\partial_t u(\cdot, T)\|_{L^2}^2 + \int_0^T \|\partial_t u(\cdot, t)\|_{L^2}^2 dt + a^2\frac{1}{2}\|\partial_x u(\cdot, t)\|_{L^2}^2 - \frac{1}{2}\|\partial_t u(\cdot, 0)\|_{L^2}^2 - a^2\frac{1}{2}\|\partial_x u(\cdot, 0)\|_{L^2}^2.$$

This gives

$$\frac{1}{2}\|\partial_t u(\cdot, T)\|_{L^2}^2 + \int_0^T \|\partial_t u(\cdot, t)\|_{L^2}^2 dt + a^2\frac{1}{2}\|\partial_x u(\cdot, t)\|_{L^2}^2 = \frac{1}{2}\|u_1\|_{L^2}^2 + a^2\frac{1}{2}\|\partial_x u_0\|_{L^2}^2.$$
