M602: Methods and Applications of Partial Differential Equations
Final Exam, May 5th
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

$$\mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{2}$$

$$\mathcal{F}(\frac{2\alpha}{x^2+\alpha^2})(\omega) = e^{-\alpha|\omega|}, \tag{3}$$

Question 1

Let $\Omega = \{(t,x) \in \mathbb{R}^2 : \ t > 0, \ x \geq -t\}$. Let Γ be defined by the following parameterization $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s), s \in \mathbb{R}\}$, with $x_\Gamma(s) = s$ and $t_\Gamma(s) = -s$ if $s \leq 0$, $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s \geq 0$. Solve the following PDE (give the implicit and explicit representations):

$$u_t + 2u_x + u = 0, \quad \text{in } \Omega, \qquad u(x,t) = u_\Gamma(x,t) := \begin{cases} 1 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases} \quad \text{for all } (x,t) \text{ in } \Gamma.$$

We define the characteristics by

$$\frac{dx(t,s)}{dt} = 2, \quad x(t_\Gamma(s), s) = x_\Gamma(s).$$

This gives $x(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s))$. Upon setting $\phi(t,s) = u(x(t,s),t)$, we observe that $\partial_t\phi(t,s) + \phi(t,s) = 0$, which means

$$\phi(t,s) = ce^{-t}.$$

The initial condition implies $\phi(t_\Gamma(s), s) = u_\Gamma(x_\gamma(s), t_\Gamma(s))$; as a result $c = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{t_\Gamma(s)}$.

$$\phi(t,s) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{t_\Gamma(s)-t}.$$

The implicit representation of the solution is

$$u(x(t,s),t) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{t_\Gamma(s)-t}, \qquad x(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s)).$$

Now we give the explicit representation.
Case 1: If $s \leq 0$, $x_\Gamma(s) = s$, $t_\Gamma(s) = -s$, and $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 2$. This means $x(t,s) = s+2(t+s)$ and we obtain $s = \frac{1}{3}(x - 2t)$, which means

$$u(x,t) = 2e^{-\frac{1}{3}(x-2t)-t}, \qquad \text{if } x - 2t < 0.$$

Case 2: If $s \geq 0$, $x_\Gamma(s) = s$, $t_\Gamma(s) = 0$, and $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 1$. This means $x(t,s) = s + 2t$ and we obtain $s = x - 2t$, which means

$$u(x,t) = e^{-t}, \qquad \text{if } x - 2t > 0.$$

Question 2

Consider the following conservation equation

$$\partial_t \rho + \partial_x (q(\rho)) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} 3 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2+\rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

The characteristics are defined by

$$\frac{dX(t)}{dt} = q'(\rho) = 2(1 + \rho(x(t),t)), \quad X(0) = X_0.$$

Set $\phi(t) = \rho(X(t),t)$, then we obtain that ϕ is constant, i.e., ρ is constant along the characteristics: $\rho(X(t),t) = \rho(X_0,0) = \rho_0(X_0)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t) = 2(1+\rho_0(X_0))t + X_0$. We then have two cases depending whether X_0 is positive or negative.

1. $X_0 < 0$, then $\rho_0(X_0) = 3$ and $X(t) = 2(1+3)t + X_0 = 8t + X_0$. This means

$$\rho(x,t) = 3 \quad \text{if} \quad x < 8t.$$

2. $X_0 > 0$, then $\rho_0(X_0) = 1$ and $X(t) = 2(1+1)t + X_0 = 4t + X_0$. This means

$$\rho(x,t) = 1 \quad \text{if} \quad x > 4t.$$

We see that the characteristics cross in the region $\{8t > x > 4t\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock:

$$\frac{dx_s(t)}{dt} = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{15-3}{3-1} = 6, \qquad x_s(0) = 0.$$

In conclusion, $x_s(t) = 6t$ and

$$\begin{aligned} \rho = 3, &\quad x < x_s(t) = 6t, \\ \rho = 1, &\quad x > x_s(t) = 6t. \end{aligned}$$

Question 3

Consider the following conservation equation

$$\partial_t u + u\partial_x u = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad u(x,0) = u_0(x) := \begin{cases} 1 & \text{if } x \leq 0, \\ 1-x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 \leq x. \end{cases}$$

(i) Solve this problem using the method of characteristics for $0 \leq t \leq 1$.

The characteristics are defined by

$$\frac{dX(t,x_0)}{dt} = u(X(t,x_0),t), \qquad X(0,x_0) = x_0.$$

From class we know that $u(X(t,x_0),t)$ does not depend on time, that is to say

$$X(t,x_0) = u(X(0,x_0),0)t + x_0 = u(x_0,0)t + x_0 = u_0(x_0)t + x_0.$$

Case 1: If $x_0 \leq 0$, we have $u_0(x_0) = 1$ and $X(t,x_0) = t + x_0$; as a result, $x_0 = X - t$, and

$$u(x,t) = 1, \qquad \text{if } x \leq t.$$

Case 2: If $0 \leq x_0 \leq 1$, we have $u_0(x_0) = 1 - x_0$ and $X(t,x_0) = t(1-x_0) + x_0$; as a result $x_0 = (X-t)/(1-t)$, and

$$u(x,t) = 1 - (t-x)/(t-1), \qquad \text{if } 0 \leq x - t \leq 1 - t,$$

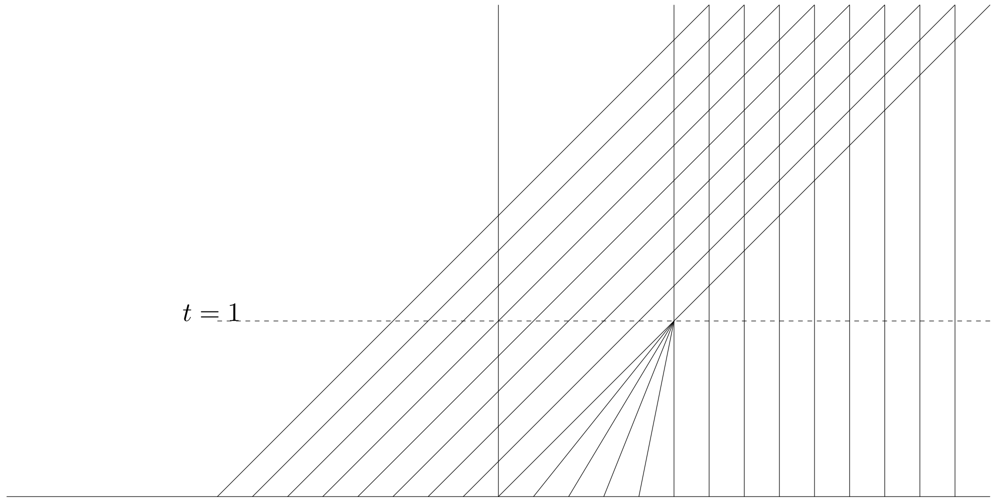
which can also be re-written

$$u(x,t) = \frac{x-1}{t-1}, \qquad \text{if } t \leq x \leq 1.$$

case 3: If $1 \leq x_0$, we have $u_0(x_0) = 0$ and $X(t,x_0) = x_0$; as a result

$$u(x,t) = 0, \qquad \text{if } 1 \leq x.$$

(ii) Draw the characteristics for all $t > 0$ and all $x \in \mathbb{R}$.



(iii) At $t = 1$ we have $u(x, 1) = 1$ if $x < 1$ and $u(x, 1) = 0$ if $x > 1$. Solve the problem for $t > 1$.

Denote by $u_1(x)$ the solution at $t = 1$. The characteristics are $X(t, x_0) = u_1(x_0)(t - 1) + x_0$.
Case 1: If $x_0 < 1$, $u_1(x_0) = 1$ and $X(t, x_0) = t - 1 + x_0$; as a result,

$$u(x,t) = 1, \qquad \text{If } x < t.$$

Case 2: If $1 < x_0$, $u_1(x_0) = 0$ and $X(t, x_0) = x_0$; as a result,

$$u(x,t) = 0, \qquad \text{If } 1 < x.$$

The characteristics cross in the domain $\{1 < x < t\}$; as a result we have a shock. The speed of the shock is given by the Rankin-Hugoniot relation (recall that $q(u) = u^2/2$):

$$\frac{dx_s(t)}{dt} = \frac{q^+ - q^-}{u^+ - u^-} = \frac{1/2 - 0}{1 - 0} = \frac{1}{2}, \qquad x_s(1) = 1,$$

Which gives $x_s(t) = \frac{1}{2}(t + 1)$. In conclusion,

$$u(x,t) = \begin{cases} 1 & \text{If } t > 1 \text{ and } x < \frac{1}{2}(t+1), \\ 0 & \text{If } t > 1 \text{ and } \frac{1}{2}(t+1) < x. \end{cases}$$

Question 4

Consider the following wave equation

$$\partial_{tt} w - 4\partial_{xx} w = 0, \quad x > 0,\ t > 0$$
$$w(x,0) = x(1+x^2)^{-1}, \quad x > 0, \qquad \partial_t w(x,0) = 0, \quad x > 0, \quad \text{and} \quad w(0,t) = 0, \quad t > 0.$$

(a) Solve the equation.

We define $f(x) = x(1+x^2)^{-1}$ and its odd extension $f_o(x)$. Let w_0 be the solution to the wave equation over the entire real line with f_o as initial data:

$$\partial_{tt} w_o - 4\partial_{xx} w_o = 0, \quad x \in \mathbb{R},\ t > 0$$
$$w_o(x,0) = f_o(x), \quad x > 0, \qquad \partial_t w_o(x,0) = 0, \quad x \in \mathbb{R}.$$

The solution to this problem is given by the D'Alembert formula

$$w_o(x,t) = \frac{1}{2}(f_o(x-2t) + f_o(x+2t)), \qquad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0.$$

Let x be positive. Then $w(x,t) = w_o(x,t)$ (since by construction $w_o(0,t) = 0$ for all times).

Case 1: If $x - 2t > 0$, $f_o(x-2t) = f(x-2t)$; as a result

$$w(x,t) = \frac{1}{2}(f(x-2t) + f(x+2t).$$

Case 2: If $x - 2t < 0$, $f_o(x-2t) = -f(-x+2t)$; as a result

$$w(x,t) = \frac{1}{2}(-f(-x+2t) + f(x+2t)).$$

Note that actually $f_0(x) = x(1+x^2)^{-1}$; as a result, the solution can also be re-written as follows:

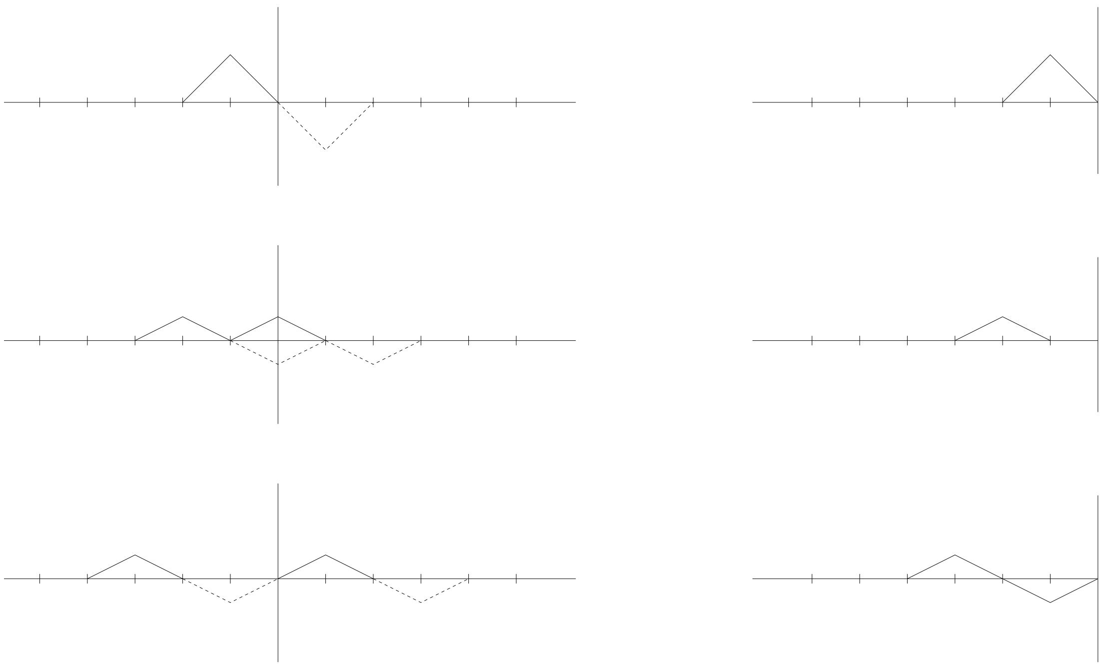
$$w(x,t) = \frac{1}{2}((x-2t)(1+(x-2t)^2)^{-1} + (x+2t)(1+(x+2t)^2)^{-1}).$$

Question 5

Consider the wave equation

$$\partial_{tt}w - \partial_{xx}w = 0, \quad x < 0,\ t > 0$$
$$w(x,0) = f(x), \quad x < 0, \qquad \partial_t w(x,0) = 0, \quad x < 0, \quad \text{and} \quad w(0,t) = 0, \quad t > 0.$$

where $f(x) = -x$, if $x \in [-1, 0]$, $f(x) = 2 + x$, if $x \in [-2, -1]$, and $f(x) = 0$ otherwise. Give a graphical solution to the problem at $t = 0$, $t = 1$, and $t = 2$ (draw three different graphs and explain what you do)



Initial data + odd extension Solution

Question 6

Solve the following integral equation $\int_{-\infty}^{+\infty} \frac{g(y)}{(x-y)^2+1} dy = \frac{1}{x^2+4}$ for all $x \in (-\infty, +\infty)$, i.e. find the function g that solves the above equation.

The left-hand side of the equation is a convolution; hence,

$$(\frac{1}{z^2+1} * g)(x) = \frac{1}{x^2+4}.$$

By taking the Fourier transform, we obtain

$$2\pi \frac{1}{2} e^{-|\omega|} \hat{g}(\omega) = \frac{1}{4} e^{-2|\omega|}.$$

That yields

$$\hat{g}(\omega) = \frac{1}{4\pi} e^{-|\omega|}.$$

By taking the inverse Fourier transform, we deduce

$$g(x) = \frac{1}{2\pi} \frac{1}{1+x^2}.$$

Question 7

Let f be a smooth function in $[0,1]$. Consider the PDE

$$u - \partial_{xx}u = f(x), \quad x \in (0,1), \qquad \partial_x u(1) + u(1) = 2, \ -\partial_x u(0) + u(0) = 1.$$

What PDE and which boundary conditions must satisfy the Green function, $G(x,x_0)$, (DO NOT compute the Green function)? Give the integral representation of u assuming $G(x,x_0)$ is known. Fully justify your answer.

Multiply the equation by $G(x,x_0)$ and integrate over $(0,1)$:

$$\begin{aligned}
\int_0^1 f(x)G(x,x_0)dx &= \int_0^1 (u(x) - \partial_{xx}u(x))G(x,x_0)dx \\
&= \int_0^1 u(x)G(x,x_0) + \partial_x u(x)\partial_x G(x,x_0)dx - \partial_x u(1)G(1,x_0) + \partial_x u(0)G(0,x_0) \\
&= \int_0^1 u(x)(G(x,x_0) - \partial_{xx}G(x,x_0))dx + u(1)\partial_x G(1,x_0) - u(0)\partial_x G(0,x_0) \\
&- \partial_x u(1)G(1,x_0) + \partial_x u(0)G(0,x_0) \\
&= \int_0^1 u(x)(G(x,x_0) - \partial_{xx}G(x,x_0))dx + u(1)\partial_x G(1,x_0) - u(0)\partial_x G(0,x_0) \\
&(u(1)-2)G(1,x_0) + (u(0)-1)G(0,x_0) \\
&= \int_0^1 u(x)(G(x,x_0) - \partial_{xx}G(x,x_0))dx \\
&+ u(1)(G(1,x_0) + \partial_x G(1,x_0)) + u(0)(G(0,x_0) - \partial_x G(0,x_0)) - 2G(1,x_0) - G(0,x_0)
\end{aligned}$$

If we define $G(x,x_0)$ so that

$$G(x,x_0) - \partial_{xx}G(x,x_0) = \delta(x-x_0), \qquad G(1,x_0) + \partial_x G(1,x_0) = 0, \ G(0,x_0) - \partial_x G(0,x_0) = 0,$$

then $u(x_0)$, $x_0 \in (0,1)$, has the following representation

$$u(x_0) = \int_0^1 f(x)G(x,x_0)dx + 2G(1,x_0) + G(0,x_0), \qquad \forall x_0 \in (0,1).$$
