M602: Methods and Applications of Partial Differential Equations. Final TEST, May, 2014. Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}\mathrm{d}x, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}\mathrm{d}\omega, \qquad \mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{1}$$

$$\mathcal{F}(S_\lambda(x)) = \frac{1}{\pi}\frac{\sin(\lambda\omega)}{\omega}, \quad \text{where} \quad S_\lambda(x) = \begin{cases} 1 & \text{if } |x| \le \lambda \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The implicit representation of the solution to the equation $\partial_t v + \partial_x q(v) = 0$, $v(x,0) = v_0(x)$, is

$$X(s,t) = q'(v_0(s))t + s; \quad v(X(s,t),t) = v_0(s). \tag{3}$$

Question 1: Let $y(x,t) = x\cos(2t + \log(|x|))$. Compute $\partial_{tt}y + x^2\partial_{xx}y - x\partial_x y + 6y$.

Solution: This exercise is meant to check whether you understand the notion of partial derivatives and the chain rule

$$\partial_{tt}y(x,t) = -4x\cos(2t + \log(|x|)) = -4y(x,t),$$
$$\partial_x y(x,t) = \cos(2t + \log(|x|)) - x\sin(2t + \log(|x|))\frac{1}{x} = \cos(2t + \log(|x|)) - \sin(2t + \log(|x|)),$$
$$\partial_{xx}y(x,t) = -\frac{1}{x}\sin(2t + \log(|x|)) - \frac{1}{x}\cos(2t + \log(|x|))$$

In conclusion

$$\begin{aligned}\partial_{tt}y + x^2\partial_{xx}y - x\partial_x y + 6y &= -4x\cos(2t + \log(|x|)) - x\sin(2t + \log(|x|)) \\ &\quad - x\cos(2t + \log(|x|)) - x\cos(2t + \log(|x|)) + x\sin(2t + \log(|x|)) + 6x\cos(2t + \log(|x|)) \\ &= 0,\end{aligned}$$

that is to say, $y(x,t)$ solve the PDE $\partial_{tt}y + x^2\partial y - x\partial_x y + 6y = 0$.

Question 2: Let μ and c be two positive numbers. Consider the following system of coupled partial differential equations with dependent variables $\rho(x,t)$ and $u(x,t)$: $\partial_t\rho + \partial_x u = 0$, $\partial_t u - \mu\partial_{xx}u + c^2\partial_x\rho = 0$, $x \in (0,L)$, $t > 0$, with initial data $\rho(x,0) = \rho_0$, $u(x,0) = u_0$, and boundary conditions $u(0,t) = u(L,t) = 0$. (i) Show that the quantity $E(t) = \int_0^L \frac{1}{2}(\rho^2(x,t) + c^{-2}u(x,t))\mathrm{d}x$ decreases in time. (Hint: Use the energy argument: Use ρ for the first equation, u/c^2 for the second, add the results, do not panic and proceed as usual.)

Solution: We proceed as in the hint. We multiply the first equation by ρ and integrate over the domain,

$$0 = \int_0^L (\rho(x,t)\partial_t\rho(x,t) + \rho(x,t)\partial_x u(x,t))\mathrm{d}x = \int_0^L (\partial_t(\frac{1}{2}\rho^2(x,t)) + \rho(x,t)\partial_x u(x,t))\mathrm{d}x$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{1}{2}\rho^2(x,t)\mathrm{d}x + \int_0^L \rho(x,t)\partial_x u(x,t)\mathrm{d}x.$$

We proceed similarly for the second equation. We multiply the second equation by $c^{-2}u$ and integrate over the domain,

$$0 = \int_0^L (c^{-2}u(x,t)\partial_t u(x,t) - \mu c^{-2}u(x,t)\partial_{xx}u(x,t) + u(x,t)\partial_x\rho(x,t))\mathrm{d}x$$

We integrate by parts the viscous term and we obtain $-\int_0^L \mu c^{-2}u(x,t)\partial_{xx}u(x,t)\mathrm{d}x = \int_0^L \mu c^{-2}(\partial_x u(x,t))^2\mathrm{d}x$, owing to the boundary conditions $u(0,t) = u(L,t) = 0$,

$$0 = \int_0^L (c^{-2}\partial_t(\frac{1}{2}u^2(x,t)) + \mu c^{-2}(\partial_x u(x,t))^2 + u(x,t)\partial_x\rho(x,t))\mathrm{d}x$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{1}{2}c^{-2}u^2(x,t)\mathrm{d}x + \int_0^L (\mu c^{-2}(\partial_x u(x,t))^2 + u(x,t)\partial_x\rho(x,t))\mathrm{d}x.$$

Adding the two equations together and using the product rule gives

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{1}{2}\rho^2(x,t)\mathrm{d}x + \int_0^L \rho(x,t)\partial_x u(x,t)\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{1}{2}c^{-2}u^2(x,t)\mathrm{d}x + \int_0^L (\mu c^{-2}(\partial_x u(x,t))^2 + u(x,t)\partial_x\rho(x,t))\mathrm{d}x$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{1}{2}(\rho^2(x,t) + c^{-2}u^2(x,t))\mathrm{d}x + \int_0^L \mu c^{-2}(\partial_x u(x,t))^2\mathrm{d}x + \int_0^L (\rho(x,t)\partial_x u(x,t) + u(x,t)\partial_x\rho(x,t))\mathrm{d}x$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{1}{2}(\rho^2(x,t) + c^{-2}u^2(x,t))\mathrm{d}x + \int_0^L \mu c^{-2}(\partial_x u(x,t))^2\mathrm{d}x + \int_0^L \partial_x(\rho(x,t)u(x,t))\mathrm{d}x.$$

The Fundamental Theorem of Calculus gives $\int_0^L \partial_x(\rho(x,t)u(x,t))\mathrm{d}x = 0$, owing to the boundary conditions on u. In conclusion

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{1}{2}(\rho^2(x,t) + c^{-2}u^2(x,t))\mathrm{d}x = -\int_0^L \mu c^{-2}(\partial_x u(x,t))^2\mathrm{d}x \le 0,$$

which proves that the quantity $E(t) = \int_0^L \frac{1}{2}(\rho^2(x,t) + c^{-2}u(x,t))\mathrm{d}x$ decreases in time.

(ii) What how does $E(t)$ behaves when $\mu = 0$?

Solution: The above computation shows that $\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0$ when $\mu = 0$; as a result, $E(t)$ is constant in time when $\mu = 0$.

Solve the PDE $\partial_t u + 4\partial_x u + 12u = 0$ in $\Omega = \{(x,t) \in \mathbb{R}^2 \mid x \geq 0,\ x + 11t \geq 0\}$ with boundary conditions $u(x,0) = x + 4$, $u(-11t, t) = t + 4$ for $t > 0$.

Solution: (1) The boundary of Ω, say Γ, is parametrized as follows: $\Gamma = \{(x_\Gamma(s), t_\Gamma(s)) \mid s \in \mathbb{R}\}$ where

$$x_\Gamma(s) = \begin{cases} 11s & \text{if } s < 0, \\ s & \text{if } s > 0. \end{cases} \qquad t_\Gamma(s) = \begin{cases} -s & \text{if } s < 0, \\ 0 & \text{if } s > 0. \end{cases}$$

(2) The characteridtics are defined by the ODE

$$\partial_t X(s,t) = 4, \quad X(s, t_\Gamma(s)) = x_\Gamma(s).$$

The solution is $X(s,t) = 4(t - t_\Gamma(s)) + x_\Gamma(s)$.
(3) We make a change of variable: $\phi(s,t) = u(X(s,t),t)$ and compute $\partial_t \phi(s,t)$,

$$\begin{aligned}\partial_t \phi(s,t) &= \partial_t u(X(s,t),t) + \partial_x u(X(s,t),t)\partial_t X(s,t) = \partial_t u(X(s,t),t) + 4\partial_x u(X(s,t),t) \\ &= -12u(X(s,t),t) = -12\phi(s,t),\end{aligned}$$

with initial data $\phi(s, t_{\Gamma(s)}) = u(x_{\Gamma(s)}, t_{\Gamma(s)})$. The solution is

$$\phi(s,t) = \phi(s, t_{\Gamma(s)})e^{-12(t - t_{\Gamma(s)})} = u(x_{\Gamma(s)}, t_{\Gamma(s)})e^{-12(t - t_{\Gamma(s)})}.$$

i.e., the implicit representation of the solution is

$$u(X(s,t),t) = u(x_{\Gamma(s)}, t)e^{-12(t - t_{\Gamma(s)})}, \quad X(s,t) = 4(t - t_\Gamma(s)) + x_\Gamma(s).$$

(4) The explicit representation is obtained as follows:

Case 1 ($s < 0$): $X(s,t) = 4(t+s) + 11s$, then $s = (X - 4t)/15$ and

$$\begin{aligned}u(x,t) &= u(11s, -s)e^{-12(t+s)} = (-s+4)e^{-12(t+s)} = (4 - \frac{(x-4t)}{15})e^{-12(t + \frac{x-4t}{15})} \\ &= (4 + \frac{(4t-x)}{15})e^{-4\frac{11t+x}{5}}, \qquad \text{if } x < 4t.\end{aligned}$$

Case 2 ($s > 0$): $X(s,t) = 4t + s$, then $s = X - 4t$, and

$$u(x,t) = u(s,0)e^{-12t} = (s+4)e^{-12t} = (4 + x - 4t)e^{-12t} \qquad \text{if } 4t < x.$$

Question 3: Consider the telegraph equation $\partial_{tt}u + 2\alpha\partial_t u + \alpha^2 u - c^2\partial_{xx}u = 0$ with $\alpha \geq 0$, $u(x,0) = 0$, $\partial_t u(x,0) = g(x)$, $x \in \mathbb{R}$, $t > 0$ and boundary condition at infinity $u(\pm\infty, t) = 0$. Solve the equation by the Fourier transform technique. (Hint: the solution to the ODE $\phi''(t) + 2\alpha\phi'(t) + (\alpha^2 + \lambda^2)\phi(t) = 0$ is $\phi(t) = \mathrm{e}^{-\alpha t}(a\cos(\lambda t) + b\sin(\lambda t))$)

Solution: Applying the Fourier transform with respect to x to the equation, we infer that

$$\begin{aligned} 0 &= \partial_{tt}\mathcal{F}(u)(\omega,t) + 2\alpha\partial_t\mathcal{F}(u)(\omega,t) + \alpha^2\mathcal{F}(u)(\omega,t) - c^2(-i\omega)^2\mathcal{F}(u)(\omega,t) \\ &= \partial_{tt}\mathcal{F}(u)(\omega,t) + 2\alpha\partial_t\mathcal{F}(u)(\omega,t) + (\alpha^2 + c^2\omega^2)\mathcal{F}(u)(\omega,t) \end{aligned}$$

Using the hint, we deduce that

$$\mathcal{F}(u)(\omega,t) = \mathsf{e}^{-\alpha t}(a(\omega)\cos(\omega ct) + b(\omega)\sin(\omega ct)).$$

The initial condition implies that $a(\omega) = 0$ and $\mathcal{F}(g)(\omega) = \omega c b(\omega)$; as a result, $b(\omega) = \mathcal{F}(g)(\omega)/(\omega c)$ and

$$\mathcal{F}(u)(\omega,t) = \mathsf{e}^{-\alpha t}\mathcal{F}(g)\frac{\sin(\omega ct)}{\omega c}.$$

Then using (2), we have

$$\mathcal{F}(u)(\omega,t) = \frac{\pi}{c}\mathsf{e}^{-\alpha t}\mathcal{F}(g)\mathcal{F}(S_{ct}).$$

The convolution theorem implies that

$$u(x,t) = \mathsf{e}^{-\alpha t}\frac{1}{2c}g * S_{ct} = \mathsf{e}^{-\alpha t}\frac{1}{2c}\int_{-\infty}^{\infty} g(y)S_{ct}(x-y)\mathsf{d}y.$$

Finally the definition of S_{ct} implies that $S_{ct}(x-y)$ is equal to 1 if $-ct < x - y < ct$ and is equal zero otherwise, which finally means that

$$u(x,t) = \mathsf{e}^{-\alpha t}\frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\mathsf{d}y.$$

Question 4: Consider the equation $-\partial_x((1+x)\partial_x u(x)) + \partial_x u(x) = f(x)$, $x \in (0,1)$ with $u(0) = \alpha$ and $-2\partial_x u(1) + u(1) = \beta$. Let $G(x, x_0)$ be the Green's function. (i) Give the integral representation of $u(x_0)$ for all $x_0 \in (0,1)$ in terms of G, f, α and β and give the equation and boundary conditions that G must satisfy. Do not compute G at this question. (Hint: The differential operator is not self-adjoint. You should find $G(0, x_0) = 0$, $\partial_x G(1, x_0) = 0$).

Solution: We multiply the PDE by $G(x, x_0)$ and integrate by parts,

$$\begin{aligned}\int_0^1 f(x)G(x,x_0)\mathsf{d}x &= \int_0^1 ((1+x)\partial_x u(x)\partial_x G(x,x_0) - u(x)\partial_x G(x,x_0))\,\mathsf{d}x + [(-(1+x)\partial_x u(x) + u(x))G(x,x_0)]_0^1 \\ &= \int_0^1 ((1+x)\partial_x u(x)\partial_x G(x,x_0) - u(x)\partial_x G(x,x_0))\,\mathsf{d}x + \beta G(1,x_0) - (-\partial_x u(0) + \alpha)G(0,x_0).\end{aligned}$$

Since $\partial_x u(0)$ is not known, we must have $G(0, x_0) = 0$. Then

$$\begin{aligned}\int_0^1 f(x)G(x,x_0)\mathsf{d}x &= \int_0^1 (-u(x)\partial((1+x)\partial_x G(x,x_0)) - u(x)\partial_x G(x,x_0))\,\mathsf{d}x + \beta G(1,x_0) + [(1+x)u(x)\partial_x G(x,x_0)]_0^1 \\ &= \int_0^1 -u(x)\left(\partial((1+x)\partial_x G(x,x_0)) + \partial_x G(x,x_0)\right)\mathsf{d}x + \beta G(1,x_0) + 2u(1)\partial_x G(1,x_0) - \alpha\partial_x G(0,x_0).\end{aligned}$$

Since $u(1)$ is not known, we must have $\partial_x G(1, x_0) = 0$. Finally G must satisfy

$$-\partial_x((1+x)\partial_x G(x,x_0)) - \partial_x G(x,x_0) = \delta(x-x_0), \quad G(0,x_0) = 0, \quad \partial_x G(1,x_0) = 0,$$

and we have the following representation for $u(x_0)$,

$$u(x_0) = \int_0^1 f(x)G(x,x_0)\mathsf{d}x - \beta G(1,x_0) + \alpha\partial_x G(0,x_0).$$

(ii) Compute $G(x,x_0)$ such that $-\partial_x((1+x)\partial_x G(x,x_0)) - \partial_x G(x,x_0) = \delta(x-x_0)$, $G(0,x_0)=0$, $\partial_x G(1,x_0)=0$, for all $x,x_0 \in (0,1)$. (Hint: observe that $(1+x)\phi'(x)+\phi(x) = ((1+x)\phi(x))'$.)

Solution: For all $x \neq x_0$ we have

$$-\partial((1+x)\partial_x G(x,x_0)) - \partial_x G(x,x_0) = -\partial((1+x)\partial_x G(x,x_0) + G(x,x_0)) = 0$$

Using the hint, this implies that

$$(1+x)\partial_x G(x,x_0) + G(x,x_0) = \partial_x((1+x)G(x,x_0)) = a,$$

In conclusion

$$G(x,x_0) = \begin{cases} \frac{ax+b}{1+x} & \text{if } x \leq x_0 \\ \frac{cx+d}{1+x} & \text{if } x_0 \leq x. \end{cases}$$

The boundary condition at 0 gives

$$G(0,x_0) = 0 = b,$$

i.e., $b=0$, $G(x,x_0) = \frac{ax}{1+x}$ if $x \leq x_0$. The boundary condition at 1 gives

$$\partial_x G(0,x_0) = \frac{c(1+0) - (c\times 0 + d)\times 1}{(1+0)^2} = 0,$$

i.e., $c=d$. $G(x,x_0)=c$, if $x_0 \leq x$. We need to impose the continuity of $G(x,x_0)$ at x_0,

$$\frac{ax_0}{1+x_0} = c,$$

which gives $ax_0 = c(1+x_0)$. The jump condition gives

$$\begin{aligned} 1 &= \lim_{\epsilon\to 0}\int_{x_0-\epsilon}^{x_0+\epsilon} (-\partial_x((1+x)\partial_x G(x,x_0)) + \partial_x G(x,x_0))\,\mathsf{d}x = (1+x_0)(\partial_x G(x_0^-,x_0) - \partial_x G(x_0^+,x_0)) \\ &= (1+x_0)(\frac{a(1+x_0)-ax_0}{(1+x_0)^2}) = \frac{a}{1+x_0}. \end{aligned}$$

In conclusion $a = (1+x_0)$ and $c = x_0$, Then

$$G(x,x_0) = \begin{cases} \frac{x(1+x_0)}{1+x} & \text{if } x \leq x_0 \\ x_0 & \text{if } x_0 \leq x. \end{cases}$$

Question 5: Consider the conservation equation $\partial_t \rho + \partial_x(\sin(\frac{\pi}{2}\rho)) = 0$, $x \in \mathbb{R}$, $t > 0$, with initial data $\rho_0(x) = 1$ if $x < 0$ and $\rho_0(x) = 0$ if $x > 0$. Draw the characteristics and give the explicit representation of the solution.

Solution: The implicit representation of the solution to the equation $\partial_t \rho + \partial_x q(\rho) = 0$, $\rho(x, 0) = \rho_0(x)$, is

$$X(s,t) = q'(\rho_0(s))t + s; \quad \rho(X(s,t),t) = \rho_0(s). \tag{4}$$

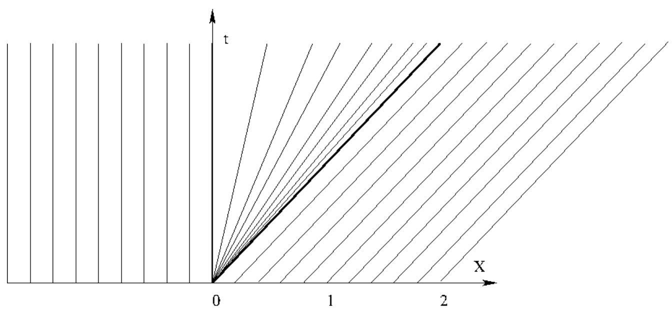
The explicit representation is obtained by expressing s in terms of X and t.
Case 1: $s < 0$, we have $\rho_0(s) = 1$, $q'(\rho_0(s)) = \frac{\pi}{2}\cos(\frac{\pi}{2}) = 0$, which implies $X = s$. Then

$$\rho(x,t) = 1 \text{ if } x < 0.$$

Case 2: $0 < s$, we have $\rho_0(s) = 0$, $q'(\rho_0(s)) = \frac{\pi}{2}\cos(0) = \frac{\pi}{2}$, $X = \frac{\pi}{2}t + s$, which means $s = X - \frac{\pi}{2}t$. Then

$$\rho(x,t) = 0 \text{ if } \frac{\pi}{2}t < x.$$

Case 3: $s = 0$. Note that there is no characteristics in the region $0 < x < \frac{\pi}{2}t$; this means that there is an expansion wave in this region.



The solution is given by setting $s = 0$ in the expression defining te characteristics: $X = q'(\rho_0)t + 0 = \frac{\pi}{2}\cos(\frac{\pi}{2}\rho_0)t$ with the constraint that $0 < \rho_0(s) < 1$. This means that

$$\rho_0 = \frac{2}{\pi}\arccos(\frac{2X}{\pi t}), \quad \text{for} \quad 0 \le X \le \frac{\pi}{2}t.$$

Finally

$$\rho(x,t) = \begin{cases} 1 & \text{if } x < 0, \\ \frac{2}{\pi}\arccos(\frac{2x}{\pi t}), & 0 \le x \le \frac{\pi}{2}t, \\ 0 & \text{if } \frac{\pi}{2}t < x. \end{cases} \tag{5}$$

Question 6: Consider the conservation equation with flux $q(\rho) = \rho^3$. Assume that the initial data is $\rho_0(x) = 2$, if $x < 0$, $\rho_0(x) = 1$, if $0 < x < 1$, and $\rho_0(x) = 0$, if $1 < x$. (i) Draw the characteristics

Solution: There are three families of characteristics.

Case 1: $s < 0$, $X(s,t) = 12t + s$. In the x-t plane, these are lines with slope $\frac{1}{12}$.

Case 2: $0 < s < 1$, $X(s,t) = 3t + s$. In the x-t plane, these are lines with slope $\frac{1}{3}$.

Case 3: $1 < s$, $X(s,t) = s$. In the x-t plane, these are vertical lines.

One shock forms between the two black characteristics and another forms between the two red characteristic (see figure).

(ii) Describe qualitatively the nature of the solution.

Solution: We have two shocks moving to the right. One shock forms between the two black characteristics and another forms between the two red characteristic (see figure).

(iii) When and where does the left shock catch up with the right one?

Solution: The speeds of the shocks are

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \frac{2^3 - 1}{2 - 1} = 7, \quad \text{and} \quad \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \frac{1 - 0}{1 - 0} = 1.$$

The location of the left shock at time t is $x_1(t) = 7t$ and that of the right shock is $x_2(t) = t + 1$. The two shocks are at the same location when $7t = t + 1$, i.e., $t = \frac{1}{6}$; the two shocks merge at $x = \frac{7}{6}$.

(iv) What is the speed of the shock once the two shocks have merged and what is the position of the shock as a function of time?

Solution: When the shocks have merged the left state is $\rho = 2$ and the right state is $\rho = 0$; as a result the speed of the shock is

$$\frac{\mathrm{d}x_3(t)}{\mathrm{d}t} = \frac{2^3 - 0}{2 - 0} = 4,$$

and the shock trajectory is $x_3(t) = 4(t - \frac{1}{6}) + \frac{7}{6}$.

(v) Draw precisely all the characteristics of the solution.

Solution: The three shocks are shown in color.

Question 7: Consider the following Cauchy-Euler problem: Let f be a smooth function in $[0, \frac{\pi}{2}]$ and find u such that $13u - 5x\partial_x u + x^2\partial_{xx}u = f(x)$, $x \in (1, e^{\frac{\pi}{2}})$, $u(1) = 0$, $u(e^{\frac{\pi}{2}}) = 0$. (i) Does this problem have a unique solution. (Hint: The solution to the homogeneous equation (i.e., $f = 0$) is $v(x) = ax^3\cos(2\log(|x|)) + bx^3\sin(2\log(|x|))$.)

Solution: Let us compute the null space of the operator

$$L : \{v \in \mathcal{C}^2(0,\pi) |\ v(\frac{\pi}{2}) = 0,\ v(0) = 0\} \ni u \longmapsto 13u - 5x\partial_x u + x^2\partial_{xx}u \in \mathcal{C}^0(0,\pi).$$

Let $N(L)$ be the null space. Let $v \in N(L)$, then

$$13v - 5x\partial_x v + x^2\partial_{xx}v = 0$$

which means that $v = ax^3\cos(2\log(|x|)) + bx^3\sin(2\log(|x|))$. The boundary conditions imply that

$$0 = a\cos(2\log(1)) + bx^3\sin(2\log(1)) = a, \quad 0 = b(\pi/2)^3\sin(2\log(e^{\frac{\pi}{2}})) = b(\pi/2)^3\sin(\pi),$$

$a = 0$ and b is an arbitrary real number; as a result $N(L) = \text{span}(x^3\sin(2\log(|x|)))$, i.e., $N(L)$ is the one-dimensional vector space spanned by the function $x^3\sin(2\log(|x|))$. Assuming that the problem has a solution, it cannot be unique. We are in the second case of Fredholm's alternative. Once $N(L^T)$ is charaterized, the above problem has a solution only if $\int_1^{\frac{\pi}{2}} vf(x)\mathrm{d}x = 0$ for all $v \in N(L^T)$. Question: What is $N(L^T)$?