M602: Methods and Applications of of Partial Differential Equations
Mid-Term TEST, September 21, 2009
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1

Let u solve $\partial_t u - \partial_x((3x+1)\partial_x u) = -3$, $x \in (0,L)$, with $\partial_x u(0,t) = 1$, $\partial_x u(L,t) = \alpha$, $u(x,0) = f(x)$.
(a) Compute $\int_0^L u(x,t)dx$ as a function of t.

Integrate the equation over the domain $(0,L)$:

$$\begin{aligned} d_t \int_0^L u(x,t)dx &= \int_0^L \partial_t u(x,t)dx = \int_0^L \partial_x((3x+1)\partial_x u)dx - 3L \\ &= (3L+1)\partial_x u(L,t) - \partial_x u(0,t) - 3L = (3L+1)\alpha - 1 - 3L \\ &= (3L+1)(\alpha-1) \end{aligned}$$

That is $d_t \int_0^L u(x,t)dx = (3L+1)(\alpha-1)$. This implies $\int_0^L u(x,t)dx = (3L+1)(\alpha-1)t + \int_0^L f(x)dx$.

(b) For which value of α the quantity $\int_0^L u(x,t)dx$ does not depend on t?

The above computation yields $\int_0^L u(x,t)dx = (3L+1)(\alpha-1)t + \int_0^L f(x)dx$. This is independent of t if $(3L+1)(\alpha-1) = 0$, meaning $\alpha = 1$.

Question 2

Consider the differential equation $-\frac{d^2\phi}{dt^2} = \lambda\phi$, $t \in (0, \pi)$, supplemented with the boundary conditions $\phi(0) = 0$, $3\phi(\pi) = -\phi'(\pi)$.

(a) Prove that it is necessary that λ be positive for a non-zero solution to exist.

(i) Let ϕ be a non-zero solution to the problem. Multiply the equation by ϕ and integrate over the domain.

$$\int_0^{\pi} (\phi'(t))^2 dt - \phi'(\pi)\phi(\pi) + \phi'(0)\phi(0) = \lambda \int_0^{\pi} \phi^2(t)dt.$$

Using the BCs, we infer

$$\int_0^{\pi} (\phi'(t))^2 dt + 3\phi(\pi)^2 = \lambda \int_0^{\pi} \phi^2(t)dt,$$

which means that λ is non-negative since ϕ is non-zero.

(ii) If $\lambda = 0$, then $\int_0^{\pi} (\phi'(t))^2 dt = 0$ and $\phi(\pi)^2 = 0$, which implies that $\phi'(t) = 0$ and $\phi(\pi) = 0$. The fundamental theorem of calculus implies $\phi(t) = \phi(\pi) + \int_{\pi}^{t} \phi'(\tau)d\tau = 0$. Hence, ϕ is zero if $\lambda = 0$. Since we want a nonzero solution, this implies that λ cannot be zero.

(iii) In conclusion, it is necessary that λ be positive for a nonzero solution to exist.

(b) Find the equation that λ must solve for the above problem to have a nonzero solution.

Since λ is positive, ϕ is of the following form

$$\phi(t) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The other boundary condition $\phi'(\pi) = -3\phi(\pi)$ implies $\sqrt{\lambda}c_2 \cos(\sqrt{\lambda}\pi) = -3c_2 \sin(\sqrt{\lambda}\pi)$. The constant c_2 cannot be zero since we want ϕ to be nonzero; as a result, λ must solve the following equation

$$\sqrt{\lambda} \cos(\sqrt{\lambda}\pi) + 3 \sin(\sqrt{\lambda}\pi) = 0,$$

for a nonzero solution ϕ to exist.

Question 3

Consider the Laplace equation $-\Delta u = 0$ in the rectangle $\{(x,y);\ x \in [0,L], y \in [0,H]\}$ with the boundary conditions $u(0,y) = 0$, $u(L,y) = 3\cos(\frac{5}{2}\pi\frac{y}{H})$, $\partial_y u(x,0) = 0$, $u(x,H) = 0$. Solve the equation using the method of separation of variables. (Give all the details.)

Let $u(x) = \phi(x)\psi(y)$. Then, provided ψ and ϕ are non zero functions, this implies $\frac{\phi''(x)}{\phi(x)} = -\frac{\psi''(y)}{\psi(y)} = \lambda$. Observe that $\psi'(0) = 0$ and $\psi(H) = 0$. The energy technique applied to $-\psi''(y) = \lambda\psi(y)$ gives

$$\int_0^H -\psi''(y)\psi(y)dy = \int_0^H \psi'(y)^2 dy - \psi'(H)\psi(H) + \psi'(0)\psi(0) = \lambda \int_0^H \psi(y)^2 y,$$

which implies $\int_0^H \psi'(y)^2 dy = \lambda \int_0^H \psi(y)^2 y$ since $\psi'(0) = 0$ and $\psi(H) = 0$. This in turn implies that λ is nonnegative. Actually λ cannot be zero since it would mean that $\psi = 0$, which would contradict the fact that the solution u is nonzero ($\lambda = 0 \Rightarrow \psi'(y) = 0 \Rightarrow \psi(y) = \psi(H) = 0$ for all $y \in [0,H]$). As a result λ is positive and $\psi(y) = a\cos(\sqrt{\lambda}y) + b\sin(\sqrt{\lambda}y)$. The Neumann condition at $y = 0$ gives $b = 0$. The Dirichlet condition at H implies $\cos(\sqrt{\lambda}H) = 0$, which implies $\sqrt{\lambda}H = (n + \frac{1}{2})\pi$, where n is any integer. This means that $\psi(y) = a\cos((n+\frac{1}{2})\pi\frac{y}{H})$. The fact that λ is positive implies $\phi(x) = c\cosh(\sqrt{\lambda}x) + d\sinh(\sqrt{\lambda}x)$. The boundary condition at $x = 0$ implies $c = 0$. Then

$$u(x,y) = A\cos((n+\tfrac{1}{2})\pi\frac{y}{H})\sinh((n+\tfrac{1}{2})\pi\frac{x}{H}).$$

The boundary condition at $x = L$ gives

$$3\cos(\frac{5}{2}\pi\frac{y}{H}) = A\cos((n+\tfrac{1}{2})\pi\frac{y}{H})\sinh((n+\tfrac{1}{2})\pi\frac{L}{H}),$$

which implies $n = 2$ and $A = \sinh^{-1}\left(\frac{\frac{5}{2}\pi L}{H}\right)$, i.e.,

$$u(x,y) = 3\frac{\sinh\left(\frac{\frac{5}{2}\pi x}{H}\right)}{\sinh\left(\frac{\frac{5}{2}\pi L}{H}\right)}\cos(\frac{5}{2}\pi\frac{y}{H}).$$

Question 4

Let $k : [-1, +1] \longrightarrow \mathbb{R}$ be such that $k(x) = 1$, if $x \in [-1, 0]$ and $k(x) = 2$ if $x \in (0, 1]$. Solve the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = 0$ with $T(-1) = 0$ and $\partial_x T(1) = 1$.
(i) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1, 0]$ and T^+ the solution on $[0, +1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 1$ and $k^+(0) = 2$.

(ii) Solve the problem, i.e., find $T(x)$, $x \in [-1, +1]$.

On $[-1, 0]$ we have $k^-(x) = 1$, which implies $\partial_{xx} T^-(x) = 0$. This in turn implies $T^-(x) = a + bx$. The Dirichlet boundary condition at $x = -1$ implies $T^-(-1) = 0 = a - b$. This gives $a = b$ and $T^-(x) = a(1 + x)$.

We proceed similarly on $[0, +1]$ and we obtain $T^+(x) = c + dx$. The Neumann boundary condition at $x = +1$ gives $\partial_x T^+(+1) = 1 = d$ and $T^+(x) = c + x$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give

$$a = c, \qquad \text{and} \qquad a = 2.$$

In conclusion

$$T(x) = \begin{cases} 2(1 + x) & \text{if } x \in [-1, 0], \\ 2 + x & \text{if } x \in [0, +1]. \end{cases}$$