Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let u solve $\partial_t u - \partial_x((\sin(x)+2)\partial_x u) = g(x)e^{-t}$, $x \in (0,L)$, with $\partial_x u(0,t) = \sin(L)+2$, $\partial_x u(L,t) = 2$, $u(x,0) = f(x)$, where f and g are two smooth functions.
(a) Compute $\frac{d}{dt}\int_0^L u(x,t)dx$ as a function of t.

Integrate the equation over the domain $(0,L)$ and apply the fundamental Theorem of calculus:

$$\begin{aligned}\frac{d}{dt}\int_0^L u(x,t)dx = \int_0^L \partial_t u(x,t)dx &= \int_0^L \partial_x((\sin(x)+2)\partial_x u)dx + e^{-t}\int_0^L g(x)dx \\ &= (\sin(L)+2)\partial_x u(L) - (\sin(0)+2)\partial_x u(0) + e^{-t}\int_0^L g(x)dx \\ &= (\sin(L)+2)2 - 2(\sin(L)+2) + e^{-t}\int_0^L g(x)dx \\ &= e^{-t}\int_0^L g(x)dx.\end{aligned}$$

That is

$$\frac{d}{dt}\int_0^L u(x,t)dx = e^{-t}\int_0^L g(x)dx.$$

(b) Use (a) to compute $\int_0^L u(x,t)dx$ as a function of t.

Applying the fundamental Theorem of calculus again gives

$$\begin{aligned}\int_0^L u(x,T)dx &= \int_0^L u(x,0)dx + \int_0^T \frac{d}{dt}\int_0^L u(x,t)dxdt \\ &= \int_0^L f(x)dx + (1-e^{-T})\int_0^L g(x)dx.\end{aligned}$$

(c) What is the limit of $\int_0^L u(x,t)dx$ as $t \to +\infty$?

The above formula gives

$$\lim_{T\to+\infty}\int_0^L u(x,T)dx = \int_0^L f(x)dx + \int_0^L g(x)dx.$$

Question 2: Consider the eigenvalue problem $-\frac{d^2}{dt^2}\phi(t) + 2\frac{d}{dt}\phi(t) = \lambda\phi(t)$, $t \in (0,\pi)$, supplemented with the boundary condition $\phi(0) = 0$, $\phi(\pi) = 0$. (Hint: $2\phi(t)\frac{d}{dt}\phi(t) = \frac{d}{dt}\phi^2(t)$.)
(a) Prove that it is necessary that λ be positive for a non-zero solution to exist.

(i) Let ϕ be a non-zero solution to the problem. Multiply the equation by ϕ and integrate over the domain. Use the fundamental Theorem of calculus and use the hint to obtain

$$\int_0^\pi (\phi'(t))^2 dt - \phi'(\pi)\phi(\pi) + \phi'(0)\phi(0) + \int_0^\pi \frac{d}{dt}(\phi^2(t))dt = \lambda \int_0^\pi \phi^2(t)dt$$
$$\int_0^\pi (\phi'(t))^2 dt - \phi'(\pi)\phi(\pi) + \phi'(0)\phi(0) + \phi^2(\pi) - \phi^2(0) = \lambda \int_0^\pi \phi^2(t)dt$$

Using the boundary conditions, we infer

$$\int_0^\pi (\phi'(t))^2 dt = \lambda \int_0^\pi \phi^2(t)dt,$$

which means that λ is non-negative since ϕ is non-zero.

(ii) If $\lambda = 0$, then $\int_0^\pi (\phi'(t))^2 dt = 0$ and $\phi(\pi)^2 = 0$, which implies that $\phi'(t) = 0$. The fundamental theorem of calculus implies $\phi(t) = \phi(0) + \int_0^t \phi'(\tau)d\tau = 0$. Hence, ϕ is zero if $\lambda = 0$. Since we want a nonzero solution, this implies that λ cannot be zero.

(iii) In conclusion, it is necessary that λ be positive for a nonzero solution to exist.

(b) The general solution to $-\phi'' + 2\phi' = \lambda\phi$ is $\phi(t) = e^t(c_1\cos(t\sqrt{\lambda-1}) + c_2\sin(t\sqrt{\lambda-1}))$ for $\lambda \geq 1$. Find all the eigenvalues $\lambda \geq 1$ and the associated nonzero eigenfunctions.

Since $\lambda \geq 1$ by hypothesis, ϕ is of the following form

$$\phi(t) = e^t(c_1\cos(\sqrt{\lambda-1}t) + c_2\sin(\sqrt{\lambda-1}t)).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The other boundary condition implies $\phi(\pi) = 0 = e^\pi c_2 \sin(\sqrt{\lambda-1}\pi)$. The constant c_2 cannot be zero since we want ϕ to be nonzero; as a result, $\sqrt{\lambda-1} = n$, $n = 1, 2, \ldots$. In conclusion

$$\lambda = n^2 + 1, \quad n = 1, 2, \ldots, \qquad \phi(t) = ce^t\sin(nt)$$

Question 3: Consider the equation $-\Delta u = 0$ in the rectangle $\{(x,y);\ x \in [0,L], y \in [0,H]\}$ with the boundary conditions $u(0,y) = 0$, $u(L,y) = -5\cos(\frac{3}{2}\pi\frac{y}{H})$, $\partial_y u(x,0) = 0$, $u(x,H) = 0$. Solve the equation using the method of separation of variables. (Give all the details.)

Let $u(x) = \phi(x)\psi(y)$. Then, provided ψ and ϕ are non zero functions, this implies $\frac{\phi"(x)}{\phi(x)} = -\frac{\psi"(y)}{\psi(y)} = \lambda$. Observe that $\psi'(0) = 0$ and $\psi(H) = 0$. The energy technique applied to $-\psi"(y) = \lambda\psi(y)$ gives

$$\int_0^H -\psi"(y)\psi(y)dy = \int_0^H \psi'(y)^2 dy - \psi'(H)\psi(H) + \psi'(0)\psi(0) = \lambda \int_0^H \psi(y)^2 y,$$

which implies $\int_0^H \psi'(y)^2 dy = \lambda \int_0^H \psi(y)^2 dy$ since $\psi'(0) = 0$ and $\psi(H) = 0$. This in turn implies that λ is nonnegative. Actually λ cannot be zero since it would mean that $\psi = 0$, which would contradict the fact that the solution u is nonzero ($\lambda = 0 \Rightarrow \psi'(y) = 0 \Rightarrow \psi(y) = \psi(H) = 0$ for all $y \in [0,H]$). As a result λ is positive and

$$\psi(y) = a\cos(\sqrt{\lambda}y) + b\sin(\sqrt{\lambda}y).$$

The Neumann condition at $y = 0$ gives $b = 0$. The Dirichlet condition at H implies $\cos(\sqrt{\lambda}H) = 0$, which implies $\sqrt{\lambda}H = (n+\frac{1}{2})\pi$, where n is any integer. This means that $\psi(y) = a\cos((n+\frac{1}{2})\pi\frac{y}{H})$. The fact that λ is positive implies $\phi(x) = c\cosh(\sqrt{\lambda}x) + d\sinh(\sqrt{\lambda}x)$. The boundary condition at $x = 0$ implies $c = 0$. Then

$$u(x,y) = A\cos((n+\tfrac{1}{2})\pi\frac{y}{H})\sinh((n+\tfrac{1}{2})\pi\frac{x}{H}).$$

The boundary condition at $x = L$ gives

$$-5\cos(\frac{3}{2}\pi\frac{y}{H}) = A\cos((n+\tfrac{1}{2})\pi\frac{y}{H})\sinh((n+\tfrac{1}{2})\pi\frac{L}{H}),$$

which, by identification, implies $1 = 2$ and $A = -5\sinh^{-1}\left(\frac{\frac{3}{2}\pi L}{H}\right)$, i.e.,

$$u(x,y) = -5\frac{\sinh\left(\frac{\frac{3}{2}\pi x}{H}\right)}{\sinh\left(\frac{\frac{3}{2}\pi L}{H}\right)}\cos(\frac{3}{2}\pi\frac{y}{H}).$$

Question 4: Let $k : [-1,+1] \longrightarrow \mathbb{R}$ be such that $k(x) = 1$, if $x \in [-1,0]$ and $k(x) = 2$ if $x \in (0,1]$. Solve the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = 0$ with $\partial_x T(-1) = T(-1)$ and $\partial_x T(1) = 1$.

(i) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1,0]$ and T^+ the solution on $[0,+1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 1$ and $k^+(0) = 2$.

(ii) Solve the problem, i.e., find $T(x)$, $x \in [-1,+1]$.

On $[-1,0]$ we have $k^-(x) = 1$, which implies $\partial_{xx}T^-(x) = 0$. This in turn implies $T^-(x) = a + bx$. The Robin boundary condition at $x = -1$ implies $\partial_x T^{-1}(-1) - T^-(-1) = 0 = 2b - a$. This gives $a = 2b$ and $T^-(x) = b(2 + x)$.

We proceed similarly on $[0,+1]$ and we obtain $T^+(x) = c + dx$. The Neumann boundary condition at $x = +1$ gives $\partial_x T^+(+1) = 1 = d$ and $T^+(x) = c + x$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give

$$2b = c, \qquad \text{and} \qquad b = 2.$$

In conclusion

$$T(x) = \begin{cases} 2(2+x) & \text{if } x \in [-1,0], \\ 4 + x & \text{if } x \in [0,+1]. \end{cases}$$

Question 5: Consider $(1+x)\phi(x)+\partial_x\phi(x)-\partial_x((2+x^2)\partial_x\phi) = f(x)$, $x \in (0,1)$ with $\phi(0)=1$, $\partial_x\phi(1) = 2$. Assume that ϕ_1 and ϕ_2 are two smooth solutions ($\phi_1 \in \mathcal{C}^2([0,1])$ and $\phi_2 \in \mathcal{C}^2([0,1])$). Use the energy argument to prove that $\phi_1 = \phi_2$. (Hint: $2\phi(x)\partial_x\phi(x) = \partial_x\phi^2(x)$.)

The difference $\phi := \phi_1 - \phi_2$ satisfies

$$(1+x)\phi(x)+\partial_x\phi(x)-\partial_x((2+x^2)\partial_x\phi) = 0, \quad x \in (0,1), \quad \text{and} \quad \phi(0)=1, \quad \partial_x\phi(1)=0.$$

Multiply this equation by ϕ and integrate over the domain to obtain

$$\begin{aligned}0 &= \int_0^1 (1+x)\phi(x)^2 dx + \int_0^1 \phi(x)\partial_x\phi(x)dx - \int_0^1 \phi(x)\partial_x((2+x^2)\partial_x\phi(x))dx \\ &= \int_0^1 (1+x)\phi(x)^2 dx + \frac{1}{2}\int_0^1 \partial_x(\phi(x)^2)dx + \int_0^1 (2+x^2)\partial_x\phi(x)\partial_x\phi(x)dx - [(2+x^2)\partial_x\phi(x)]|_0^1.\end{aligned}$$

The boundary term $[(2+x^2)\partial_x\phi(x)]|_0^1$ is zero owing to the boundary conditions. We apply the fundamental Theorem of calculus one more time to obtain

$$\begin{aligned}0 &= \int_0^1 (1+x)\phi(x)^2 dx + \frac{1}{2}[\phi(x)^2]|_0^1 + \int_0^1 (2+x^2)[\partial_x\phi(x)]^2 dx \\ &= \int_0^1 (1+x)\phi(x)^2 dx + \frac{1}{2}\phi(1)^2 + \int_0^1 (2+x^2)[\partial_x\phi(x)]^2 dx.\end{aligned}$$

Since all the terms are non negative, this means that all the terms are zero:

$$0 = \int_0^1 (1+x)\phi(x)^2 dx, \quad 0 = \frac{1}{2}\phi(1), \quad 0 = \int_0^1 (2+x^2)[\partial_x\phi(x)]^2 dx.$$

This means in particular that

$$0 = (1+x)\phi(x)^2, \quad \text{for all } x \in (0,1).$$

Since $1+x \geq 1$ in the interval $(0,1)$, this implies that $\phi(x)=0$ for all $x \in (0,1)$. In conclusion $\phi_1 = \phi_2$.

Question 6: Consider the disk D centered at $(0,0)$ of radius 1. Let $f(x,y) = x^2 - y^2 + 4y - 3$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ solve $-\nabla^2 u = 0$ in D and $u|_{\partial D} = f$. Compute $\min_{(x,y)\in\overline{D}} u(x,y)$ and $\max_{(x,y)\in\overline{D}} u(x,y)$.

We use the maximum principle (u is harmonic and has the required regularity). Then

$$\min_{(x,y)\in\overline{D}} u(x,y) = \min_{(x,y)\in\partial D} f(x,y), \quad \text{and} \quad \max_{(x,y)\in\overline{D}} u(x,y) = \max_{(x,y)\in\partial D} f(x,y).$$

A point (x,y) is at the boundary of D if and only if $x^2 + y^2 = 1$; as a result, the following holds for all $(x,y) \in \partial D$:

$$f(x,y) = x^2 - y^2 + 4y - 3 = 1 - y^2 - y^2 + 4y - 3 = -2(1-y)^2.$$

We obtain

$$\min_{(x,y)\in\partial D} f(x,y) = \min_{-1\le y\le 1} -2(1-y)^2 = -8, \quad \max_{(x,y)\in\partial D} f(x,y) = \max_{-1\le y\le 1} -2(1-y)^2 = 0.$$

In conclusion

$$\min_{(x,y)\in\overline{D}} u(x,y) = -8, \quad \max_{(x,y)\in\overline{D}} u(x,y) = 0$$
