Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. **Answers with no justification will not be graded**.

Question 1: Let u be a solution to the PDE $\partial_t u(x,t)+\frac{1}{2}\partial_x u^2(x,t)-\nu\partial_{xx}u(x,t)=0$, $x\in(-\infty,+\infty)$, $t>0$. (a) Let $\psi(x,t)=\int_{-\infty}^{x}\partial_t u(\xi,t)\mathrm{d}\xi+\frac{1}{2}u^2(x,t)-\nu\partial_x u(x,t)$. Compute $\partial_x\psi(x,t)$.

The definition of ψ implies that

$$\begin{aligned}\partial_x\psi(x,t) &= \partial_x(\int_{-\infty}^{x}\partial_t u(\xi,t)\mathsf{d}x+\frac{1}{2}u^2(x,t)-\nu\partial_x u(x,t))\\ &= \partial_t u(x,t)+\frac{1}{2}\partial_x u^2(x,t)-\nu\partial_{xx}u(x,t)=0\end{aligned}$$

i.e., $\partial_x\psi(x,t)=0$. This means that ψ depends on t only.

(b) Let $\phi(x,t):=\mathrm{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x}u(\xi,t)\mathrm{d}\xi}$. Compute $\partial_t\phi$, $\partial_x\phi$, and $\partial_{xx}\phi$.

The definition of ϕ, together with the chain rule, implies that

$$\begin{aligned}\partial_t\phi(x,t) &= \partial_t\left(-\frac{1}{2\nu}\int_{-\infty}^{x}u(\xi,t)\mathsf{d}\xi\right)\mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x}u(\xi,t)\mathsf{d}\xi}\\ &= \left(-\frac{1}{2\nu}\int_{-\infty}^{x}\partial_t u(\xi,t)\mathsf{d}\xi\right)\mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x}u(\xi,t)\mathsf{d}\xi}\end{aligned}$$

and

$$\begin{aligned}\partial_x\phi(x,t) &= \partial_x\left(-\frac{1}{2\nu}\int_{-\infty}^{x}u(\xi,t)\mathsf{d}\xi\right)\mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x}u(\xi,t)\mathsf{d}\xi}\\ &= \left(-\frac{1}{2\nu}u(x,t)\right)\mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x}u(\xi,t)\mathsf{d}\xi}\end{aligned}$$

and

$$\partial_{xx}\phi(x,t)=\left(-\frac{1}{2\nu}\partial_x u(x,t)\right)\mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x}u(\xi,t)\mathsf{d}\xi}+\left(-\frac{1}{2\nu}u(x,t)\right)^2\mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x}u(\xi,t)\mathsf{d}\xi}$$

(c) Compute $\partial_t\phi - \nu\partial_{xx}\phi$, assuming $\psi(x,t) = 0$.

The above computations give

$$-\nu\partial_{xx}\phi(x,t) = -\frac{1}{2\nu}\left(-\nu\partial_x u(x,t) + \frac{1}{2}u^2(x,t)\right)\mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathsf{d}\xi}$$

In conclusion

$$\begin{aligned}\partial_t\phi - \nu\partial_{xx}\phi &= -\frac{1}{2\nu}\left(\int_{-\infty}^{x}\partial_t u(\xi,t)\mathsf{d}\xi + \frac{1}{2}u^2(x,t) - \nu\partial_x u(x,t)\right)\mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathsf{d}\xi} \\ &= -\frac{1}{2\nu}\psi(x,t)\mathsf{e}^{-\frac{1}{2\nu}\int_{-\infty}^{x} u(\xi,t)\mathsf{d}\xi}.\end{aligned}$$

This means $\partial_t\phi - \nu\partial_{xx}\phi = 0$.

Question 2: Consider the vibrating beam equation $\partial_{tt}u(x,t) + \partial_{xxxx}u(x,t) = 0$, $x \in (-\infty,+\infty)$, $t > 0$ with $u(\pm\infty,t) = 0$, $\partial_x u(\pm\infty,t) = 0$, $\partial_{xx}u(\pm\infty,t) = 0$. Use the energy method to compute $\partial_t\int_{-\infty}^{+\infty}([\partial_t u(x,t)]^2 + [\partial_{xx}u(x,t)]^2)\mathrm{d}x$. (Hint: test the equation with $\partial_t u(x,t)$).

Using the hint we have

$$0 = \int_{-\infty}^{+\infty}(\partial_{tt}u(x,t)\partial_t u(x,t) + \partial_{xxxx}u(x,t)\partial_t u(x,t))\mathsf{d}x$$

Using the product rule, $a\partial_t a = \frac{1}{2}\partial_t a^2$ where $a = \partial_t u(x,t)$, and integrating by parts two times (i.e., applying the fundamental theorem of calculus) we obtain

$$\begin{aligned}0 &= \int_{-\infty}^{+\infty}(\frac{1}{2}\partial_t(\partial_t u(x,t))^2 - \partial_{xxx}u(x,t)\partial_t\partial_x u(x,t))\mathsf{d}x \\ &= \int_{-\infty}^{+\infty}(\partial_t\frac{1}{2}(\partial_t u(x,t))^2 + \partial_{xx}u(x,t)\partial_t\partial_{xx}u(x,t))\mathsf{d}x.\end{aligned}$$

We apply again the product rule $a\partial_t a = \frac{1}{2}\partial_t a^2$ where $a = \partial_{xx}u(x,t)$,

$$0 = \int_{-\infty}^{+\infty}(\partial_t\frac{1}{2}(\partial_t u(x,t))^2 + \frac{1}{2}\partial_t(\partial_{xx}u(x,t))^2)\mathsf{d}x.$$

Switching the derivative with respect to t and the integration with respect to x, this finally gives

$$0 = \frac{1}{2}\partial_t\int_{-\infty}^{+\infty}([\partial_t u(x,t)]^2 + [\partial_{xx}u(x,t)]^2)\mathsf{d}x.$$

Question 3: Let $k, f : [-1,+1] \longrightarrow \mathbb{R}$ be such that $k(x) = 2$, $f(x) = 0$ if $x \in [-1, 0]$ and $k(x) = 1$, $f(x) = 2$ if $x \in (0, 1]$. Consider the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = f(x)$ with $T(-1) = -2$ and $T(1) = 2$.

(a) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1, 0]$ and T^+ the solution on $[0,+1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 2$ and $k^+(0) = 1$.

(b) Solve the problem, i.e., find $T(x)$, $x \in [-1,+1]$.

On $[-1, 0]$ we have $k^-(x) = 2$ and $f^-(x) = 0$ which implies $-\partial_{xx}T^-(x) = 0$. This in turn implies $T^-(x) = ax + b$. The Dirichlet condition at $x = -1$ implies that $T^-(-1) = -2 = -a + b$. This gives $a = b + 2$ and $T^-(x) = (b + 2)x + b$.

We proceed similarly on $[0,+1]$ and we obtain $-\partial_{xx}T^-(x) = 2$, which implies that $T^+(x) = -x^2 + cx + d$. The Dirichlet condition at $x = 1$ implies $T^+(1) = 2 = -1 + c + d$. This gives $c = 3 - d$ and $T^-(x) = -x^2 + (3 - d)x + d$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give $b = d$ and $2(b + 2) = 3 - d$, respectively. In conclusion $b = -\frac{1}{3}$, $d = -\frac{1}{3}$ and

$$T(x) = \begin{cases} \frac{5}{3}x - \frac{1}{3} & \text{if } x \in [-1, 0], \\ -x^2 + \frac{10}{3}x - \frac{1}{3} & \text{if } x \in [0, 1]. \end{cases}$$

Question 4: Let $\mathrm{CS}(f) = \frac{\pi^2}{6} - 2(\cos(x) - \frac{\cos(2x)}{2^2} + \frac{\cos(3x)}{3^2} - \frac{\cos(4x)}{4^2} \ldots)$ be the Fourier cosine series of the function $f(x) := \frac{1}{2}x^2$ defined over $[0,+\pi]$.

(a) For which values of x in $[0,+\pi]$ does this series coincide with $f(x)$? (Explain).

The Fourier cosine series coincides with the function $f(x)$ over the entire interval $[0,+\pi]$ since f is smooth over $[0,+\pi]$ (recall that the Fourier cosine series is the Fourier series of the even extension of f over $[-\pi,+\pi]$).

(b) Compute the Fourier sine series, $\mathrm{SS}(x)$, of the function $g(x) := x$ defined over $[0,+\pi]$.

We know from class that it is always possible to obtain a Fourier sine series by differentiating term by term a Fourier cosine series, in other words

$$\mathsf{SS}(x) = \partial_x \mathsf{CS}(\frac{1}{2}x^2) = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} \ldots \frac{\sin(nx)}{n} \ldots\right).$$

(c) For which values of $x \in [0, +\pi]$ does the Fourier sine series of g coincide with $g(x)$?.

The Fourier sine series coincides with the function $g(x) := x$ over the interval $[0, +\pi)$ since g is smooth over $[0, +\pi]$and $g(0) = 0$. The Fourier sine series of g is zero at $+\pi$, and thus differs from $g(+\pi)$.

Question 5: Using cylindrical coordinates and the method of separation of variables, solve the equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0, \frac{3}{2}\pi],\ r \in [0, 3]\}$, subject to the boundary conditions $u(r, 0) = 0$, $u(r, \frac{3}{2}\pi) = 0$, $u(3, \theta) = 18\sin(2\theta)$. (Give all the details.)

(1) We set $u(r, \theta) = \phi(\theta)g(r)$. This means $\phi'' = -\lambda\phi$, with $\phi(0) = 0$ and $\phi(\frac{3}{2}\pi) = 0$, and $r\frac{\mathsf{d}}{\mathsf{d}r}(r\frac{\mathsf{d}}{\mathsf{d}r}g(r)) = \lambda g(r)$.

(2) The usual energy argument applied to the two-point boundary value problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = 0, \qquad \phi(\frac{3}{2}\pi) = 0,$$

implies that λ is non-negative. If $\lambda = 0$, then $\phi(\theta) = c_1 + c_2\theta$ and the boundary conditions imply $c_1 = c_2 = 0$, i.e., $\phi = 0$, which in turns gives $u = 0$ and this solution is incompatible with the boundary condition $u(3, \theta) = 18\sin(2\theta)$. Hence $\lambda > 0$ and

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta).$$

(3) The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The boundary condition $\phi(\frac{3}{2}\pi) = 0$ implies $\sqrt{\lambda}\frac{3}{2}\pi = n\pi$ with $n \in \mathbb{N} \setminus \{0\}$. This means $\sqrt{\lambda} = \frac{2}{3}n$, $n = 1, 2, \ldots$.

(4) From class we know that $g(r)$ is of the form r^α, $\alpha \geq 0$. The equality $r\frac{\mathsf{d}}{\mathsf{d}r}(r\frac{\mathsf{d}}{\mathsf{d}r}r^\alpha) = \lambda r^\alpha$ gives $\alpha^2 = \lambda$. The condition $\alpha \geq 0$ implies $\frac{2}{3}n = \alpha = \sqrt{\lambda}$. The boundary condition at $r = 3$ gives $18\sin(2\theta) = c_2 3^{\frac{2}{3}n}\sin(\frac{2}{3}n\theta)$ for all $\theta \in [0, \frac{3}{2}\pi]$. This implies $n = 3$ and $c_2 = 2$.

(5) Finally, the solution to the problem is

$$u(r, \theta) = 2r^2\sin(2\theta).$$

Question 6: Let $p, q : [-1, +1] \longrightarrow \mathbb{R}$ be smooth functions. Assume that $p(x) \geq 0$ and $q(x) \geq q_0$ for all $x \in [-1, +1]$, where $q_0 \in \mathbb{R}$. Consider the eigenvalue problem $-\partial_x(p(x)\partial_x\phi(x)) + q(x)\phi(x) = \lambda\phi(x)$, supplemented with the boundary conditions $\phi(-1) = 0$ and $\phi(1) = 0$.
(a) Prove that it is necessary that $\lambda \geq q_0$ for a non-zero (smooth) solution, ϕ, to exist. (Hint: $q_0 \int_{-1}^{+1} \phi^2(x)\mathrm{d}x \leq \int_{-1}^{+1} q(x)\phi^2(x)\mathrm{d}x$.)

As usual we use the energy method. Let (ϕ, λ) be an eigenpair, then

$$\int_{-1}^{+1} (-\partial_x(p(x)\partial_x\phi(x))\phi(x) + q(x)\phi^2(x))\mathrm{d}x = \lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

After integration by parts and using the boundary conditions, we obtain

$$\int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q(x)\phi^2(x))\mathrm{d}x = \lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

which, using the hint, can also be re-written

$$\int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q_0\phi^2(x))\mathrm{d}x \leq \lambda \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

Then

$$\int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x \leq (\lambda - q_0) \int_{-1}^{+1} \phi^2(x)\mathrm{d}x.$$

Assume that ϕ is non-zero, then

$$\lambda - q_0 \geq \frac{\int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x}{\int_{-1}^{+1} \phi^2(x)\mathrm{d}x} \geq 0,$$

which proves that it is necessary that $\lambda \geq q_0$ for a non-zero (smooth) solution to exist.

(b) Assume that $p(x) \geq p_0 > 0$ for all $x \in [-1, +1]$ where $p_0 \in \mathbb{R}_+$. Show that $\lambda = q_0$ cannot be an eigenvalue, i.e., prove that $\phi = 0$ if $\lambda = q_0$. (Hint: $p_0 \int_{-1}^{+1} \psi^2(x)\mathrm{d}x \leq \int_{-1}^{+1} p(x)\psi^2(x)\mathrm{d}x$.)

Assume that $\lambda = q_0$ is an eigenvalue. Then the above computation shows that

$$p_0 \int_{-1}^{+1} (\partial_x\phi(x))^2\mathrm{d}x \leq \int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathrm{d}x = 0,$$

which means that $\int_{-1}^{+1} (\partial_x\phi(x))^2\mathrm{d}x = 0$ since $p_0 > 0$. As a result $\partial_x\phi = 0$, i.e., $\phi(x) = c$ where c is a constant. The boundary conditions $\phi(-1) = 0 = \phi(1)$ imply that $c = 0$. In conclusion $\phi = 0$ if $\lambda = q_0$, thereby proving that (ϕ, q_0) is not an eigenpair.

Question 7: Use the Fourier transform technique to solve $\partial_t u(x,t)+\sin(t)\partial_x u(x,t)+(2+3t^2)u(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = u_0(x)$. (Use the shift lemma: $\mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)e^{i\omega\beta}$ and the definition $\mathcal{F}(f)(\omega) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx$)

Applying the Fourier transform to the equation gives

$$\partial_t \mathcal{F}(u)(\omega,t) + \sin(t)(-i\omega)\mathcal{F}(u)(\omega,t) + (2+3t^2)\mathcal{F}(u)(\omega,t) = 0$$

This can also be re-written as follows:

$$\frac{\partial_t \mathcal{F}(u)(\omega,t)}{\mathcal{F}(u)(\omega,t)} = i\omega\sin(t) - (2+3t^2).$$

Then applying the fundamental theorem of calculus between 0 and t, we obtain

$$\log(\mathcal{F}(u)(\omega,t)) - \log(\mathcal{F}(u)(\omega,0)) = -i\omega(\cos(t)-1) - (2t+t^3).$$

This implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{-i\omega(\cos(t)-1)}e^{-(2t+t^3)}.$$

Then the shift lemma gives

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0(x+\cos(t)-1)(\omega)e^{-(2t+t^3)}.$$

This finally gives

$$u(x,t) = u_0(x+\cos(t)-1)e^{-(2t+t^3)}.$$
